STRONG LINEAR PRESERVERS OF GW-MAJORIZATION ON $\mathbf{M}_n$

A. ARMANDNEJAD AND A. SALEMI*
DEPARTMENT OF MATHEMATICS
VALIASR UNIVERSITY OF RAFSANJAN
7713936417, RAFSANJAN
IRAN
E-MAIL: ARMANDNEJAD@MAIL.VRU.AC.IR
*DEPARTMENT OF MATHEMATICS,
SHAHID BAHONAR UNIVERSITY OF KERMAN
7619614111, KERMAN
IRAN
E-MAIL: SALEMI@MAIL.UK.AC.IR

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Abstract. Let $\mathbf{M}_n$ be the set of all $n \times n$ matrices whose all entries are in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. In this paper we introduce the relation gw-majorization on $\mathbf{M}_n$. Also we characterize all linear operators that strongly preserve gw-majorization on $\mathbf{M}_n$.

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1. Introduction

A nonnegative matrix $R \in M_n$ which all it's row sums are equal one is said to be a row stochastic matrix.

The definition of matrix majorization was introduced by Dahl in [5], and this notion generalized the definition of multivariate majorization. The matrix B is said to be matrix majorized by A if there exists an n×n row stochastic matrix R such that B=RA , and denoted by A $\succ$B, for more details see [8]. Also, several characterizations of matrix majorization was given in [5].

In [2], Beasley, S.-G. Lee and Y.H Lee proved that, if a linear operator T:$\mathbf{M}_n \rightarrow \mathbf{M}_n$ strongly preserves matrix majorization then, there exist a permutation P and an invertible matrix $M \in \mathbf{M}_n$ such that $T(X) = PXM$, $\forall\ X \in\ span\{\mathbf{R}_n\}$, where

$\mathbf{R}_n$ is the set of all $n \times n$ row stochastic matrices. For more information about majorization see [3] and [7].

Now, we introduce gw-majorization on $\mathbf{M}_n$ as follows:

A matrix $R \in \mathbf{M}_n$ which all it's row sums are equal one is said to be a g-row stochastic matrix. A matrix D is said to be a g-doubly stochastic matrix if both D and D^t are g-row stochastic matrices. For more details see [4].

Definition 1.1. Let $A, B \in \mathbf{M}_n$. The matrix B is said to be gw-majorized (gs-majorized) by A if there exists an n×n g-row (g-doubly) stochastic matrix R such that B=RA and denoted by A$\succ_{gw}$B (A$\succ_{gs}$B).

In [1] authors showed that a linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ strongly preserves gs-majorization if and only if $T(X) = AXR + JXS$ for some $R, S \in \mathbf{M}_m$ and g-doubly stochastic invertible matrix A, such that, R and $R + nS$ are invertible . In this paper, we will to show that the strong linear preservers of gw-majorization on $\mathbf{M}_n$ are in the following form :

$$T(X) = RXM,$$

where R and M are invertible and $R \in \mathbf{GR}_n$.

Throughout this paper, $\mathbf{GR}_n$ is the set of g-row stochastic matrices and $\mathbf{J}$ is the $n \times n$ matrix that all entries are equal one.

2. GW-MAJORIZATION ON $\mathbf{M}_n$ AND ITS STRONG LINEAR PRESERVERS

In this section, we will investigate linear operators on $\mathbf{M}_n$ that strongly preserves gw-majorization.

Let $\sim$ be a relation on $\mathbf{M}_n$. A linear operator T:$\mathbf{M}_n \longrightarrow \mathbf{M}_n$ is said to be a linear strong preserver of $\sim$ whenever:

$$x \sim y \quad \Longleftrightarrow \quad T(x) \sim T(y).$$

Proposition 2.1. *Let T :$\boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear operator that strongly preserves gw-majorization . Then T is invertible.*

Proof. Suppose T(A)=0. Since T is linear and 0$\succ_{gw}$T(A), T(0) $\succ_{gw}$T(A). Therefore, 0$\succ_{gw}$A because T strongly preserves gw-majorization. Then, there exists an n×n g-row stochastic matrix R such that A=R0. then, A=0 and hence T is invertible. □

Remark 2.2. Let A,B be two g-row stochastic matrices then, AB and A^{-1} (If A is invertible) are g-row stochastic matrices.

