HIGHER MULTIFRACTAL ANALYSIS FOR SIMULTANEOUS ACTIONS OF FUCHSIAN GROUPS

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ABSTRACT. We consider groups $\Gamma_1, \Gamma_2, ..., \Gamma_d$ acting on the hyperbolic plane H^2 and by the method of Series for coding hyperbolic geodesics we can do simultaneous multifractal decompositions from the action of these groups. We obtain a variational formula for a dimension of the multifractal sets which generalizes that presented in our previous article for $d = 1$.

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1. INTRODUCTION

In the area of Dynamical Systems lies an important subject which is the study of the complexity of sets. For this is necessary to introduce suitable "characteristic dimensions", known example of such invariants are in metric spaces, are the Hausdorff dimension and the topological entropy. The formulation of the *dimension theory of dynamical systems* is due to Pesin [8]. He considered a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a collection of subsets of X. This family satisfies some special conditions and it is called a $C-$structure (C for Carathéodory). Thus the examples mentioned are particular cases obtained by considering special $C-$structures on metric spaces.

A relevant place in dimension theory of dynamical systems is occupied by the *multifractal analysis.* This discipline raised in Physics in order to study the behavior of physical measures supported on strange attractors. Invariant sets with a complex mathematical structure are encountered when studying chaotic dynamical

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behaviors, a *fractal decomposition* of such invariant sets is what mainly treats the multifractal analysis.

This following presentation of the general multifractal analysis appeared in [1]:

Let X be a set and $g : X \to [-\infty, +\infty]$, now the sets in which X is partitioned are:

$$K_\alpha = K_\alpha(g) = \{x : g(x) = \alpha\}.$$

Let G be a function defined on sets, and let $F(\alpha) = G(K_\alpha)$, the map F is called *the multifractal spectrum* specified by the pair (g, G). The most used spectra are defined by a limit process, for instance:

$g(x) = D_\mu(x) := \lim_{r \to 0} \dfrac{\log(\mu(B_r(x)))}{-\log r}$, the *point-wise dimension of the measure* μ, and $F(\alpha) = \dim_H K_\alpha$. This is called the point-twise *dimension spectrum.*

Another example is: Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map, let $d_n(x, y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0, 1, ..., n-1\right\}$ and we denote by $B_{n,\varepsilon}(x)$ the ball of centre x and radius ε in the metric d_n. If μ a $f-$invariant measure, the *upper and lower local entropies* are

$$\overline{h_\mu}(x, f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup -\frac{1}{n} \log \mu(B_{n,\varepsilon}(x))$$

$$\underline{h_\mu}(x, f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf -\frac{1}{n} \log \mu(B_{n,\varepsilon}(x)).$$

Then (Brin-Katok theorem[3]), the local entropy does exist, i.e.
$\overline{h_\mu}(x, f) = \underline{h_\mu}(x, f) := h_\mu(x, f)$, for $\mu - a.e.\ x \in X$. Now the multifractal spectrum is given by

$g(x) = h(f, x)$, $F(\alpha) = \mathcal{E}(\alpha) := h_{top}(f, K_\alpha)$. This is called the *local entropies spectrum of* f. The Hausdorff dimension and the topological entropy are really particular cases of the mentioned general dimension theory in dynamical systems due to Pesin.

In [7] we have considered finitely generated groups Γ acting on the hyperbolic disc H^2. For a class of such a groups is possible the assignation to any point $\xi \in \Lambda$ (the limit set of Γ) a unique reduced sequence $e_0 e_1 ... e_n ...$ in the generators of Γ. This is done by mean of the technique by C. Series [10],[11] to code geodesics in H^2/Γ whose ends are in the limit set of Γ. One case in which the assignation is a group generated by isometries pairing the sides of a polygon inscribed in H^2. Another interesting case is the modular group $SL_2(\mathbf{Z})$,in this case the assigned sequence to a point $\xi \in \Lambda$ is given by the corresponding numbers in its continued fraction expansion. In these conditions we can define a family $\varphi_n(\xi) := d_h(O, e_0 e_1 ... e_n(O))$, $\xi \in \Lambda$, (d_h is the hyperbolic metric). Now for the fractal decomposition sets

$$K_\alpha := \left\{\xi : \lim_{n \to \infty} \frac{\varphi_n(\xi)}{n} = \alpha\right\} \tag{1}$$

we have done a variational description of the multifractal entropy spectrum $\mathcal{E}(\alpha) := h_{top}(f, K_\alpha)$, where h_{top} is the topological entropy for non-compact sets and $f = f_\Gamma : \partial H^2 \to \partial H^2$ is a *boundary hyperbolic map* or *Bowen-Series map.* In [7] was obtained a variational formula for $h_{top}(f, K_\alpha)$.

One interesting extension of the mentioned result is to consider a multiparameter $\alpha = (\alpha_1, ..., \alpha_d)$ and simultaneous decompositions
$K_{\alpha_i} = \left\{ \xi : \lim_{n\to\infty} \frac{\varphi_n^i(\xi)}{n} = \alpha_i \right\}$, $i = 1, 2, ..., d$ and to analyze the multiparametric spectrum
$K_\alpha = \bigcap_{i=1}^{d} \left\{ \xi : \lim_{n\to\infty} \frac{\varphi_n^i(\xi)}{n} = \alpha_i \right\}$. The multifractal analysis for $d > 1$ is called multifractal analysis in higher dimensions or multidimensional multifractal analysis. New phenomena appear in dimension $d > 1$, one is for instance that the domain of the multifractal spectrum may be not necessarily connected like in the one-dimensional case in which it is always an interval. Besides this difference with the one dimensional case there are interesting applications to number theory of the multidimensional spectra as is pointed out in detail in [2], where is also emphasized, that the extension from $d = 1$ to higher dimensions of the multifractal analysis is non trivial and the results in $d > 1$ cannot be directly obtained from known results for the one dimensional case

2. Fuchsian groups and boundary hyperbolic maps

In this section we review some concepts about Fuchsian groups acting on the hyperbolic disc H^2 and the construction of the boundary hyperbolic maps. We sketch here the main aspects, for more details see [4], [10],[11],[7] and references therein.

