

EXCEPTIONAL DYNAMICALLY INVARIANT SECTIONS

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ABSTRACT. In [33] we developed a notion of a Green's section that generalized the of Green's currents that arise in higher dimensional holomorphic dynamical systems. In such systems there may be totally invariant currents other than the Green's currents. These are called exceptional currents. We thus define a corresponding notion of an *exceptional section* for a continuous self map $f: X \rightarrow X$ of a topological space X .

Our notion requires the use of sheaves on X and cohomorphisms of these sheaves. For holomorphic endomorphisms for which the methods of [33] give Green's currents we show that corresponding exceptional hypersurfaces are attracting and give rise to a function, plurisubharmonic off the attractor, which measures the rate of contraction. We also show that if such an exceptional hypersurface can be perturbed to a smooth current then the Green's current is supported in the closure of the basin of attraction.

Lastly, we give conditions under which exceptional sections in more general settings are necessarily attracting.

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1. INTRODUCTION

Invariant objects are an essential tool in the study of dynamical systems. In a previous paper, [33], we gave a method for constructing invariant objects for a continuous self map $f: X \rightarrow X$ of a topological space X . These objects are supported on sets on which f is "cohomologically expanding" and generalize the Green's currents that have been useful in studying holomorphic dynamical systems to other systems as well as extending the known results for holomorphic systems.

The Julia set of a rational self map of $\overline{\mathbb{C}}$ is an essential example of such an invariant current. These have been studied extensively, some sample sources are

[15], [14], [30], [31], [29]. In higher complex dimensions the study of invariant closed positive (p, p) currents has been a substantial tool for automorphisms, such as complexified Hénon maps [19], [20], [25], [26], [7], [8], [9], [5], [10], [11], [12] and for automorphisms of K3 surfaces or K3 surfaces [13], [28]. For endomorphisms of $\mathbb{P}^n$ a great deal of work has been done relating a special invariant closed positive $(1, 1)$ current to its dynamics [27], [21], [22], [23], [3], [2], [32], [18], [34], [35]. In the case of holomorphic automorphisms and endomorphisms of Kähler manifolds, nice results showing the existence of invariant closed positive $(1, 1)$ currents and some of their basic properties have been obtained in [16].

In all of the above cases, the current under study is distinguished by the fact that it is not only invariant, but that the successive pullbacks of many other (p, p) currents, suitably rescaled, will converge to the given invariant current, i.e. the current is attracting under pullbacks. Such a current is called a *Greens* current. In [16], a special weighted average of the pullbacks of a given current is used instead, so the currents so constructed may not be attracting in the same sense, (but this allows them to obtain invariant currents in a much broader set of circumstances).

Whether a Greens current attracts all holomorphic varieties for which there is no cohomological obstruction has been a question of interest for some time. It is well known that the successive preimages of a point will converge to the Julia set of a rational map R for all but at most two points of $\overline{\mathbb{C}}$. Any points which do not have this property are called exceptional and must be totally invariant (for $R(R(z))$). Favre and Jonsson have shown the corresponding result for endomorphisms of $\mathbb{P}^2$, the 2-dimensional analogue of the Riemann sphere ([18]). In this case, the Green's current is a closed positive bidegree $(1, 1)$ current, so the result of Favre and Jonsson concerns pullbacks of hypersurfaces, not pullbacks of points. Again, it was shown that an irreducible hypersurface which does not converge to the Greens current must be totally invariant (under some iterate). By taking a generalized "alternating product" of the Green's current with itself one obtains higher degree Green's currents. Previously, Briend and Duval [3] showed for holomorphic endomorphisms of $\mathbb{P}^n$ that successive preimages of points converge to the closed positive bidegree (n, n) Green's current (which they also showed is the unique measure of maximal entropy). Again their result held for all points outside of a totally invariant holomorphic variety. Bedford & Smillie [7] showed that for Hénon maps of $\mathbb{C}^2$ that the successive preimages of any smooth algebraic curve converge to a Green's current.