For i and j ($1 \leq i \leq n$, $1 \leq j \leq m$) E_{ij} is the n×m matrix whose all entries are equal zero except the $(i,j)^{th}$ entry which is equal one.

Lemma 2.3. *Let A be an invertible matrix.Then, $A^{-1}RA \in \boldsymbol{GR}_n$ for every R$\in$ $\boldsymbol{GR}_n$ if and only if there exists a non zero scalar α, such that $\alpha A \in \boldsymbol{GR}_n$.*

Proof. Let A=(a_{ij}) , $A^{-1} = (b_{ij})$ and $A^{-1}RA \in \mathbf{GR}_n$, for all $R \in \mathbf{GR}_n$. For $1 \leq i < j \leq n$, define $U_{ij} := I - E_{jj} + E_{ji}$. It is clear that $U_{ij} \in \mathbf{GR}_n$, then by hypostasis,

$A^{-1}U_{ij}A = I - A^{-1}E_{jj}A + A^{-1}E_{ji}A \in \mathbf{GR}_n$, for $1 \leq i < j \leq n$. Therefore, the row sums of $A^{-1}E_{jj}A$ are equal to the row sums of $A^{-1}E_{ji}A$. Since A is invertible so there exists $1 \leq k \leq n$ such that $b_{kj} \neq 0$. The k^{th} row sum of $A^{-1}E_{ji}A$ is equal to the k^{th} row sum of $A^{-1}E_{ji}A$. Then
$a_{i1}b_{kj} + \cdots + a_{in}b_{kj} = a_{j1}b_{kj} + \cdots + a_{jn}b_{kj}$, thus $a_{i1} + \cdots + a_{in} = a_{j1} + \cdots + a_{jn}$, for $1 \leq i < j \leq n$. Therefore, the row sums of A are equal to a scalar denoted by β. The scalar β is not zero because the columns of A are linearly independent. Let $\alpha = \frac{1}{\beta}$, it is clear that $\alpha A \in \mathbf{GR}_n$. Conversely, let $\alpha A \in \mathbf{GR}_n$ for some scalar $\alpha \neq 0$, then $(\frac{1}{\alpha}A^{-1}) \in \mathbf{GR}_n$ and hence $A^{-1}RA = (\frac{1}{\alpha}A^{-1})R(\alpha A) \in \mathbf{GR}_n$, for all $R \in \mathbf{GR}_n$. □

Lemma 2.4. *Let A and B be two invertible matrices. Then the linear operator $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ defined by $T(X) = AX^tB$, for all $X \in \boldsymbol{M}_n$, is not a strong linear preserver of gw-majorization.*

Proof. Let T be a strong linear preserver of gw-majorization. Since $E_{11} \succ_{gw} E_{n1}$ thus, $T(E_{11}) = AE_{11}^tB \succ_{gw} AE_{n1}^tB = T(E_{n1})$ then, $AE_{11} \succ_{gw} AE_{1n}$ and hence $RAE_{11} = AE_{1n}$, for some $R \in \mathbf{GR}_n$. Therefore, the first column of A is zero, a contradiction. □

Theorem 2.5. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear operator that strongly preserves gw-majorization. Then T strongly preserves the set of singular matrices.*

Proof. Let A be a singular matrix, then there exist scalars $r_1, ..., r_n$, such that $\sum_{i=1}^n r_iA_i = 0$ and $r_k \neq 0$, for some $1 \leq k \leq n$, where A_i is the i^{th} row of A. Let $r = \sum_{i=1}^n r_i$. Now, we consider two cases,

Case 1; Let $r \neq 0$. Define, R:=$\frac{1}{r}\begin{pmatrix} r_1 & \dots & r_n \\ \vdots & & \vdots \\ r_1 & \dots & r_n \end{pmatrix}$, so RA=0 then, $A \succ_{gw} 0$ and hence $T(A) \succ_{gw} 0$. Therefore $T(A)$ is singular.