For a class of finitely generated groups acting on the hyperbolic disc H^2 there is a way of coding geodesics in H^2/Γ. The limit set of Γ can be identified with a finite type subshift and in particular to any point in the limit set is associated an infinite reduced word in the generators of Γ. An example, mentioned in the earlier section, in which the Series method can be applied is the following: let R be a polygon inscribed in H^2 and let γ_1, γ_2 be isometries pairing the sides of R such that the group generated by γ_1, γ_2 acts discretely on H^2 and has R as a fundamental domain. The action is also considered such that the gluing of the sides of the polygon gives a surface which has funnels flaring-off to infinity. Cutting off these funnels along a geodesic in the homotopy class of circles rounding each funnel is obtained a surface M whose fundamental group is the same of that of a punctured torus. By the Poincaré theorem the group generated by those isometries is a free group $\Gamma = F(\gamma_1, \gamma_2)$ and so any element of Γ can be written as a reduced word in the generators. There is a relevant subshift of finite type $\sum$, whose alphabet is precisely $S = \left\{\gamma_1^{\pm1}, \gamma_2^{\pm1}\right\}$. The words in the generators of Γ are identified with paths in the Cayley graph of Γ, in fact the word $e_0e_1...e_n$ is identified with the path joining the vertices O, $e_0(O)$, $e_0e_1(O), ...,$ $e_0e_1...e_n(O)$. Thus in this particular case the points in the limit set Λ of Γ are in correspondence with infinite reduced sequences, and the ends of the graph, a tree, is identified with Λ.Moreover there is a bijection $p : \sum \to \Lambda$. Recall that ξ is a limit point of Γ if and only there exists a point $w \in H^2$ such that the $\Gamma-$orbit $\Gamma(w) = \{\gamma w : \gamma \in \Gamma\}$ accumulates at ξ. The set Λ of limit points is called the *limit set of* Γ. Because Γ acts discontinuously $\Lambda \subset \partial H^2$, besides Λ is a closed subset, so compact, of ∂H^2 and $\Gamma-$invariant; moreover it is the smallest non-empty, $\Gamma-$invariant closed set in ∂H^2.

More generally the techniques of Series of assigning a sequence to any point in the limit set of a group Γ can be applied to groups with a fundamental region R which has all its vertices in ∂H^2. When the region has some of its vertices in $int(H^2)$ the results of the example may not hold. Now for the technique can be extended for regions with one or more vertices in $int(H^2)$ it must be added conditions on the fundamental regions[10],[11]. One the hypothesis is the following: a side of R is extended to a complete geodesic α in H^2 and if Υ is the tessellation of H^2 obtained by images of R under Γ, then α lies in Υ, this property is known as R has *even corners.* Other condition is that R have at least five sides. Additionally it shall be considered groups Γ such that $vol(H^2/\Gamma) < \infty$. In these more general situations

there is also a relevant subshift space $\sum$, but the alphabet, in general, does not agree with the generators of the group. The relation between the alphabet for $\sum$ and the set of generators for Γ is finite-to-one. For the special case of the modular group $SL_2(\mathbf{Z})$, to a point $\xi \in \Lambda$ is assigned a sequence given by the numbers in its continued fraction expansion, and the alphabet for the corresponding shift is infinite. This occurs when the group contains parabolic elements.

Finally we recall the definitions of *boundary hyperbolic maps, or Bowen-Series maps* [4], [10],[11],. Let $e \in S$ (the set of generators for Γ) and each side of R is labelled by elements of S, by $T(e)$ is denoted the half-plane in H^2 determined by the extension of the side $C(e)$ (the side labelled by $C(e)$) which does not contain R to ∂H^2. In each intersection $T(e_i) \cap T(e_j)$ ($\{e_i, e_j\} \subset S$), it chooses arbitrary $e_k \in \{e_i, e_j\}$ and defines

$f\mid_{T(e_i)\cap T(e_j)}(x) = e_k^{-1}x$, whereas in the remaining part of $T(e_i)$ the map is defined by $f = e_i^{-1}$. Any triple of half-planes has non-empty intersection [4].

The following notation is used:

$B(e_i) := \left\{x \in T(e_i) : f(x) = e_i^{-1}x\right\}$

$I(e_i) := \partial H^2 \cap \overline{B(e_i)}$

$B(e_0 e_1 ... e_n) := \bigcap\limits_{l=0}^{n} f^{-l}(B(e_l))$

$I(e_0 e_1 ... e_n) := \partial H^2 \cap \overline{B(e_0 e_1 ... e_n)} = \bigcap\limits_{l=0}^{n} f^{-l}(I(e_l))$.

Theorem A[10],[11]**:**

*i)*There is a one-side finite type subshift $\sum$ and a map $p : \sum \to \Lambda$ continuous and bijective possibly except at a countable set of points, such that $p \circ \sigma = f \circ p$ ($\sigma : \sum \to \sum$ is the Bernoulli shift).

ii) each $\gamma \in \Gamma$ has an unique representation $\gamma = e_0 e_1 ... e_n$, $e_i \in \Gamma_1$, with $int(I(e_0 e_1 ... e_n)) \neq \oslash$.

iii) $\lim\limits_{n\to\infty} |I(e_0 e_1 ... e_n)| = 0$ ($|.|$ is the euclidean norm).

The boundary ∂H^2 is partitioned into, a most countable, number of arcs J_i and the alphabet for the Markov system $\sum$ is precisely $\{J_i\}$. An important fact is that the partition $\{J_i\}$ has the *Markov property* with respect to (Λ, f) i.e. $f(J_i) = \bigcup_{\ell=1}^{k} J_{i_\ell}$ for any i. As we mentioned the cardinal of $\{J_i\}$ is infinite if and only if Γ has parabolic elements.

3. Carathéodory-Pesin dimensions

The general dimension theory of sets was introduced by Carathéodory [5] to define a family of measures on a metric space. The Carathéodory construction was generalized by Pesin who produced general characteristic dimensions called the *Carathéodory dimensions*. The most known special cases of such a dimensions are the Hausdorff dimension of sets and the topological entropy of non-compact sets (a dynamical quantity). A detailed account of the general Carathéodory-Pesin constructions can be found in [8], we briefly review here the main aspects of this theory:

Let X be a set and $\mathcal{F}$ be a collection of subsets of X, let also consider two functions $\eta, \psi : \mathcal{F} \to \mathbf{R}^+$ such that

i) $\emptyset \in \mathcal{F}$, $\eta(\emptyset) = 0$, $\eta(U) > 0$ for any $U \in \mathcal{F}$.

ii) for any $\delta > 0$, there is an $\varepsilon > 0$ such that $\eta(U) \leq \delta$ or any $U \in \mathcal{F}$ with $\psi(U) \leq \varepsilon$.

iii) for any $\varepsilon > 0$ there is a, at most countable, subcollection $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{G}$ covers X, i.e. $\bigcup_{U\in\mathcal{G}} U = X$, and $\psi(\mathcal{G}) := \sup\{\psi(U) : U \in \mathcal{G}\} \leq \varepsilon$.