Totally invariant varieties are sometimes called *exceptional varieties*. The existence of an exceptional variety is what distinguishes polynomial self maps of the Riemann sphere from rational self maps of the Riemann sphere. The added structure provided by the existence of an exceptional variety has been very significant in this case, e.g. through the use of external rays.

For holomorphic endomorphisms for which the methods of [33] give bidegree $(1, 1)$ Green's currents we show that any corresponding exceptional hypersurfaces are attracting and give rise to a function, which measures the rate of contraction. We also show that if such an exceptional hypersurface can be perturbed to a smooth current then this function is plurisubharmonic off the hypersurface and the Green's current is supported in the closure of the basin of attraction.

Having developed Green's sections in [33], it is natural to define exceptional sections in the same generality (e.g. nonholomorphic endomorphisms). We do so, and state some basic results. However, we note that typically the application of these methods requires the solution of an auxiliary problem of finding a section which is "invariant on a gross scale". Favre and Jonsson [17] have done a thorough study of exceptional fixed points and other exceptional sets in two complex dimensions, and their work indicates that the auxiliary problem may be very difficult to solve in most cases. Hence, we expect nonholomorphic or higher codimension applications to be quite specialized.

2. COHOMOMORPHISMS

2.1. Cohomomorphism Basic Facts. Here we provide some basic facts about cohomomorphisms of sheaves. We will use the functorial definition of a sheaf ([24],[36]) as we will need topologies on our sheaves that would clash with the topological definition of a sheaf.

Assume for this section that we are given a continuous map $f: X \rightarrow Y$ of topological spaces as well as sheaves $\mathcal{A}$ and $\mathcal{B}$ on X and Y respectively.

The following is adapted from [6] page 14–15.

A *cohomomorphism* is a generalized pullback from $\mathcal{B}$ to $\mathcal{A}$. More formally:

Definition 2.1. An "f-cohomomorphism" $k: X: \mathcal{B} \rightarrow \mathcal{A}$ is a collection of homomorphisms $k_U: \mathcal{B}(U) \rightarrow \mathcal{A}(f^{-1}(U))$, for U open in Y , compatible with restrictions.

An id_X cohomomorphism of sheaves on X agrees with the standard notion of "a morphism of sheaves on X ".

2.2. Cohomomorphisms and Γ . The functor Γ is a standard functor on topological spaces with sheaves, which returns the global sections of the given sheaf. For a sheaf cohomomorphism $F: \mathcal{B} \rightarrow \mathcal{A}$ then $\Gamma(F)$ is the induced homomorphism $\mathcal{B}(Y) \rightarrow \mathcal{A}(X)$ (see e.g., [6]).

Fix a continuous self map $f: X \rightarrow X$ of a topological space X . We let $f_{\mathcal{A}}: \mathcal{A} \rightsquigarrow \mathcal{A}$ be the default notation for an f -cohomomorphism of some sheaf $\mathcal{A}$ on X .

Assume that

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

is a short exact sequence of sheaves on X . Assume further that we are given f self cohomomorphisms of each of these sheaves and that

$$(1) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\quad p \quad} & \mathcal{B} & \xrightarrow{\quad q \quad} & \mathcal{C} \\ \downarrow f_{\mathcal{A}} & & \downarrow f_{\mathcal{B}} & & \downarrow f_{\mathcal{C}} \\ \mathcal{A} & \xrightarrow{\quad p \quad} & \mathcal{B} & \xrightarrow{\quad q \quad} & \mathcal{C} \end{array}$$

commutes. Such a construction will be called an f self-cohomomorphism of the short exact sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$.

We note that sheaves of $\mathbb{R}$ modules on a topological space X have enough injectives (see e.g. [6] page 41) and thus we can speak of the long exact sequence of $\mathbb{R}$ modules derived from the functor Γ .

