THE DYNAMICS OF NON-HYPERBOLIC TORAL AUTOMORPHISMS

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Abstract. The dynamics of non hyperbolic automorphisms on the two-dimensional torus are classified up to topological conjugacy.

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1. INTRODUCTION

In [2] Devaney introduces hyperbolic toral automorphisms and shows that the underlying dynamics under iteration are chaotic on the whole torus. This raises the question of what happens if the maps are not hyperbolic. In this paper we completely classify the dynamics in the non-hyperbolic case up to topological conjugacy using elementary means.

Although it turns out that none of the non-hyperbolic automorphisms are chaotic in the sense of Devaney, we show that these maps share at least some of the dynamic structure of their chaotic counterparts. In some cases we find that these non-hyperbolic maps exhibit sensitive dependence on initial conditions, have periodic points of all periods in addition to having a dense set of periodic points. Other non-hyperbolic automorphisms act like rotations about a fixed point. It is shown that these rotations have period one, two, three, four or six. While exhibiting these properties, we briefly describe a interesting connection to number theory.

It should be noted that in [1] that Adler, Tresser and Worfolk consider the more general question of classifying all toral automorphisms up to topological conjugacy. This is a much harder problem than the one we consider and, not surprisingly, the mathematics required to solve the general problem is more sophisticated.

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2. The torus and automorphisms

2.1. **The torus.** In order to describe the torus, we begin with the plane. All points whose coordinates differ by integers we consider identical. For example, the point (x, y) in the plane is considered the same as the points $(x+1, y)$, $(x+6, y+3)$ and in general $(x+M, y+N)$, where M and N represent integers. Let $[x, y]$ denote the set of all points equivalent to (x, y) under this relation. More formally, the relation $(x, y) \sim (x', y')$ if and only if $x - x'$ and $y - y'$ are integers, this gives an equivalence relation on the points in the plane. The torus is the set of all equivalence classes under this relation. This is because there is a one-to-one correspondence between equivalence classes and points in the unit square if we also identify points that lie in the opposite edges. Figure 1. illustrates the geometric interpretation of the construction of a torus from the unit square by first gluing one set of opposite edges and then the other.

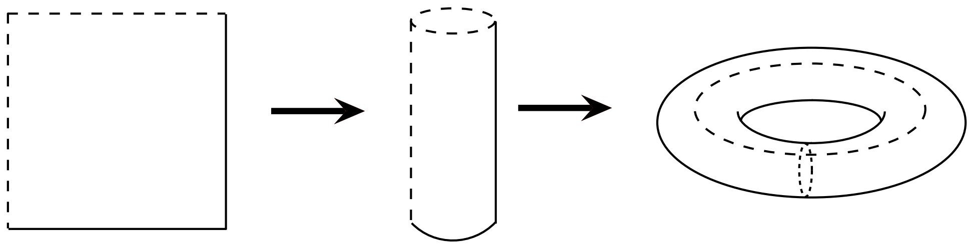

Figure 1. Construction of a torus from a unit square

2.2. **Automorphisms.** Let T denote the torus, and let π be the natural projection of $\mathbf{R}^2$ onto T, i.e. $\pi(x, y) = [x, y] = \pi(x+M, y+N)$. Certain dynamical systems on the torus can be most efficiently described in the plane and then projected onto the torus. As an example, suppose $F : \mathbf{R}^2 \to \mathbf{R}^2$ has the property that $F\begin{pmatrix} x \\ y \end{pmatrix} - F\begin{pmatrix} x+M \\ y+N \end{pmatrix}$ belongs to the integer lattice for all points in the plane and all integers M and N. It follows that $\pi \circ F\begin{pmatrix} x \\ y \end{pmatrix} = \pi \circ F\begin{pmatrix} x+M \\ y+N \end{pmatrix}$ so that F induces a well-defined map $\hat{F}$ on the torus. As another example, if L is a linear map whose matrix representation is an integer matrix with $det(L) = \pm 1$, then both $\hat{L}$ and $\hat{L}^{-1}$ are well-defined on T. We will restrict our attention to these automorphisms that come from matrices and call $\hat{L}$ and $\hat{L}^{-1}$ *toral automorphisms*.

The set of matrices that have integer entries and determinant either plus one or minus one is denoted $GL(2, \mathbb{Z})$. These are exactly the set of matrices that correspond to toral automorphisms.