A triple $(\mathcal{F}, \eta, \psi)$ is called a *C-structure* on X. If X is endowed with a $C-$structure, then for a each at most countable subcollection $\mathcal{G}$ of $\mathcal{F}$ is associated a "free energy" $F(q, \mathcal{G}) := \sum_{U\in\mathcal{G}} \eta(U)^q$. Next for any $Z \subset X$ is defined:

$$M_C(Z, q, \varepsilon) = \inf F(q, \mathcal{G}), \tag{2}$$

where the infimum is taken over all the at most countable collections $\mathcal{G}$ which cover Z and with $\psi(\mathcal{G}) \leq \varepsilon$. This quantity is well defined because the condition *iii)* in the definition of $C-$structures. Then there exists the limit

$$M_C(Z, q) = \lim_{\varepsilon\to 0} M_C(Z, q, \varepsilon). \tag{3}$$

For any $Z \subset X$, the function $q \longmapsto M_C(Z, q)$ has the following property [8]: there is a critical value $q_c \in [-\infty, \infty]$ such that $M_C(Z, q)$ jumps from ∞ to 0, then the *dimension of a set* Z, in the context of the Carathéodory-Pesin theory, is

$$\dim_C Z := q_c = \sup\{q : M_C(Z, q) = +\infty\} = \inf\{q : M_C(Z, q) = 0\} \tag{4}$$

If X is endowed with a metric d, the *Hausdorff dimension* of a set $Z \subset X$ is the particular characteristic dimension obtained with $\mathcal{F}$ a collection of open sets, $\eta(U) = diam(U)$ (the diameter of U), and

$$M_H(Z, q, \varepsilon) = \inf\left\{\sum_{U\in\mathcal{G}} diam(U)^q\right\} \tag{5}$$

where the infimum is taken over all the at most countable coverings of Z by open sets with diameter$\leq \varepsilon$

The characteristic dimension verifies [8]:
-$\dim_C \varnothing \leq 0$
- If $Z_1 \subset Z_2$ then $\dim_C Z_1 \leq \dim_C Z_2$.
-$\dim_C (\cup Z_i) = \sup_i \dim_C (Z_i)$, $i = 0, 1, ...,$ $Z_i \subset X$.

A "dynamical" $C-$structure can be given as follows:

Let $f : X \to X$ be a continuous map and (X, d) a compact metrizable space. Let $\mathcal{U} = \{U_1, ..., U_N\}$ be finite covering of X and a string is defined as a sequence $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ such that $\{U_{\ell_0}, ..., U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, ...N\}$. The length of the string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap ... \cap f^{-n+1}(U_{\ell_{n-1}}), \tag{6}$$

if $Z \subset X$ it says that $\Pi = \{L = (\ell_0, \ell_1, ..., \ell_{n-1})\}$ covers Z if

$$Z \subset \bigcup_{L \in \Pi} X(L). \tag{7}$$

For any real number s:

$$M_h(Z, \mathcal{U}, s, n) = \inf_{\Gamma} \sum_{L \in \Pi} \exp(-sn(L)), \tag{8}$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which covers Z .

Let

$$M_h(\mathcal{U}, Z, s) = \lim_{n \to \infty} M_h(\mathcal{U}, Z, s, n). \tag{9}$$

There is an unique number $\overline{s}$ such that $M(\mathcal{U}, Z, .)$ jumps from $+\infty$ to 0, now let (10)

$$h_{top}(f, Z, \mathcal{U}) = \overline{s} = \sup\{s : M_h(\mathcal{U}, Z, s) = +\infty\} = \inf\{s : M_h(\mathcal{U}, Z, s) = 0\}.$$

Finally the number

$$h_{top}(f, Z) = \lim_{\Delta(\mathcal{U}) \to 0} h_{top}(f, Z, \mathcal{U}), \ \Delta(\mathcal{U}) = \text{diameter of } \mathcal{U} \tag{11}$$

is the *topological entropy* of f restricted to Z. Indeed the topological entropy is a special case of characteristic dimension for the following $C-$structure:

$$\mathcal{F}_\varepsilon = \{B_{n,\varepsilon}(x) : x \in X, \ \varepsilon > 0\}, \ \ \eta(B_{n,\varepsilon}(x)) = e^{-n}, \ \ \psi(B_{n,\varepsilon}(x)) = \frac{1}{n}.$$

The characteristic dimension verifies [8]:
-$\dim_C \varnothing \leq 0$
- If $Z_1 \subset Z_2$ then $\dim_C Z_1 \leq \dim_C Z_2$.

-$\dim_C(\cup Z_i) = \sup_i \dim_C(Z_i)$, $i = 0, 1, ..., Z_i \subset X$.

Let μ be a finite measure on a set X with a $C-$structure $\mathcal{F}$. Let us assume that for any $\mu - a.e.$ point $x \in X$ and for $\varepsilon > 0$, any set $U \in \mathcal{F}$ with $\psi(U) \le \varepsilon$ satisfies $\mu(U) > 0$. Besides we shall also assume that the following condition stronger than iii), holds: there is a $\varepsilon > 0$ such that for any $\varepsilon_0 > \varepsilon > 0$ exists a finite subcollection $\mathcal{G}$ of $\mathcal{F}$ which covers and with $\psi(U) = \varepsilon$ for any $U \in \mathcal{G}$. Thus for any $x \in X$ and arbitrary $\varepsilon > 0$ there exists a set

$U(x,\varepsilon) \ni x$ such that $\psi(U(x,\varepsilon)) = \varepsilon$, thus we have the subcollection

$\mathcal{F}' = \{U(x,\varepsilon) \in \mathcal{F},\ x \in X,\ \varepsilon > 0\}$. The *upper and lower Carathéodory pointwise dimensions* of the measure μ are respectively

$$\overline{d_{C,\mu}}(x) = \overline{\lim_{\varepsilon\to 0}} \frac{\log U(x,\varepsilon)}{\eta(U(x,\varepsilon))}$$

$$\overline{d_{C,\mu}}(x) = \underline{\lim_{\varepsilon\to 0}} \frac{\log U(x,\varepsilon)}{\eta(U(x,\varepsilon))}.$$

By the made assumptions this quantities are well defined $\mu - a.e.$ for any x.

Theorem B ([8]):

i) If $\underline{d_{C,\mu}}(x) \le D$, $\mu - a.e$ for any $x \in Z$, then $\dim_C Z \le D$.

ii) If $\overline{\underline{d_{C,\mu}}}(x) = \overline{d_{C,\mu}}(x) = D$, for any x, then $\dim_C \mu := \inf\{\dim_C Z : \mu(Z) = 1\}$ $= D$.

A "dynamical" $C-$structure can be given as follows:

Let $f : X \to X$ be a continuous map and (X, d) a compact metrizable space. Let $\mathcal{U} = \{U_1, ..., U_N\}$ be finite covering of X and a string is defined as a sequence $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ such that $\{U_{\ell_0}, ..., U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, ...N\}$. The length of the string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap ... \cap f^{-n+1}(U_{\ell_{n-1}}),$$

if $Z \subset X$ it says that $\Pi = \{L = (\ell_0, \ell_1, ..., \ell_{n-1})\}$ covers Z if

$$Z \subset \bigcup_{L\in\Pi} X(L).$$

For any real number s:

$$M_h(Z,\mathcal{U}, s, n) = \inf_{\Gamma} \sum_{L\in\Pi} \exp(-sn(L)),$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which covers Z .