Applying the functor Γ and its right derived functors to (1) yields the usual long exact sequence on each row. The resulting diagram of $\mathbb{R}$ modules is commutative ([6] pages 62–63).

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\mathcal{A}) & \xrightarrow{\Gamma p} & \Gamma(\mathcal{B}) & \xrightarrow{\Gamma q} & \Gamma(\mathcal{C}) \xrightarrow{\delta} H^1(X, \mathcal{A}) \longrightarrow \dots \\ & & \downarrow \Gamma f_{\mathcal{A}} & & \downarrow \Gamma f_{\mathcal{B}} & & \downarrow \Gamma f_{\mathcal{C}} \quad \downarrow H^1 f_{\mathcal{A}} \\ 0 & \longrightarrow & \Gamma(\mathcal{A}) & \xrightarrow{\Gamma p} & \Gamma(\mathcal{B}) & \xrightarrow{\Gamma q} & \Gamma(\mathcal{C}) \xrightarrow{\delta} H^1(X, \mathcal{A}) \longrightarrow \dots \end{array}$$

This arises naturally for dynamical systems by letting $\mathcal{C}$ be a sheaf of objects of dynamical interest, letting $\mathcal{B}$ be local potentials for the members of $\mathcal{C}$, and letting $\mathcal{A}$ be the kernel of $\mathcal{B} \rightarrow \mathcal{C}$. Often $\mathcal{C}$ is some sheaf of currents and $\mathcal{B}$ is potentials for those currents under d or dd^c . Typically $\mathcal{B}$ will be a soft sheaf.

3. INVARIANT GLOBAL SECTIONS

The approach in [33], while relatively elementary, allows for considerable generality in the type of invariant object considered. It is not required, for instance, to be a current. The basic data for the central theorem in [33] is a continuous self map $f: X \rightarrow X$ of a topological space X , a short exact sequence of sheaves

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

of $\mathbb{R}$ vector spaces, and a self-cohomomorphism of this exact sequence of sheaves. A self-cohomomorphism is the construction that tells us how to pull back a given sheaf through a self map of the base space. The main theorem of [33] gives conditions under which one can take an invariant member of $H^1(X, \mathcal{A})$ (i.e. an invariant $\mathcal{A}$ bundle) and construct a unique corresponding invariant global section of $\mathcal{C}$. As the existence of Greens currents under reasonably general circumstances is a necessary backdrop to motivate the study of exceptional sections, we recall the basic result from [33]. We also recall a relevant notion of expansiveness of a map $f: X \rightarrow X$ about a point $x \in X$.

Definition 3.1. Given a bundle $a \in H^1(X, \mathcal{A})$ such that $f_{\mathcal{A}}(a) = r \cdot a$ for some real number r , and given that there is some $c \in \Gamma(\mathcal{C})$ with $\delta(c) = a$, then we say that f is cohomologically expansive at x for a if for any open neighborhood U of x and any $c \in \Gamma(\mathcal{C})$ such that $\delta(c) = a$, the set U intersects the support of $f_{\mathcal{C}}^k(c)$ for all sufficiently large k .

We will refer to the following condition as well.

Postulate 1. For the given member $v \in H^1(X, \mathcal{A})$, for every $x \in X$ there is some $c \in \Gamma(\mathcal{C})$ such that $\delta(c) = v$ and x is not in the support of c .

Theorem 3.2. *We assume the following all hold:*

- *We are given a topological space X with a self map $f: X \rightarrow X$.*
- *We are given an f self cohomomorphism of the short exact sequence of sheaves on X ,*