Occasionally we will restrict our attention to the set of matrices that have integer entries and determinant one. This set we will denote by $SL(2, \mathbb{Z})$.

A basic result, stated below, shows that the set of periodic points of a toral automorphism is dense in the torus.

Theorem 1. *Per* $(\hat{L})$ *is dense in* T.

This theorem appears in [1]. The proof from Devaney looks at the rational coordinates p of T, which may be represented as $[\frac{x}{k}, \frac{y}{k}]$, where x, y and k are all integers. The claim is that p is periodic with period less than or equal to k^2. We first note there are exactly k^2 points in T of the form $[\frac{x}{k}, \frac{y}{k}]$ with $0 \leqslant x, y < k$. Moreover, the image of any such point under $\hat{L}$ may also be written in this form, since the entries of A are integers, this means that $\hat{L}$ permutes these points. Therefore there exist integers i and j such that $\hat{L}^i(p) = \hat{L}^j(p)$ and $|i - j| \leqslant k^2$. Applying $\hat{L}^{-i}$ shows that p is periodic of period less than or equal to k^2

Definition 1. *An invertible linear map is non-hyperbolic if its eigenvalues lie on the unit circle.*

Definition 2. *Let* $A \in GL(2, \mathbb{Z})$ *and* A *is non-hyperbolic. Then the map induced on the torus by* A *is called a non-hyperbolic toral automorphism, and will be denoted by* L_A.

An important aspect to understanding the dynamics of these maps is the concept of topological conjugacy. If L_A and L_B are maps induced on the torus by matrices A and B. Then L_A and L_B are topologically conjugate over $\mathbb{Z}$ if there exists a homeomorphism f of the torus such that $L_A = f^{-1} \circ L_B \circ f$. Mappings which are topologically conjugate are equivalent in terms of their dynamics.

A related concept is matrix similarity. A matrix A is similar to a matrix B over the integers if there exists a matrix $C \in GL(2, \mathbb{Z})$ such that $A = C^{-1}BC$. In this case, the map L_C is a homeomorphism and so we have L_A is topologically conjugate to L_B. But it should be noted that not all continuous maps on the torus come from matrices and so though matrix similarity implies topological conjugacy the converse implication is not necessarily true.

Later we will use the fact that maps with different dynamics cannot be topologically conjugate and so that the underlying matrices cannot be similar.

3. The characteristic equations of non-hyperbolic automorphisms

It is well known that the characteristic polynomial of A has the form $x^2 - trace(A)x + det(A) = 0$. From the above definitions we know all non-hyperbolic automorphisms have either two real eigenvalues equal to one of the following pairs $(1, 1)$, $(1, -1)$, or $(-1, -1)$; or two complex eigenvalues where $\sqrt{trace(A)^2 - 4(\pm 1)}$ cannot be real. In the complex case we must have $det(A) = 1$ and this implies that $trace(A) = 0$ or $trace(A) = \pm 1$. From the above, we obtain the six characteristic equations for all the non-hyperbolic toral automorphisms. Which are:

1. $x^2 - 2x + 1 = 0$
2. $x^2 + 2x + 1 = 0$
3. $x^2 - 1 = 0$
4. $x^2 + 1 = 0$
5. $x^2 + x + 1 = 0$
6. $x^2 - x + 1 = 0$

We note that x^2+1 is a factor of x^4-1, that x^2+x+1 is a factor of x^3-1 and that x^2-x+1 is a factor of x^6-1. (The equations (4), (5) and (6) are cyclotomic polynomials.) So that if a matrix A has characteristic equation (3), (4), (5) or (6) it will satisfy $A^2=I$, $A^4=I$, $A^3=I$, $A^6=I$, respectively.

4. Similarity classes of non-hyperbolic matrices

4.1. **Some basic matrices.** The next step in our classification is to look at matrices that satisfy these characteristic equations and to classify them up to similarity over $GL(2,\mathbb{Z})$. We give an elementary proof. A more general theory involving algebraic number theory can be found in [4]. (There is a close connection to number theory which we illustrate with an example when we look at the characteristic equation $x^2+1=0$.)

We will introduce various matrices that will be used throughout the rest of the paper. First, let $J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Notice that $J^{-1}\begin{pmatrix}a&b\\c&d\end{pmatrix}J=\begin{pmatrix}a&-b\\-c&d\end{pmatrix}$. So conjugating by J preserves the sign of the entries on the main diagonal and changes the sign of the other entries.