Let

$$M_h(\mathcal{U}, Z, s) = \lim_{n\to\infty} M_h(\mathcal{U}, Z, s, n).$$

There is an unique number $\overline{s}$ such that $M(\mathcal{U}, Z, .)$ jumps from $+\infty$ to 0, now let

$$h_{top}(f, Z, \mathcal{U}) = \overline{s} = \sup\{s : M_h(\mathcal{U}, Z, s) = +\infty\} = \inf\{s : M_h(\mathcal{U}, Z, s) = 0\}.$$

Finally the number

$$h_{top}(f, Z) = \lim_{\Delta(\mathcal{U})\to 0} h_{top}(f, Z, \mathcal{U}), \ \Delta(\mathcal{U}) = \text{diameter of } \mathcal{U}$$

is the *topological entropy* of f restricted to Z. Indeed the topological entropy is a special case of characteristic dimension for the following $C-$structure:

$$\mathcal{F}_\varepsilon = \{B_{n,\varepsilon}(x) : x \in X, \ \varepsilon > 0\}, \ \ \eta(B_{n,\varepsilon}(x)) = e^{-n}, \ \ \psi(B_{n,\varepsilon}(x)) = \frac{1}{n}.$$

As a characteristic dimension the topological entropy has these properties :

- If $Z_1 \subset Z_2$ then $h_{top}(f, Z_1) \le h_{top}(f, Z_2)$.
- If μ is a $f-$invariant measure and Z is a set with $\mu(Z) = 1$ then $h_{top}(f, Z) \ge h_\mu(f)$.
- If $Z = \cup Z_i$ then $h_{top}(f, Z) = \sup_i h_{top}(f, Z_i)$.

Notice also that for the $C-$structure considered results: $\underline{d_{C,\mu}}(x) = \underline{h_\mu(f,x)}$ and $\overline{d_{C,\mu}}(x) = \overline{h_\mu}(f, x)$. Of course if the local entropy exists (c.f. Brin-Katok theorem) then $h_\mu(f, x) = d_{C,\mu}(x)$ for every x, $\mu - a.e.$

4. Multifractal analysis with simultaneous actions of groups

We begin by recalling some facts about the non-additive thermodynamic formalism developed by L. Barreira. Let (X, d) be a compact metric space and let $f : X \in X$ be a continuous map. Let $\mathcal{U} = \{U_1, ..., U_N\}$ be finite covering of X and a string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ such that $\left\{U_{\ell_0}, ..., U_{\ell_{n-1}}\right\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, ...N\}$. As before let us denote by $n(L) = n$ the length of the string L and by $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$. For a sequence $\Phi = \{\varphi_n : X \to \mathbf{R}\}$ of potentials let us introduce the quantity

$$\nu_n(\Phi, \mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in L, \ \text{some } L \in W_n(\mathcal{U})\}. \tag{12}$$

For a string set $\varphi(L) := \begin{cases} \sup\{\varphi_n\} & \text{if } X(L) \neq \varnothing \\ -\infty & \text{if } X(L) = \varnothing \end{cases}$. For any real number s and $Z \subset X$ let

$$M(\Phi, Z, \mathcal{U}, s, n) = \inf_\Gamma \sum_{L\in\Pi} \exp(-sn(L) + \varphi(L)),$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which covers Z . Let now

$$M(\Phi, \mathcal{U}, Z, s) = \lim_{n\to\infty} M(\Phi, Z, \mathcal{U}, s, n). \tag{13}$$

There is an unique number $\overline{s}$ such that $M(\Phi,\mathcal{U},Z,s)$ jumps from $+\infty$ to 0, now let
(14)
$$P(\Phi,\mathcal{U},Z)=\overline{s}=\sup\{s:M(\Phi,\mathcal{U},Z,s)=+\infty\}=\inf\{s:M(\Phi,\mathcal{U},Z,s)=0\}.$$

It is assumed that $\lim\limits_{\Delta(\mathcal{U})\to 0}\limsup\limits_{n\to\infty}\nu_n(\Phi,\mathcal{U})=0$. Thus the quantity $P(\Phi,Z)=\lim\limits_{\Delta(\mathcal{U})\to 0}P(\Phi,\mathcal{U},Z)$ is the *topological pressure* of the sequence Φ restricted to the set Z. When Z is the whole space X is written directly $P(\Phi)$.

A sequence of potentials $\Phi=\{\varphi_n\}$ is called *additive* if $\varphi_{n+m}(x)=\varphi_n(x)+\varphi_m(f^n(x))$, for every x,n,m. One example of additive sequences is
$\varphi_n(x)=S_n(\varphi)(x):=\sum\limits_{i=0}^{n-1}\varphi(f^i(x))$, $\varphi:X\to\mathbf{R}$. In the special case when $\Phi=\{\varphi_n=S_n(\varphi)\}$, it has $P(\Phi)=P(\varphi)$, the usual topological pressure for a potential φ. Thus the Ruelle thermodynamic formalism is recovered for the additive case.

We will say that a sequence $\Phi=\{\varphi_n\}$ has a *limit* ψ , with respect to the dynamics $f:X\to X$ if $\varphi_{n+1}-\varphi_n\circ f$ converges uniformly to ψ in X. The variational principle in this non-additive case, and for a sequence Φ with a limit ψ, was established by Barreira and reads:

$$P(\Phi)=\sup\left\{h_\mu(f)+\int\psi d\mu:\mu\in\mathcal{M}_f(X)\right\}, \tag{15}$$

here $\mathcal{M}_f(X)$ denotes the $f-$invariant probability measures on X.

The classical variational principle is obtained for the additive sequence $\varphi_n=S_n(\varphi)$, whose limit is φ.

We recall a few concepts from topological dynamics:

A homeomorphism $f:X\to X$ is called *expansive* if there is a constant $\delta>0$ (the expansiveness constant), such that $d(f^n(x),f^n(y))<\delta$, for any integer n implies $x=y$. A set $E\subset X$ is $(n,\varepsilon)-$ *separated* if the $n-$orbits of all of its points have orbits can be distinguished with precision ε, i.e. if any $x,y\in E$, $x\neq y$, then $d_n(x,y)>\varepsilon$. If X is compact any separated subset of X is finite.