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

- W is a finite dimensional $\mathbb{R}$ vector space, and there are $\mathbb{R}$ -linear maps $\iota: W \rightarrow H^1(X, \mathcal{A})$ and an invertible $\mathbb{R}$ -linear map $g: W \rightarrow W$ such that $f_A \iota = \iota g$,
- $\Gamma(\mathcal{B})$ is a Banach space over $\mathbb{R}$.
- $\Gamma(\mathcal{C})$ is a $\mathbb{R}$ vector space with a topology compatible with the vector space structure
- The maps $f_{\mathcal{B}}: \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{B})$, $f_{\mathcal{C}}: \Gamma(\mathcal{C}) \rightarrow \Gamma(\mathcal{C})$ and $q^*: \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{C})$ are all continuous.
- If a sequence c_i of members of $\Gamma(\mathcal{C})$ converge to another member c_∞ then the support of c_∞ is contained in the closure of the union of the supports of c_i .
- There exists some $\alpha, \lambda, c, d > 0$ with $\alpha < \lambda$ such that
 - (1) $\|g^{-k}(w)\| \leq c\lambda^{-k}\|w\|$ for $k \in \mathbb{N}$, $w \in W$ for some fixed norm $\|\cdot\|$ on W .
 - (2) $\|f_{\mathcal{B}}^k(b)\| \leq d \cdot \alpha^k \|b\|$ for $k \in \mathbb{N}$, $b \in \Gamma(\mathcal{B})$.

Then there is a unique $\mathbb{R}$ linear map $\kappa: W \rightarrow \Gamma(\mathcal{C})$ such that $\delta \circ \kappa = \iota$ and $f_{\mathcal{C}} \circ \kappa = \kappa \circ g$. Moreover, given any $c \in \Gamma(\mathcal{C})$ such that $\delta(c) = \iota(w)$ for some $w \in W$, and if $g: W \rightarrow W$ is dominated by a single simple real eigenvalue $r > 0$ with eigenvector v then $f_{\mathcal{C}}^k(c)/r^k$ converges to $\alpha\kappa(v)$ in $\Gamma(\mathcal{C})$, where αv is the component of w in the v direction when c decomposed into subspaces of W associated to eigenvalues. Finally, given any eigenvector $v \in W$ for which $\iota(v)$ satisfies Postulate 1 the support of the invariant member $\kappa(v) \in \Gamma(\mathcal{C})$ is contained in the set of points $x \in X$ on which f is cohomologically expansive for $\kappa(v)$.

4. EXCEPTIONAL SECTIONS

In this section we will make the following standing assumptions:

- We are given a topological space X with a self map $f: X \rightarrow X$.
- We are given an f self cohomomorphism $\tilde{f}$ of the short exact sequence of sheaves on X ,

$$\mathcal{A} \xrightarrow{p} \mathcal{B}' \xrightarrow{q} \mathcal{C}'$$

- There are subsheaves $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{C} \subset \mathcal{C}'$ such that

$$\mathcal{A} \xrightarrow{p} \mathcal{B} \xrightarrow{q} \mathcal{C}$$

is a short exact sequence of sheaves and such that the restriction of $\tilde{f}$ to $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ gives an f self cohomomorphism of this exact sequence of sheaves.

- $\Gamma(\mathcal{B})$ is a Banach space over $\mathbb{R}$.

4.1. Exceptional Sections. While we are really interested in exceptional sections in the context of Green's sections, it is more convenient to define them in the reverse order, in such a way that exceptional sections always correspond to a Green's section.

Definition 4.1. Assume that $\Gamma(\mathcal{B})$ is a Banach space and that $\Gamma(q): \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{C})$ is continuous, we will say that $c' \in \Gamma(\mathcal{C}')$ is an *exceptional section* if $c' \notin \Gamma(\mathcal{C})$,

$(\Gamma f_{\mathcal{C}})(c') = \lambda c'$ for some $\lambda \in \mathbb{R}$ and if the operator $\frac{1}{\lambda} \Gamma f_{\mathcal{B}}$ is uniformly contracting on $\Gamma(\mathcal{B})$.

There is always an associated Green's section.