Next, let $K=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Now $K^{-1}\begin{pmatrix}a&b\\c&d\end{pmatrix}K=\begin{pmatrix}d&c\\b&a\end{pmatrix}$. So conjugating by K flips the entries on both diagonals.

Finally we introduce a one-parameter family of matrices $M_n=\begin{pmatrix}1&n\\0&1\end{pmatrix}$. Notice that $M_n^{-1}\begin{pmatrix}a&b\\c&d\end{pmatrix}M_n=\begin{pmatrix}a-nc&na+b-n^2c-nd\\c&nc+d\end{pmatrix}$. We will also use the transpose of these matrices.

$(M_n^T)^{-1}\begin{pmatrix}a&b\\c&d\end{pmatrix}M_n^T=\begin{pmatrix}a+nb&b\\-na-n^2b+c+nd&-nb+d\end{pmatrix}$.

4.2. **Characteristic equation $\mathbf{x^2-2x+1=0}$.** We will show that any matrix in $GL(2,\mathbb{Z})$ that has characteristic equation $x^2-2x+1=0$ is similar over the integers to M_n for some $n\geq 0$. We will then show that if $m\neq n$ that M_n is not similar to M_m.

Suppose $A\in GL(2,\mathbb{Z})$ has characteristic equation $x^2-2x+1=0$. Since the trace is 2 we can write $A=\begin{pmatrix}k&l\\m&(2-k)\end{pmatrix}$. Calculating the determinant gives $k(2-k)-lm=1$. Re-arranging, gives $ml=-(k-1)^2$.

If $k=1$ then at least one of m and l must equal zero. The following two identities show that A must be similar to one of the M_n: $J^{-1}\begin{pmatrix}1&-n\\0&1\end{pmatrix}J=\begin{pmatrix}1&n\\0&1\end{pmatrix}$ and $K^{-1}\begin{pmatrix}1&0\\n&1\end{pmatrix}K=\begin{pmatrix}1&n\\0&1\end{pmatrix}$.

If $k\neq 1$ then neither m nor l can equal zero. Let $g=(m,l)$. Then we can find integers c and d with $(c,d)=1$ such that either $l=gd^2$, $m=-gc^2$ or such that $l=-gd^2$, $m=gc^2$ depending on which of l and m is positive. In what follows we will assume that m is negative. The proof of the other case is similar.

Since $(c, d) = 1$ we can find a and b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Now, $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad + dcg - bc & gd^2 \\ -gc^2 & ad - dcg - bc \end{pmatrix}$. If this last matrix is A we are done. If it is not A then it must be $\begin{pmatrix} (2-k) & l \\ m & k \end{pmatrix}$. But note that in this case
$\begin{pmatrix} d & b \\ c & a \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} ad - dcg - bc & gd^2 \\ -gc^2 & ad + dcg - bc \end{pmatrix} = A.$

Thus A is similar to $\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$.

In the final section we will see that $L_{M_n} : T \to T$ for $n > 0$ has exactly n circles of fixed points given on the unit square by $\{(x, k/n) | x \in [0, 1]\}$ for $1 \leq k \leq n$. So if $n \neq m$ then L_{M_n} is dynamically distinct from L_{M_m}. This shows that L_{M_n} is not topologically conjugate to L_{M_m} and so M_n and M_m cannot be similar.

4.3. **Characteristic equation $\mathbf{x^2 + 2x + 1 = 0}$.** Now $-M_n = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$, for non-negative integers n, each of which has characteristic equation $x^2 + 2x + 1 = 0$. An argument very similar to the previous case shows that any matrix in $GL(2, \mathbb{Z})$ that satisfies the characteristic equation $x^2 + 2x + 1 = 0$ is similar over the integers to $-M_n$ for some n.

To observe that $-M_n$ is not similar to $-M_m$, we note $-M_n^2 = M_{2n}$ and $-M_m^2 = M_{2m}$. Since M_{2m} is not similar to M_{2n} if $m \neq n$, we see that $-M_m$ is not similar to $-M_n$ if $m \neq n$

4.4. **Characteristic equation $\mathbf{x^2 - 1 = 0}$.** We define a family of matrices $E_n = \begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix}$, for non-negative integers n, each of which has characteristic equation $x^2 - 1 = 0$. We will begin by showing that any matrix in $GL(2, \mathbb{Z})$ that satisfies this characteristic equation is similar over the integers to E_n for some n. We will then show any matrix satisfying this characteristic equation is similar over the integers to either $E_0 = J$ or E_1. Finally we will show that J is not similar to E_1.