The *specification property* is a notion introduced by R. Bowen[**?**], naively it says that specified orbit trajectories of a map $f:X\to X$ can be approximated by adequate chosen periodic orbits of f. This condition ensures abundance of periodic points. The formal definition is:

Let $I_1,...,I_k$ be a finite disjoint collection of integer intervals $I_1,...,I_k$, then for $\varepsilon>0$, there is a integer $M(\varepsilon)$

and a function $\phi:I=\cup I_i\to X$, such that:

i) $dist(I_i,I_j)>M(\varepsilon)$ (Euclidean distance)

ii) $f^{n_1-n_2}(\phi(n_1))=\phi(n_2)$

iii) $d(f^n(x),\phi(n))<\varepsilon$, for some $x:f^m(x)=x$, with $m\geq M(\varepsilon)+length(I)$ and for every $n\in I$.

By $\nu_f(X)$ will be denoted the class of potentials ψ which satisfy this distortion condition:

There are constants $\varepsilon,K>0$ such that

$$d_n(x,y) < \varepsilon \Longrightarrow |S_n(\varphi)(x) - S_n(\varphi)(y)| < K.$$

This class includes the Hölder continuous maps on hyperbolic sets of Riemannian manifolds.

An *equilibrium state* for a sequence $\Phi = \{\varphi_n\}$ with a limit ψ with respect to to the dynamics $f : X \to X$ is a measure $\mu = \mu_\Phi \in \mathcal{M}_f(X)$ such that

$$P(\Phi) = h_\mu(f) + \int \psi d\mu. \tag{16}$$

If X is a compact metric space and $f : X \to X$ is an expansive homeomorphism which have the specification property then any sequence $\Phi = \{\varphi_n\}$ with a limit ψ belonging to the class $\nu_f(X)$ has a unique equilibrium state μ_Φ, which is a Gibbs state. Besides it is ergodic [6]-[9].

The set of *tangent functionals* to the pressure $P = P(\Phi, f)$ at Φ is

$$\tau_\Phi = \left\{\mu : P(\Phi+\Psi) - P(\Psi) \geq \mu(\rho) := \int \rho d\mu : \text{ for any potential } \Psi\right\}. \tag{17}$$

Here ρ denotes the limit of Ψ. Thus in the one-dimensional case we have that for $q, q' \in \mathbf{R}$ holds
$\dfrac{P((q+q')\Phi) - P(q\Phi)}{q'} \geq \int \psi d\mu_{q'}$, here $q\Phi$ means $\{q\varphi_n\}$. This uniqueness of equilibrium states leads to $\tau_\Phi = \mathcal{M}_f$, this results known for the additive case[14], can be immediately extended to the sequences with limit. So that the equilibrium states are in correspondence with the tangents to the graphics of the map $q \mapsto P(q\Phi)$, therefore the uniqueness of equilibrium state under the imposed conditions, implies that this map is differentiable and $\dfrac{dP(q\Phi)}{dq} = \int \psi d\mu_q$, where μ_q is the equilibrium state for the potential $q\Phi$.

A $f-$invariant measure μ_Φ is a *Gibbs state* associated to a sequencel $\Phi = \{\varphi_n\}$ and dynamics f if sufficiently small $\varepsilon > 0$, there are constants A_ε, $B_\varepsilon > 0$, such that for any $x \in X$ and for any positive integer n:

$$A_\varepsilon(\exp(\varphi_n(x)) - nP(\Phi)) \leq \mu_\Phi(B_{n,\varepsilon}(x)) \leq B_\varepsilon(\exp(\varphi_n(x)) - nP(\Phi)).$$

Taking a decomposition of X in its ergodic components it may be considered in the formalism ergodic measures. In this way it may be assumed that $\lim_{n\to\infty} \dfrac{1}{n}\varphi_n(x) = \int \psi d\mu$, $\mu - a.e.$, for any sequence Φ with limit ψ. Since the Gibbs states are ergodic we have that $h_{\mu_\Phi}(x,f) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\dfrac{1}{n}\log\mu_\Phi(B_{n,\varepsilon}(x)) = P(\Phi) - \int \psi d\mu_\Phi$. But by the ergodicity of the measure μ_Φ we have $h_{\mu_\Phi}(x,f) = h_{\mu_\Phi}(f)$, $\mu_\Phi - a.e.$ and so any Gibbs state is an equilibrium state. Now under the conditions of expansiveness and specification for the dynamics and bounded distortion for the limit ψ any sequence $\Phi = \{\varphi_n\}$ has an unique Gibbs state μ_Φ.

We consider the following situation: let X be a compact metric space and be a homeomorphism $f : X \to X$, let $\Phi_i = \{\varphi_n^i : X \to \mathbf{R}\}$, $i = 1, 2, ..., d$, where each sequence $\{\varphi_n^i\}$ has a limit ψ_i, with respect to f. We denote $\Psi = \{\psi_1, \psi_2, ..., \psi_d\}$ and we term this vector as a limit of $\Phi = \{\Phi_i\}_{i=1,2,...,d}$. Let us denote by $(\mathbf{q},\Phi)$, $\mathbf{q} =(q_1, ..., q_d) \in \mathbf{R}^d$, the potential $\sum_{i=1}^{d} q_i\psi_i$ and consider a "free energy" $T(\mathbf{q}) = P((\mathbf{q},\Phi)) = P(\sum_{i=1}^{d} q_i\psi_i)$. Therefore if $\psi_1, \psi_2, ..., \psi_d \in \nu_f\,(X)$ and f has the properties of expansiveness and specification then $\Phi_{\mathbf{q}} := (\mathbf{q},\Phi)$ has an unique Gibbs state $\mu_{\mathbf{q}}$, for every $\mathbf{q} \in \mathbf{R}^d$ and so that the map $\mathbf{q} \longmapsto T(\mathbf{q})$ is C^1 for any $\mathbf{q} \in \mathbf{R}^d$ with

$$\nabla T(\mathbf{q}) = \left(\int \psi_i d\mu_{\mathbf{q}}\right)_{i=1,2,..,d} . \tag{18}$$

Next we do the following simultaneous multifractal decompositions for the spectrum of local entropies: Let $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{R}^d$ and

$$K_\alpha = K_\alpha^\Phi = \bigcap_{i=1}^{d} \left\{x \in X : \lim_{n\to\infty} \frac{1}{n}\varphi_n^i(x) = \alpha_i\right\}. \tag{19}$$

The family of measures $\{\mu_{\mathbf{q}}\}_{\mathbf{q}=(q_1,...,q_d)\in\mathbf{R}^d}$, of Gibbs states for the potential $(\mathbf{q},\Phi)$, describes this spectrum. This means that there is a level parametrization $\alpha(\mathbf{q})$ with $\mu_{\mathbf{q}}(K_{\alpha(\mathbf{q})}) = 1$, $K_{\alpha(\mathbf{q})} = \bigcap_{i=1}^{d} K_{\alpha_i(q_i)}$. This is inmediately seen taking $\alpha(\mathbf{q}) = -\nabla T(\mathbf{q})$ and using the "ergodic" assumption $q_i \lim_{n\to\infty} \frac{1}{n}\varphi_n^i(x) = q_i \int \psi_i d\mu$.