Lemma 4.2. *Given an exceptional section $c' \in \Gamma(\mathcal{C}')$, there is a unique associated $c \in \Gamma(\mathcal{C})$ such that $(\Gamma f_{\mathcal{C}})(c) = \lambda c$ and $\delta(c) = \delta(c')$.*

Proof. We recall the long exact sequence with its self homomorphism:

$$(3) \quad \begin{array}{ccccc} \Gamma(\mathcal{C}') & \xrightarrow{\delta} & H^1(X, \mathcal{A}) & \xrightarrow{H^1 p} & H^1(X, \mathcal{B}) \\ & \downarrow \Gamma f_{\mathcal{C}} & \downarrow H^1 f_{\mathcal{A}} & & \downarrow H^1 f_{\mathcal{B}} \\ \Gamma(\mathcal{C}) & \xrightarrow{\delta} & H^1(X, \mathcal{A}) & \xrightarrow{H^1 p} & H^1(X, \mathcal{B}) \end{array}$$

Let $a_1 = \delta(c')$. Then by commutativity, $\Gamma f_{\mathcal{C}}(c') = \lambda c'$ implies $(H^1 f_{\mathcal{A}})(a_1) = \lambda a_1$. Since $(H^1 p)(a_1) = 0$ then from the long exact sequence there exists $c_0 \in \Gamma(\mathcal{C})$ such that $\delta(c_0) = a_1$. Then $\delta(c' - c_0) = 0$ so choose $b'_0 \in \Gamma(\mathcal{B})$ such that $(\Gamma q)(b'_0) = c' - c_0$. The choice of this b'_0 is unique up to the addition of a member of $\ker(\Gamma q) = (\Gamma p)(\Gamma \mathcal{A}) \subset \Gamma(\mathcal{B})$. Then applying Γq to $(\Gamma f_{\mathcal{B}})(b') - \lambda b'$ gives zero so $(\Gamma f_{\mathcal{B}})(b') - \lambda b' = (\Gamma p)(a_0)$ for some $a_0 \in \Gamma(\mathcal{A})$. We will show that there is a unique $b \in \Gamma(\mathcal{B})$ such that $(\Gamma f_{\mathcal{B}})(b'_0 + b) - \lambda(b'_0 + b) = 0$. This is equivalent to

$$b - \frac{1}{\lambda}(\Gamma f_{\mathcal{B}})(b) = \frac{1}{\lambda}(\Gamma p)(a_0).$$

Since $\frac{1}{\lambda} \Gamma f_{\mathcal{B}}$ is uniformly contracting on the Banach space $\Gamma(\mathcal{B})$ the operator of b on the left hand side is invertible and the result follows. $\square$

In the setting of Theorem 3.2, c is necessarily a Greens current.

4.2. Invariant Global Potentials.

Definition 4.3. Given that $\Gamma(\mathcal{B})$ is a Banach space, $\Gamma(\mathcal{C})$ is a topological space and that $\Gamma(q): \Gamma(\mathcal{B}) \rightarrow \Gamma(\mathcal{C})$ is continuous, we will say that $b' \in \Gamma(\mathcal{B}')$ is an exceptional potential if $b' \notin \Gamma(\mathcal{B})$, $(\Gamma f_{\mathcal{B}})(b') = \lambda b'$ for some $\lambda \in \mathbb{R}$, and if the operator $\frac{1}{\lambda} \Gamma f_{\mathcal{B}}$ is uniformly contracting on $\Gamma(\mathcal{B})$.

Given an exceptional section $c' \in \Gamma(\mathcal{C}')$ and the corresponding $c \in \Gamma(\mathcal{C})$ given by Lemma 4.2 the difference $\gamma = c' - c$ is a closed exceptional section, i.e. it satisfies $\delta(\gamma) = 0$. We show that a closed exceptional section can be “lifted” to an exceptional potential $b \in \Gamma(\mathcal{B})$.