Suppose $A \in GL(2, \mathbb{Z})$ with characteristic equation $x^2 - 1 = 0$. Since $trace(A) = 0$ we can write $A = \begin{pmatrix} k & l \\ m & -k \end{pmatrix}$. From the determinant we see that $k(-k) - lm = 1$. Re-arranging, gives $ml = -k^2 + 1$.

The following equation shows that A is similar to a matrix with 0 in the $a_{2,1}$ position

$$\begin{pmatrix} \ldots & \ldots \\ \frac{-m}{gcd(m,k+1)} & \frac{k+1}{gcd(m,k+1)} \end{pmatrix}\begin{pmatrix} k & 1 \\ m & -k \end{pmatrix}\begin{pmatrix} \frac{k+1}{gcd(m,k+1)} & \ldots \\ \frac{m}{gcd(m,k+1)} & \ldots \end{pmatrix} = \begin{pmatrix} \ldots & \ldots \\ 0 & \ldots \end{pmatrix}.$$

Because the *determinant* $= -1$, the $a_{1,1}, a_{2,2}$ positions in this last matrix must be $1, -1$ or $-1, 1$, conjugating with J shows that these two cases result in similar matrices. So we now have that A is similar to E_n for some n.

Finally, to show that there are only two similarity classes for this characteristic equation. We use the following equation:
$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -2k+n \\ 0 & -1 \end{pmatrix}$.
We see that E_n is similar to either E_0 or E_1 depending on the parity of n. Ultimately we have found that this characteristic equation has only two similarity classes, thus A is similar to E_0 or E_1.

Finally we observe that $L_{E_1} : T \to T$ has exactly one circle of fixed points given on the unit square by $\{(x, 0)|x \in [0, 1]\}$. Whereas $L_{E_0} : T \to T$ has two circles of fixed points given by $\{(x, 0)|x \in [0, 1]\}$ and $\{(x, \frac{1}{2})x \in [0, 1]\}$. Therefore, L_{E_1} is dynamically distinct from L_{E_0} and so E_1 and E_0 cannot be similar.

4.5. **Characteristic equation $\mathbf{x^2 + 1 = 0}$.** The proof in the cases when the characteristic equation is irreducible follows from Fine and Rosenberger given in [3].

We will show that any matrix A in $GL(2, \mathbb{Z})$ that has characteristic equation $x^2 + 1 = 0$ is similar over the integers to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let S be the set of conjugates of A within $GL(2, \mathbb{Z})$ such that $S = \{C^{-1}AC; C \in GL(2, \mathbb{Z})\}$. Since conjugation preserves trace, S consists of matrices with trace equal to 0. Let $D = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ be an element of S with $|a|$ minimal. We may assume that $a \geqslant 0$. If $a < 0$ then we can conjugate the matrix with K to interchange the two entries on the main diagonal.

Now we must show that $a = 0$. Suppose $a \neq 0$, then $-a^2 - bc = 1$ implies $-bc = a^2 + 1$ which implies that $|b||c| = a^2 + 1$. It follows that $b \neq 0$ and $c \neq 0$ and either $|b| \leqslant |a| < |c|$ or $|c| \leqslant |a| < |c|$. First assume that $|c| \leqslant |a| < |b|$. Conjugating with J changes the sign of the $(2, 1)$ entry, so we may assume that $c > 0$.

Because $|c| \leqslant |a|$ we have $0 \leqslant a - c < a$.

Now, $M_1^{-1}DM_1 = \begin{pmatrix} (a-c) & (a+b-c) \\ (c) & (c-a) \end{pmatrix}$

But $0 \leqslant a - c < a$ contradicts the fact that $|a|$ is minimal. Therefore a minimal conjugate of A must have $a = 0$ and hence $-bc = 1$. From this it follows that $b = \pm 1$ and $c = \mp 1$, therefore $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Conjugating by J shows that these two matrices are similar.

In the case where $|b| \leqslant |a| < |c|$, we use a similar argument. Again we may assume that $a > 0$ and $b > 0$ so that $0 \leqslant a - b < a$.