We make now a consideration about the domain of the map $\alpha \mapsto \mathcal{E}(\alpha) := h_{top}\,(f, K_\alpha) = h_{top}\left(f, \bigcap_{i=1}^{d} K_{\alpha_i}\right)$. For $d = 1$, the domain of $\mathcal{E}(\alpha)$ is an interval (α_i, α_s), for this range of values of α is ensured that if $\alpha \in (\alpha_i, \alpha_s)$ then $K_\alpha \neq \varnothing$[8, 12]. Let us consider the map

$$\begin{gathered} \mathcal{I}_\Phi : \mathcal{M}\,(X) \to \mathbf{R}^d \\ \mathcal{I}_\Phi(\mu) = \int \Phi d\mu := \left(-\int \psi_1 d\mu, ..., -\int \psi_d d\mu\right). \end{gathered} \tag{20}$$

Thus $\mathcal{I}_\Phi(\mathcal{M}\,(X)) = \{\alpha \in \mathbf{R}^d : \int \Phi d\mu = \alpha,\ \text{for some}\ \mu \in \mathcal{M}\,(X)\}$.
If $\alpha(\mathbf{q}) = -\nabla T(\mathbf{q}) = \int \Phi d\mu_{\mathbf{q}}$ then there exists a measure $\mu_{\mathbf{q}} \in \mathcal{M}\,(X)$, such that $\mathcal{I}_\Phi(\mu_{\mathbf{q}}) = \alpha(\mathbf{q})$. Therefore if $\alpha = \alpha(\mathbf{q}) \in int(\mathcal{I}_\Phi(\mathcal{M}\,(X)))$ then $K_\alpha \neq \varnothing$. So that it can be take as domain of $\mathcal{E}(\alpha) := h_{top}\,(f, K_\alpha)$ the set $int(\mathcal{I}_\Phi(\mathcal{M}\,(X)))$.

Now we state the following variational result, which generalizes a similar one obtained by Barreira, Saussol and Schmeling in [2]. In that article the formula was derived for a class of Hölder continuous potentials and for systems which admit a symbolic representation. The conditions (expansiveness and specification) imposed on the dynamics are weaker than the existence of Markov partitions and the class $\nu_f(X)$ contains, as mentioned earlier, the class of Hölder continuous maps on hyperbolic sets. In [13] was demostrated the additive case corresponding to the below theorem, however the approach followed herein is different.

Theorem 1: Let $f : X \to X$ be an expansive homeomorphism which have the specification property and let $\Phi_i = \left\{\varphi_n^i : X \to \mathbf{R}\right\}$, $i = 1, 2, ..., d$, where each sequence $\left\{\varphi_n^i\right\}$ has a limit ψ_i, with respect to f belonging to the class $\nu_f(X)$. Let $K_\alpha = K_\alpha^\Phi = \bigcap_{i=1}^{d} \left\{x \in X : \lim_{n\to\infty} \frac{1}{n}\varphi_n^i(x) = \alpha_i\right\}$ then

$\mathcal{E}_\Phi(\alpha) = h_{top}(f, K_\alpha) = \sup\{h_\mu(f) : \mathcal{I}(\mu) = \alpha\}$.

The proof will follow by joining the next lemmata:

Lemma 1: $\mathcal{E}(\alpha) \leq \sup\left\{\delta : P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \delta : \mathbf{q} \in \mathbf{R}^d\right\}$, with $\mathbf{q}.\alpha = \sum_{i=1}^{d} q_i\alpha_i$.

Proof: Let $\delta > 0$ such that $P(-\delta) > 0$, now by a property from the Barreira non-additive thermodynamic formalism gets

$P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \delta \geq P(-\delta, K_\alpha) \geq 0$.

Lemma 2: Let $T_{\alpha,\delta}(\mathbf{q}) = T(\mathbf{q}) - \mathbf{q}.\alpha - \delta$, with $\alpha \in int[\mathcal{M}(X)]$, then $T_{\alpha,\delta}(\mathbf{q})$ has a maximum in an open disc D_R.

Proof: We have $\nabla T_{\alpha,\delta}(\mathbf{q}) = \left(\int \psi_i d\mu_{\mathbf{q},\alpha,\delta} - \alpha_i\right)_{i=1,2,..,d}$, where $\mu_{\mathbf{q},\alpha,\delta}$ is the Gibbs state corresponding to $(\mathbf{q}, \Phi) - \mathbf{q}.\alpha - \delta$. So, by the variational principle

$T_{\alpha,\delta}(\mathbf{q}) = T(\mathbf{q}) - \mathbf{q}.\alpha - \delta \geq h_\mu(f) + \int (q, \Psi)d\mu - \mathbf{q}.\alpha - \delta$, for any invariant measure μ, where Ψ denotes the $d-$uple $(\psi_1, ..., \psi_d)$ and $(q, \Psi) := \sum_{i=1}^{d} q_i\psi_i$. Thus

$T_{\alpha,\delta}(\mathbf{q}) \geq \int (q, \Psi)d\mu - \mathbf{q}.\alpha - \delta$, for any invariant measure μ.

Let us consider an enough small $\lambda > 0$ such that $\overline{\alpha} = \alpha + \lambda s(\mathbf{q}) \in int[\mathcal{M}(X)]$, where $s(\mathbf{q}) = (s_i(q)) := (sign(q_i))$. Therefore there exists a measure μ such that $\int \psi_i d\mu = \overline{\alpha}_i$, $i = 1, 2, ..., d$.

Let $S(\mathbf{q}) = T(\mathbf{q}) - \mathbf{q}.\alpha - \delta$, so $S(\mathbf{q}) \geq \sum_{i=1}^{d} q_i(\alpha_i - \overline{\alpha}_i) = \sum_{i=1}^{d} q_i sign(q_i)\lambda - \delta \geq \lambda \sum_{i=1}^{d} |q_i| - \delta$.

Thus we have that if $|\mathbf{q}| = \sum_{i=1}^{d} q_i \geq R := \dfrac{\delta + S(\mathbf{0})}{\lambda}$ then $S(\mathbf{q}) \geq \dfrac{\delta + S(\mathbf{0})}{\lambda}\lambda - \delta = S(\mathbf{0})$. So that $S(\mathbf{q})$ has a minimum in the open disc $D_R = \left\{|\mathbf{q}| < R := \dfrac{\delta + S(\mathbf{0})}{\lambda}\right\}$.

Lemma 3: For any $\alpha \in int[\mathcal{M}(X)]$ there exists an ergodic measure μ such that $\mathcal{I}(\mu) = \alpha$ and $h_\mu(f) \geq \delta$, for some $\delta > 0$.