Proposition 4.4. *Given a closed exceptional section $\gamma \in \Gamma(\mathcal{C}')$ there exists an exceptional potential $b' \in \Gamma(\mathcal{B}')$ such that $(\Gamma q)(b') = \gamma$.*

Proof. Given a closed exceptional section $\gamma \in \Gamma(\mathcal{C}')$ there exists some $b'_0 \in \Gamma(\mathcal{B}')$ such that $(\Gamma q)(b'_0) = \gamma$. Since $\Gamma q((\Gamma f_{\mathcal{B}})(b'_0) - \lambda b'_0) = 0$ we know that $(\Gamma f_{\mathcal{B}})(b'_0) - \lambda b'_0 = (\Gamma p)(a)$ for some $a \in \Gamma(\mathcal{A})$. Let $b \in \Gamma(\mathcal{B}')$. Then $b + b'_0$ is the desired b' iff $(\Gamma f_{\mathcal{B}})(b'_0 + b) - \lambda(b'_0 + b) = 0$, or equivalently if $b - \frac{1}{\lambda}(\Gamma f_{\mathcal{B}})(b) = \frac{1}{\lambda}(\Gamma p)(a)$. Since the operator of b on the left hand side of this equation is invertible then there exists a (unique) solution $b \in \Gamma(\mathcal{B})$. $\square$

Remark 4.5. In general, given a sheaf $\mathcal{S}$ on X with an f self cohomomorphism $f_{\mathcal{S}}: \mathcal{S} \leadsto \mathcal{S}$ one has $\text{Supp}((\Gamma f_{\mathcal{S}})(s)) \subset f^{-1}(\text{Supp}(s))$.

Definition 4.6. Given a sheaf $\mathcal{S}$ on X and an f self cohomomorphism $f_{\mathcal{S}}: \mathcal{S} \leadsto \mathcal{S}$, if $\text{Supp}((\Gamma f_{\mathcal{S}})(s)) = f^{-1}(\text{Supp}(s))$ for all $s \in \Gamma(\mathcal{S})$ we say that $\mathcal{S}$ is *supportwise totally invariant*.

If $\mathcal{S}$ is the sheaf of Cartier divisors on a complex manifold and f is the usual pullback then $\mathcal{S}$ is supportwise totally invariant.

Since $f_{\mathcal{C}}$ is an f -cohomomorphism of $\mathcal{C}'$ which maps $\mathcal{C}$ into itself then there is an induced f -cohomomorphism $f_{\mathcal{C}'/\mathcal{C}}: \mathcal{C}'/\mathcal{C} \leadsto \mathcal{C}'/\mathcal{C}$. We will abuse notation and simply denote $f_{\mathcal{C}'/\mathcal{C}}$ simply by $f_{\mathcal{C}}$, the map which induces it.

Observation. It is easy to show that $\mathcal{B}'/\mathcal{B} = \mathcal{C}'/\mathcal{C}$, and the induced f self cohomomorphism of each gives the same map on both.

Definition 4.7. We define *exceptional support* of an exceptional section c' , denoted $\text{ExSupp}(c')$, to be the support of the image of c' in $\mathcal{C}'/\mathcal{C}$. Thus it is the set of points $x \in X$ which have no open neighborhood U with $c'|_U \in \mathcal{C}(U)$. One can similarly define the exceptional support of an exceptional potential b' to be the support of its image in $\mathcal{B}'/\mathcal{B}$.

Lemma 4.8. *Given an exceptional section $c' \in \Gamma(\mathcal{C}')$ and corresponding $c' \in \Gamma(\mathcal{C})$, and $b' \in \Gamma(\mathcal{B}')$ such that $(\Gamma q)(b') = c' - c$ then $\text{ExSupp}(b') = \text{ExSupp}(c')$.*

Proof. It follows from Observation 4.2 that $\text{ExSupp}(b') = \text{ExSupp}(c' - c)$. Since $c \in \Gamma(\mathcal{C})$ then $\text{ExSupp}(c' - c) = \text{ExSupp}(c')$. $\square$