If we conjugate D by M_{-1}^T we have $(M_{-1}^T)^{-1}DM_{-1}^T = \begin{pmatrix} (a-b) & (b) \\ (a-b+c) & (b-a) \end{pmatrix}$

Again $0 \leqslant a - b < a$ is contradicting the fact that $|a|$ is minimal. Therefore in a minimal conjugate of A we must have $a = 0$ and again $-bc = 1$. The rest of the argument follows from above, so that we have D is similar to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

4.5.1. *An application to number theory.* The following lemma provides an interesting connection to number theory. This is a special case of Fermat's theorem on the sum of two squares.

Lemma 1. *Let $a \in \mathbb{Z}$. Any factor of $a^2 + 1$ can be written as the sum of two squares.*

Proof. Let b be any factor of a^2+1. We can choose c such that $-bc = a^2+1$. Then $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ has characteristic equation $x^2 + 1 = 0$. Now from the above we know that $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So there exists $C = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in SL(2, \mathbb{Z})$ such that $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = C^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C = \begin{pmatrix} zy + xw & z^2 + x^2 \\ -y^2 - w^2 & -zy - xw \end{pmatrix}$. So b can be written as the sum of two squares. □

4.6. **Characteristic equation $\mathbf{x^2 + x + 1 = 0}$.** We will show that any matrix A in $GL(2, \mathbb{Z})$ that has characteristic equation $x^2 + x + 1 = 0$ is similar over the integers to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

Let S be the set of conjugates of A within $GL(2, \mathbb{Z})$ such that $S = \{C^{-1}AC; C \in GL(2, \mathbb{Z})\}$. Since conjugation preserves trace, S consists of matrices with trace equal to -1. Let $D = \begin{pmatrix} a & b \\ c & -(1+a) \end{pmatrix}$ be an element of S with $|a|$ minimal. We can assume that $a \geqslant 0$. If $a < 0$ then $K^{-1}DK = \begin{pmatrix} -(1+a) & c \\ b & a \end{pmatrix}$, but now $|1+a| < |a|$, contradicting the fact that $|a|$ was minimal. Now we must show that $a = 0$. Suppose $a \neq 0$, then $-a^2 - a - bc = 1$ implies $-bc = a^2 + a + 1$ which implies that $|b||c| = a^2 + a + 1$. It follows that $b \neq 0$ and $c \neq 0$ and either $|b| \leqslant |a| < |c|$ or $|c| \leqslant |a| < |c|$. First assume that $|c| \leqslant |a| < |b|$. From the following identity we may assume that $c > 0$.

$$J^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} J = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

Now $M_1^{-1} D M_1 = \begin{pmatrix} (a-c) & (2a+b-c+1) \\ c & (c-a-1) \end{pmatrix}$

Because $|c| \leqslant |a|$ we have $0 \leqslant a - c < a$, but this contradicts the fact that $|a|$ is minimal. Therefore a minimal conjugate of A we must have $a = 0$ and hence

$-bc = 1$. From this it follows that $b = \pm 1$ and $c = \mp 1$, therefore $D = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ or $D = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Conjugating with J shows that these two matrices are similar.

In the case where $|b| \leqslant |a| < |c|$, we use a similar argument. Again we may assume that $a > 0$ and $b > 0$ so that $0 \leqslant a - b < a$.

Then $(M_{-1}^T)^{-1} D M_{-1} = \begin{pmatrix} (a-b) & (b) \\ (2a - b + c + 1) & b - (1 + a) \end{pmatrix}$
Again $0 \leqslant a - b < a$ is contradicting the fact that $|a|$ is minimal. Therefore in a minimal conjugate of A we must have $a = 0$ and again $-bc = 1$. The rest of the argument follows from above, so that we have D is similar to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

4.7. **Characteristic equation $\mathbf{x^2 - x + 1 = 0}$.** A similar argument to the previous case shows that any matrix with characteristic equation $x^2 - x + 1 = 0$ is similar to $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

5. The dynamics of non-hyperbolic automorphisms

We now look at the dynamics of these non-hyperbolic maps and classify them up to toplogical conjugacy. We also look at Devaney's definition of chaos and see which conditions are satisfied by our maps. We begin with some preliminary definitions.

Definition 3. *$f : J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.*

Definition 4. *$f : J \to J$ has sensitive dependance on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geqslant 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

Now we come to Devaney's definition of chaotic dynamical systems.

Definition 5. *Let T be a set. $f : T \to T$ is said to be chaotic on T if*

- *f has sensitive dependance on initial conditions*
- *f is topologically transitive*
- *periodic points are dense in T.*

Now we have shown that all non-hyperbolic toral automorphisms are similar to one of the similarity classes defined above. We know from Theorem 1 that periodic points are dense in all toral automorphisms.