Proof: By the lemma 2 there is vector$\overline{\mathbf{q}} = \overline{\mathbf{q}}(\alpha, \delta) \in D_R$ such that $\nabla T_{\alpha,\delta}(\overline{\mathbf{q}}) = \mathbf{0}$, or $\int \psi_i d\mu_{\overline{\mathbf{q}}} = \alpha_i$ and so

$\mathcal{I}(\mu_{\overline{\mathbf{q}}}) = \alpha$, where $\mu_{\overline{\mathbf{q}}} = \mu_{\mathbf{q},\alpha,\delta}$ is the Gibbs state corresponding to the potential $\Phi_{\mathbf{q}} = (\mathbf{q}, \Phi) - \mathbf{q}.\alpha - \delta$.Now the measure that we are looking for will be precisely $\mu_{\overline{\mathbf{q}},\alpha,\delta}$, which being a Gibbs state is ergodic and, by the Birkhoff ergodic theorem $\mu_{\overline{\mathbf{q}}}(K_\alpha) = \mu_{\overline{\mathbf{q}}}(\bigcap_{i=1}^{d} K_{\alpha_i}) = 1$.

If δ is chosen, like in the lemma 1, such that $P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \delta \geq 0$, we will have

$0 \leq P((\mathbf{q}, \Phi)) - \overline{\mathbf{q}}.\alpha - \delta = h_{\mu_{\overline{\mathbf{q}}}}(f) + (\overline{\mathbf{q}}, \Psi) - \overline{\mathbf{q}}.\alpha - \delta$ and so $h_{\mu_{\overline{\mathbf{q}}}}(f) \geq \delta$.

Lemma 4: It holds the following variational formula

$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) = \sup_{\mu \in \mathcal{M}_E(X)} \{h_\mu(f) : \mathcal{I}(\mu) = \alpha\}$, where $\mathcal{M}_E(X)$ denotes the set of ergodic invariant measures on X.

Proof: By the above lemma we have that

$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \leq \sup_{\mu \in \mathcal{M}_E(X)} \{h_\mu(f) : \mathcal{I}(\mu) = \alpha\}$. To prove the opposite inequality we consider the ergodic measure $\mu_{\overline{\mathbf{q}}}$ determined in the lemma 3, we have that $\mu_{\overline{\mathbf{q}}}(K_\alpha) = 1$ and the Barreira variational principle gets

$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \geq h_{\mu_{\overline{\mathbf{q}}}}(f)$.

Lemma 5: It is valid: $\inf_{\mathbf{q} \in \mathbf{R}^d} \{P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \mathcal{E}(\alpha)\} = 0$.

Proof: Let $\overline{\delta}$ be such that $\inf_{\mathbf{q} \in \mathbf{R}^d} \left\{\delta : P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \overline{\delta} : \mathbf{q} \in \mathbf{R}^d\right\} = 0$. If $\overline{\delta} < \delta$ then

$\inf_{\mathbf{q} \in \mathbf{R}^d} \left\{\delta : P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \overline{\delta} : \mathbf{q} \in \mathbf{R}^d\right\} \leq \overline{\delta} - \delta < 0$. By the lemma 1 $\mathcal{E}(\alpha) \leq \overline{\delta}$ and so there exists an ergodic measure μ such that $\mathcal{I}(\mu) = \alpha$ and such that $h_\mu(f) \geq \delta$. Thus

$$\inf_{\mathbf{q} \in \mathbf{R}^d} \{P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \mathcal{E}(\alpha)\} = 0.$$

Lemma 6: $\mathcal{E}(\alpha) \geq h_\mu(f)$, for any measure $\mu \in \mathcal{M}_f(X)$ and $\mathcal{I}(\mu) = \alpha$.

Proof: Let μ any $f-$invariant measure, not necessarily ergodic, now

$0 \geq P((\mathbf{q}, \Phi)) - \mathbf{q}.\alpha - \mathcal{E}(\alpha) \geq + \int (q, \Psi) d\mu - \mathbf{q}.\alpha - \mathcal{E}(\alpha) = h_\mu(f) - \mathcal{E}(\alpha)$ and so

$h_\mu(f) \leq \mathcal{E}(\alpha)$.

We want now to apply the above computations for Fuchsian groups acting on H^2 satisfying the conditions displaying in Section 2. For this let $\Gamma_1, \Gamma_2, ..., \Gamma_d$ be finite volume groups with a fundamental region R which has all of its vertices in ∂H^2, or if not R has even corners and at least five sides. We additionally assume that if Λ_i is the limit set of the group Γ_i then $\bigcap_{i=1}^{d} \Lambda_i \neq \varnothing$. As we have commented in the introduction in [7] we treated the case $d = 1$, considering as dynamics the boundary hyperbolic maps, $f = f_\Gamma : \partial H^2 \to \partial H^2$, more precisely defined on the limit set of Γ. The situation in the case $d>1$ turns somewhat different since there are eventually different maps $f_i = f_{\ \Gamma_2}$.

The sequences to be considered are $\Phi_i = \left\{\varphi_n^i : \Lambda_i \to \mathbf{R}\right\}$, $i = 1, 2, ..., d$, given by $\varphi_n^i(\xi) = d_h\left(O, e_0^i e_1^i ... e_n^i(O)\right)$, where $e_0^i e_1^i ... e_n^i ...$ is the sequence associated to the point $\xi \in \Lambda_i$. We have established [7] that each $\Phi_i = \left\{\varphi_n^i\right\}$ has a limit $\psi_i = -\log\left|f_i^{'}\right|$. For any vector $\mathbf{q} = (q_1, q_2, ..., q_d) \in \mathbf{R}^d$, we shall denote $(\mathbf{q}, \Phi) = \sum_{i=1}^{d} q_i$ $\psi_i = -\sum_{i=1}^{d} q_i \log\left|f_i^{'}\right|$.

To see that each potential ψ_i belongs to the class $\nu_{f_i}(\Lambda_i)$ are used facts of the bounded hyperbolic map which are contained in [4], [10],[11]. Let $\left(\ell_0^i, \ell_1^i, ..., \ell_{n-1}^i\right)$ be a $n-$string and for each map f_i we consider the "cylinder"

$C_n^i = \bigcap_{j=0}^{n-1} f_i^{-1}(I_{\ell_j}^i)$, where $\left\{I_k^i\right\}$ is a Markov partition of Λ_i over which the map is conjugated to a subshift (see theorem A). There is a string $L^i = \left(\ell_0^i, \ell_1^i, ..., \ell_{n-1}^i\right)$ which determines an element $g_L^i \in Isom(H^2)$ such that

$g_L^i = g_{\ell_{n-1}^i}^i \circ ... \circ g_{\ell_0^i}^i = f_i^{n-1}(C_n^i)$. Besides each g_L^i applies C_n^i in $I_{\ell n-1}^i$. For each $i = 1, 2, ..., d$ there are numbers K_i such that for any cylinder C_n^i holds

$\left|\log(g_L^i)^{'}(\xi_1) - \log(g_L^i)^{'}(\xi_2)\right| \leq K_i \left|g_L^i(\xi_1) - g_L^i(\xi_2)\right|$, in particular there are numbers M_i, $i = 1, 2, ..., d$, such that

$$\frac{1}{M_i} \leq \left|\frac{(f_i^{n-1})^{'}(\xi_1)}{(f_i^{n-1})^{'}(\xi_2)}\right| \leq M_i, \tag{21}$$

for any $\xi_1, \xi_2 \in C_n^i$, for any n.