5. EXCEPTIONAL HYPERSURFACES FOR COMPACT COMPLEX MANIFOLDS

Exceptional hypersurfaces were significant from the beginning in holomorphic dynamics, as the existence of an exceptional point makes a polynomial self map of the Riemann sphere easier to study than a rational self map. In [3] it was shown that the union of all totally invariant algebraic subvarieties of $\mathbb{P}^n$ is again an algebraic subvariety, and that any point whose successive preimages do not converge to the closed positive (n, n) Green's current lies on such a totally invariant variety. In [1] it was shown that exceptional varieties in $\mathbb{P}^n$ are necessarily a union of finitely many linear subspaces of $\mathbb{P}^n$. Exceptional varieties are also studied in [22], and play an important role in [18]. Holomorphic maps of $\mathbb{P}^n$ with an exceptional hypersurface have been studied as well, particularly in [4]. Here we apply our results to the study of totally invariant hypersurfaces of holomorphic self maps of a compact complex manifold.

Given a holomorphic self endomorphism $f: Z \rightarrow Z$ of a compact complex manifold one can consider the exact sequence of sheaves

$$\mathcal{H} \rightarrow L_{\text{loc}}^{\infty} \xrightarrow{\text{dd}^c} \mathcal{C}$$

where $\mathcal{H}$ is the sheaf of pluriharmonic functions and $\mathcal{C}$ is the sheaf of closed $(1,1)$ currents with local dd^c potentials in L_{loc}^{∞} .

We want to consider the case of a totally invariant hypersurface. We need a sheaf of potentials for such a hypersurface, which includes the potentials for our Green's currents. Thus $\mathcal{S}$ which is the sheaf generated by functions locally of the form $\log |f| + g$ where f is meromorphic and g is locally bounded will do well. The operator dd^c is defined for such functions and its kernel is still the sheaf $\mathcal{H}$. Let the currents with local dd^c potentials in $\mathcal{S}$ be denoted by $\mathcal{C}'$.

If there is a totally invariant complex hypersurface $Y \subset Z$ then $f^*(Y) = k \cdot Y$ for some integer $k \geq 1$. We assume that $k > 1$. Choosing local plurisubharmonic potentials for Y , one sees that Y has a dd^c potential s which is a global plurisubharmonic section of some pluriharmonic bundle $w \in H^1(Z, \mathcal{H})$. The bundle w can not be a trivial pluriharmonic bundle or s is a global plurisubharmonic function on Z , which is impossible since Z is compact.

Using pullback of functions by f , and the induced pullback on $\mathcal{C}'$ obtain an f self cohomomorphism of the exact sequence of sheaves

$$\mathcal{H} \rightarrow \mathcal{S} \rightarrow \mathcal{C}'$$

which restricts to an f self cohomomorphism of

$$\mathcal{H} \rightarrow L_{\mathrm{loc}}^\infty \rightarrow \mathcal{C}.$$

Then Y satisfies the definition of an exceptional section since $\frac{1}{k}f^*$ is uniformly contracting on $L_{\mathrm{loc}}^\infty(Z)$. From Lemma 4.2 there is some unique $c \in \Gamma(\mathcal{C})$ with $\delta(c) = w$ and $(\Gamma f_{\mathcal{C}})(c) = k \cdot c$. Actually, the regularization approach in [33] gives more. Since every bundle in $H^1(Z, \mathcal{H})$ has a continuous section then one can show that c is necessarily continuous. If $w \in H^1(Z, \mathcal{H})$ has a plurisubharmonic section then by the same means it follows that c is plurisubharmonic. In either case, taking $\gamma = c - Y$ (Y considered here as a closed positive $(1,1)$ current), Proposition 4.4 allows us to conclude that we can find a function b' (locally a function continuous to $\mathbb{R} \cup \{\infty\}$ minus a continuous function to $\mathbb{R}$) such that $\mathrm{dd}^c b' = c - Y$ and $f^*(b') = k \cdot b'$.