5.1. **Characteristic equation $\mathbf{x^2 - 2x + 1 = 0}$.** We know that if A has characteristic equation $x^2 - 2x + 1 = 0$ is similar over the integers to M_n for some $n \geq 0$. Now on the torus $L_{M_n} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} x + ny \\ y \end{pmatrix}$ modulo 1.
If $n = 0$ then every point is fixed, the map is just the identity. If $n > 0$ then the map can be considered as a shear on the unit square. Notice each of the circles

described on the unit square by $\{(x, k/n)|x \in [0,1]\}$ for $1 \leq k \leq n$ consist of fixed points, and there are no other fixed points. This means that if $n \neq m$ then L_{M_n} and L_{M_m} are dynamically distinct and so cannot be topologically conjugate.

Since $L^k_{M_n} = L_{M_{nk}}$ we see that the circle given by $\{(x, 1/kn)|x \in [0,1]\}$ must be fixed under $L^k_{M_n}$ and is not fixed under a smaller number of iterations. Thus this circle consists of points of period k for L_{M_n}. This means that if $n > 1$ that L_{M_n} has periodic points of all periods.

Note also that if $n > 1$ then $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y+\epsilon \end{pmatrix}$ can be separated by an arbitrary distance up to 1/2 under a sufficiently high iteration of L_{M_n}. Note that this mapping allows two points on the torus to be separated by a maximum distance of 1/2 before they begin to wrap around the torus and the distance separating them is, once again, less than 1/2. So L_{M_n} does have sensitive dependence on initial conditions. However, since the y coordinate never changes under iteration it is clear that these maps are not topologically transitive and so not chaotic.

5.2. **Characteristic equation $\mathbf{x^2+2x+1=0}$.** Any matrix in $GL(2,\mathbb{Z})$ that satisfies the characteristic equation $x^2+2x+1=0$ is similar over the integers to $-M_n$ for some n. If $n = 0$ the map is just rotation through π. If $n \neq 0$ the the map can be considered as the composition of a shear with a rotation through π on the unit square.

To observe that L_{-M_n} is not topologically conjugate to L_{-M_m}, we note $L^2_{-M_n} = L_{M_{2n}}$ and $L^2_{-M_m} = L_{M_{2m}}$. Since $L_{M_{2m}}$ is not topologically conjugate to M_{2n} if $m \neq n$, we see that L_{-M_n} is not topologically conjugate to L_{-M_m}.

$L_{-M_n}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} -x-ny \\ -y \end{pmatrix}$ modulo 1.

Fixed points are given by solving:

$$-x-ny \equiv x \bmod (1)$$
$$-y \equiv y \bmod (1)$$

Which gives us:

$$2y \equiv 0 \bmod (1)$$
$$2x \equiv ny \bmod (1)$$

If $y = 0$ or if $y = 1/2$ and n is even then $x = 0$ or $x = 1/2$. If $y = 1/2$ and n is odd then $x = 1/4$ or $x = 3/4$. So L_{-M_n} has four fixed points. Since $L^k_{-M_n} = L_{(-M_n)^k}$ we see that L_{-M_n} has no periodic points of odd period, apart from the fixed points, and from the case above that it has periodic points of all even periods.

It is straightforward to check that So L_{-M_n} has sensitive dependence, but is not topologically transitive.

5.3. **Characteristic equation $\mathbf{x^2-1=0}$.** In this case we found above that there are only two similarity classes, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The map corresponding to the first matrix is just reflection in the line $y = 1/2$ in the unit square. The map corresponding to the second matrix is the reflection composed with a shear. Fixed points are given by:

$$x + y \equiv x \text{ mod } 1$$
$$-y \equiv y \text{ mod } 1$$
and, repectively
$$x \equiv x \text{ mod } 1$$
$$-y \equiv y \text{ mod } 1$$

From the first system of equations we see that $\{(x, 0)|x \in [0, 1]\}$ is the only circle of fixed points, where as the second system of equations we have $\{(x, 0)|x \in [0, 1]\}$ and $\{(x, \frac{1}{2})|x \in [0, 1]\}$ are circles of fixed points. Therefore since the two maps are dynamically distinct they are not topologically conjugate. Since A has characteristic equation $x^2 - 1 = 0$ it must satisfy $A^2 = I$. So every point that is not fixed is a point of period 2. Clearly these maps are not topologically transitive and don't have sensitive dependence on initial conditions.