Case 2; Let $r = 0$. Without lose of generality we can assume that $A_1 = \sum_{i=2}^n r_iA_i$ and $\sum_{i=2}^n r_i = 1$. For every $m \geq 1$ define $B_m := \begin{pmatrix} (1+\frac{1}{m})A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$, then $T(A) = T(lim_{m\to\infty}B_m) = lim_{m\to\infty}T(B_m)$. For all $m \geq 1$, B_m is singular, then, by case 1, $T(B_m)$ is singular. Thus $\{T(B_m)\}_{m=1}^{\infty}$ is a sequence of singular matrices, Therefore T(A) is singular. Conversely, let $T(A)$ be singular. By the same method as above $T^{-1}(T(A)) = A$ is singular. □

In [6] Marcus and Purves proved the following Theorem.

Theorem 2.6. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear operator. Then, T strongly preserves the set of singular matrices, if and only there exist invertible matrices A and B such that, T(X)=AXB, $\forall$ $X \in \boldsymbol{M}_n$ or T(X)=AX^tB, $\forall$ $X \in \boldsymbol{M}_n$.*

Now, we characterize linear operators on $\mathbf{M}_n$ that strongly preserves gw-majorization.

Theorem 2.7. *Let $T : \boldsymbol{M}_n \rightarrow \boldsymbol{M}_n$ be a linear operator. Then T is a strong linear preserver of gw-majorization if and only if there exist g-row stochastic invertible matrix A and invertible matrix B such that, T(X)=AXB for all X in $\boldsymbol{M}_n$.*

Proof. Let T(X)=AXB, for all $X \in \mathbf{M}_n$, where A,B are invertible and $A \in \mathbf{GR}_n$. It is easy to show that T strongly preserves gw-majorization, by Remark 2.2.

Conversely, let T strongly preserves gw-majorization. Then, by Remark 2.1 and Theorem 2.5, T is invertible and T strongly preserves the set of singular matrices. Then, by Theorem 2.6 and Lemma 2.4, there exist invertible matrices A and B such that T(X)=AXB, for all $X \in \mathbf{M}_n$. Now we consider two cases,

Case 1; Let T(I)=I, then $B = A^{-1}$. Let R be an arbitrary g-row stochastic matrix, then, $I \succ_{gw} R$ so, $T(I) = I \succ_{gw} T(R)$ therefore, $T(R) \in \mathbf{GR}_n$. Hence $ARA^{-1} \in \mathbf{GR}_n$ for all $R \in \mathbf{GR}_n$. Then, by Lemma 2.3, there exists a scalar $\alpha \neq 0$, such that $\alpha A \in \mathbf{GR}_n$. Therefore, T(X)= $(\alpha A)X(\frac{1}{\alpha}A^{-1})$, $\forall X \in \mathbf{M}_n$, where $(\alpha A), (\frac{1}{\alpha}A^{-1}) \in \mathbf{GR}_n$.

Case 2; Let $T(I) \neq I$. By Theorem 2.5, T(I) is invertible . Consider the linear operator $S : \mathbf{M}_n \rightarrow \mathbf{M}_n$, where $S(X) = T(X)(T(I))^{-1}$. It is easy to show that S strongly preserves gw-majorization and S(I)=I. Then by case 1, there exists an invertible g-row stochastic matrix A such that $S(X) = AXA^{-1}$, for all $X \in \mathbf{M}_n$. We have T(X)=S(X)T(I). Define $B := A^{-1}T(I)$. Therefore T(X)=AXB, for all $X \in \mathbf{M}_n$. □

References

[1] A. Armandnejad, A. Salemi, The structure of linear preservers of gs-majorization *Bull. Iranian Math. Soc*, Vol.32 No.2 (2006) 31-42 .

[2] L.B.Beasley, S.-G. Lee, Y.H Lee, A characterization of strong preservers of matrix majorization, Linear Algebra and Its Applications, 367 (2003) 341-346.

[3] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[4] H. Chiang and C.K. Li, Generalized Doubly Stochastic Matrices and Linear Preservers, Linear and Multilinear Algebra 53(2005), 1-11.

[5] G. Dahl, Matrix majorization, Linear Algebra Appl. 288 (1999) 53-73.

[6] M.Marcus and R. Purves, linear transformations on algebra of matricesII: The invariance of the elementary symmetric functions, Canad. J. Math. 11 (1959), 383-396

[7] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic, New York, 1972.

[8] F. Martnez Pera , P. Massey, L. Silvestre,Weak matrix majorization, Linear Algebra Appl 403 (2005), 343-368.