The above bounded distortion property for the potentials $\psi_i = -\log\left|f_i^{'}\right|$ and the fact that the dynamics, in this case the Bowen-Series maps, posses the properties of expansiveness and specification [4], [10],[11],[7], ensure that any $q_i\psi_i$ has an unique Gibbs state μ_{q_i,f_i}. For $\Phi = \Phi_i = \left\{\varphi_n^i\right\}$ and $\mathbf{q} = (q_1, q_2, ..., q_d)$ we set

$$\mu_{\Phi,\mathbf{q},\{f_i\}} = \mu_{q_1,f_1} \times ... \times \mu_{q_d,f_d}, \tag{22}$$

where is defined on $\Lambda := \bigcap_{i=1}^{d} \Lambda_i$ and each factor μ_{q_i,f_i} is supported on orbits of the f_i.

We want now to consider a characteristic dimension of the multifractal sets to be variationally described. In this vein we introduce the following dynamic ball

(23)
$$B_{n,\varepsilon}^d(x) = \left\{y : \left|f_1^i(x) - f_1^i(y)\right| < \varepsilon, ..., \left|f_d^i(x) - f_d^i(y)\right| < \varepsilon,\ i = 0, 1, ..., n-1\right\}.$$

Since each factor of the measure $\mu_{\Phi,\mathbf{q},\{f_i\}}$ is supported in orbits of the each dynamic f_i, we have that

$$\mu_{\Phi,\mathbf{q},\{f_i\}}(B^d_{n,\varepsilon}(x)) = \mu_{q_1,f_1}(B^{f_1}_{n,\varepsilon}(x))...\mu_{q_d,f_d}(B^{f_d}_{n,\varepsilon}(x)) \tag{24}$$

where $B^{f_2}_{n,\varepsilon}(x) = \left\{y : \left|f^i_j(x) - f^i_j(y)\right| < \varepsilon,\ j = 0, 1, ..., n-1\right\}$.

We now propose a variant for the definition of the topological entropy in the following way: let $\Lambda_1, ..., \Lambda_d$ be the limit sets of the groups $\Gamma_1, ..., \Gamma_d$ and let $\mathcal{U}_1, ..., \mathcal{U}_d$ be finite coverings of the $\Lambda_1, ..., \Lambda_d$ respectively. We consider strings $L_1 = \left(\ell^1_0, \ell^1_1, ..., \ell^1_{n-1}\right)$, $L_2 = \left(\ell^2_0, \ell^2_1, ..., \ell^2_{n-1}\right), ..., L_d = \left(\ell^d_0, \ell^d_1, ..., \ell^d_{n-1}\right)$ and set

$$\begin{aligned} X(L_1, L_2, , , .L_d) = U_{\ell^1_0} \cap f^{-1}\left(U_{\ell^1_1}\right) \cap ... \cap f^{-n+1}\left(U_{\ell^1_{n-1}}\right) \cap ....\cap \\ U_{\ell^d_0} \cap f^{-1}\left(U_{\ell^d_1}\right) \cap ... \cap f^{-n+1}\left(U_{\ell^d_{n-1}}\right). \end{aligned} \tag{25}$$

The "multiset" of strings $\Pi_d = \{L_1, L_2, , , .L_d\}$ covers $Z \subset \Lambda^d := \bigcap_{i=1}^{d} \Lambda_i$ if

$$Z \subset \bigcup_{(L_1, L_2, , , .L_d) \in \Pi_d} X(L_1, L_2, , , .L_d). \tag{26}$$

We shall denote the topological entropy of a set Z obtained in this manner by $h^d_{top}(Z)$. In a similar way we can introduce a local entropy for several dynamics: we set

$$h_\mu(f_1, f_2, ..., f_d; x) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\log\mu\left(B^d_{n,\varepsilon}(x)\right). \tag{27}$$

Now we state

Theorem 2: Let $\Gamma_1, \Gamma_2, ..., \Gamma_d$ be Fuchsian groups acting on H^2 fulfilling the conditions mentioned along the precedent section, with respective limit sets Λ_i and Bowen-Series maps f_i. Let us consider $\Phi_i = \left\{\varphi^i_n : \Lambda_i \to \mathbf{R}\right\}$, $i = 1, 2, ..., d$, with $\varphi^i_n(\xi) = d_h\left(O, e^i_0 e^i_1 ... e^i_n(O)\right)$, and the simultaneous multifractal decompositions $K_{\alpha_i} = \left\{\xi : \lim_{n\to\infty}\frac{\varphi_n(\xi)}{n} = \alpha_i\right\}$, $i = 1, 2, ..., d$, if $K_\alpha = \bigcap_{i=1}^{d}\left\{\xi : \lim_{n\to\infty}\frac{\varphi_n(\xi)}{n} = \alpha_i\right\}$ then

$h^d_{top}(K_\alpha) = \sum_{i=1}^{d}\sup\left\{h_\mu(f_i) : -\log\int\left|f'_i\right| = \alpha_i\right\}.$

Proof: We have, by the definition of the measures, $\mu_i = \mu_{q_i,f_i}$, and by their ergodicity, that

$h_{\mu_{\Phi,\mathbf{q},\{f_i\}}}(f_1, f_2, ..., f_d; x) = \sum_{i=1}^{d} h_{\mu_i}(f_i, x) =: \sum_{i=1}^{d} h_{top}(f_i, K_{\alpha_i})$. Thus, by *i*) of theorem B (section 3) we get

$h_{top}^d(K_\alpha) \leq H := \sum_{i=1}^{d} h_{top}(f_i)$. Since $\mu_{\Phi,\mathbf{q},\{f_i\}}(K_{\alpha(\mathbf{q})})) = 1$, applying again theorem B, in this case part ii), we
obtain

$h_{top}^d(K_\alpha) \geq H$. So that
$h_{top}^d(K_\alpha) = \sum_{i=1}^{d} h_{top}(f_i, K_{\alpha_i}) = \sum_{i=1}^{d} \sup \left\{ h_\mu(f_i) : -\log \int \left| f_i^{'} \right| = \alpha_i \right\}$.

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