The next three characteristic equations have complex eigenvalues and as in the last case we will see all points which are not fixed are periodic.

5.4. **Characteristic equation $\mathbf{x^2 + 1 = 0}$.** From above we found that there is only one similarity class $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Fixed points are given by:

$$y \equiv x \text{ mod } 1$$
$$-x \equiv y \text{ mod } 1$$

This system of equations allows us to see that $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ are the only fixed points. We notice that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. So points of period 2 are given by:

$$2y \equiv 0 \text{ mod } (1)$$
$$2x \equiv 0 \text{ mod } (1)$$

So there is one periodic orbit of period 2 containing $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. Since A has characteristic equation $x^2 + 1 = 0$ it must satisfy $A^4 = I$, so that every point which is not a fixed, or a point of period 2, is a point of period 4. Because every point is either fixed or of period 2 or 4, it is clear that these maps are not topologically transitive. Furthermore, they don't have sensitive dependance on initial conditions because any two initial points will never be further from each other than they are under the first four iterates.The dynamics of this map are more easily visualized on the unit square. The map is just rotation through $\pi/2$.

5.5. **Characteristic equation $\mathbf{x^2 + x + 1 = 0}$.** For this characteristic equation, again we have only one similarity class $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Fixed points are given by:

$$y \equiv x \text{ mod } 1$$
$$-x - y \equiv y \text{ mod } 1$$

We can see that the fixed points are $(0, 0)$, $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$. Since A has characteristic equation $x^2 + x + 1 = 0$ it must satisfy $A^3 = I$. This time every point which is not fixed must have period 3. Due to the periodic nature of these maps they can be neither topologically transitive nor have sensitive dependance on initial conditions and therefore they are not chaotic on the torus. This idea is illustrated on the unit square in Figure 2. Notice that if the unit square is divided into two triangles by cutting down a diagonal and if these triangles are then distorted into equilateral triangles, then the map rotates the two triangles by $\pi/3$ in the clockwise direction. The reader should check that everything glues up correctly to give a well-defined map of the torus.

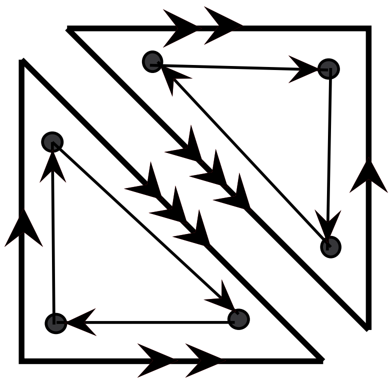

FIGURE 2. A point of period 3 on the unit square.

5.6. **Characteristic equation $\mathbf{x^2 - x + 1 = 0}$.** For this last characteristic equation we have only one similarity class $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. This time fixed points are given by:

$$-y \equiv x \bmod 1$$
$$x + y \equiv y \bmod 1$$

Solving shows that the only fixed point is $(0, 0)$. This time we note that: $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
So that there is a periodic orbit of period 2 given by $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$, and an orbit of period 3 given by $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. Because A has characteristic equation $x^2 - x + 1 = 0$ it must satisfy $A^6 = I$. This implies that every point which is not fixed, or of period 2 or 3, has period 6. For similar reasons as given for the two previous characteristic equations, these maps are not topologically transitive and do not have sensitive dependance on initial conditions and therefore are not chaotic on the torus. Figure 3. shows a point of period 6 on the unit square. The map interchanges the two triangles. After two iterations the triangles are mapped back onto themselves, but have been rotated, as in the previous example, by $\pi/3$.

Although periodic points are dense for all toral automorphisms, it is clear that none of the non-hyperbolic automorphisms onto the torus are both topologically

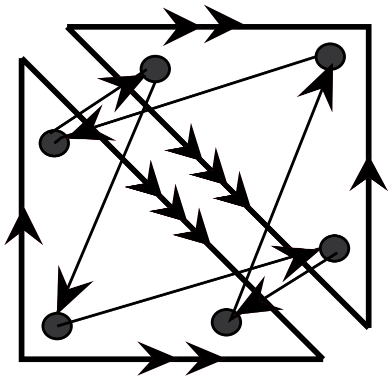

FIGURE 3. A point of period 6 on the unit square.

transitive and have sensitive dependance on initial conditions, so therefore are not chaotic.

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