This yields:

Proposition 5.1. *The global potential b' is nonnegative, continuous to $\mathbb{R} \cup \{\infty\}$ and satisfies $b' \circ f = k \cdot b'$. Thus the totally invariant set $\mathcal{U} = \{z \in Z | b'(z) > 0\}$ is an open neighborhood of Y and is precisely the set of points attracted to Y . Consequently Y is an attracting set.*

Corollary 5.2. *If there is any positive current $c_+ \in \Gamma(\mathcal{C})$ such that $\delta(c_+) = \delta(Y)$ then b' is plurisubharmonic off of Y .*

Proof. The hypothesis is equivalent to requiring that w has a plurisubharmonic section. $\square$

Corollary 5.3. *If there is any $c_0 \in \Gamma(\mathcal{C})$ such that $\delta(c_0) = \delta(Y)$, and such that $\mathrm{Supp}(c_0) \Subset \mathcal{U}$ then the map f is not cohomologically expansive for points $z \in Z \setminus \mathcal{U}$. Consequently, the invariant section c associated to c' is supported in $\bar{\mathcal{U}}$.*

Proof. This follows from Theorem 3.2. $\square$

This says that if Y can be smoothed to some closed positive $(1,1)$ current in the same cohomological class as Y and supported sufficiently near to Y then the Greens

current with the same cohomological class as Y is supported in the closure of the basin of Y .

Remark 5.4. The current $\alpha = d^c b'$ is interesting in its own right. It is a current which represents a sort of generalized set of *external rays* between the exceptional current Y and the current c , satisfying $d\alpha = c - Y$. Rays may be a bad term here, since any “rays” in this case are no longer one dimensional, but instead codimension one.

5.1. Relation to Cohomology. Both the basic result in [33] and Proposition 4.4 can be seen as describing conditions under which a certain cohomological group partially vanishes. Take a long exact sequence such as (2), and apply the functor Fixed , which returns the fixed sections, to each of the maps represented vertically. Taking cohomology of the resulting sequence, we see that Proposition 4.4 gives conditions which guarantee that a given fixed member c' of $\Gamma(\mathcal{C}')$ in the kernel of δ is the image of a fixed member $b' \in \Gamma(\mathcal{B}')$, and thus while c' is a member of $\text{Fixed}(\Gamma(\mathcal{C}'))$, it becomes zero in the cohomology of the resulting cochain sequence. We are studying the (left exact) composite functor $\text{Fixed} \circ \Gamma()$, which we anticipate will be useful in understanding dynamical systems.

5.2. When $\mathcal{B}'$ is a sheaf of functions. We make some notes here on the non-holomorphic case when $\mathcal{B}'$ is a sheaf of functions and $f_{\mathcal{B}}$ is the usual pullback of functions. In this case the invariance of an exceptional potential $b' \in \Gamma(\mathcal{B})$ gives direct control of the attraction. We assume that $\lambda > 1$. The invariance of b' states that

$$b' \circ f = \lambda \cdot b'.$$

Consequently:

Corollary 5.5. *The sets U_+ , U_- and K , on which b' is positive, negative, and zero, respectively, are totally invariant sets. Moreover if b' is continuous to the extended line then the set $b' = \infty$ is a totally invariant attracting set with basin U_+ . The analogous conclusion holds for the set $b' = -\infty$.*

Observation. If $\mathcal{B}$ is comprised of locally bounded functions then in the above case, the sets $b' = \infty$ and $b' = -\infty$ must lie in $\text{ExSupp}(b')$.

Corollary 5.6. *If there is a member $c_0 \in \Gamma(\mathcal{C})$ with $\delta(c_0) = \delta(c')$ such that $\text{Supp}(c_0) \subseteq U_+ \cup U_-$ then, in fact $\text{Supp}(c) \subset \overline{U_+ \cup U_-}$. If $\mathcal{C}$ is supportwise totally invariant then the intersection of $\text{Supp}(c)$ and $U_{\pm}$ lies in the basin of $U_{\pm} \cap \text{Supp}(c)$.*

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