

DETERMINING FUNCTIONALS FOR THE STRONGLY DAMPED NONLINEAR WAVE EQUATION

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ABSTRACT. We consider the existence of a wide collection of finite sets of functionals on the phase space $H^2(0, 1) \cap H_0^1(0, 1)$ that completely determines asymptotic behavior of solutions to the strongly damped nonlinear wave equation. The proof makes use of energy methods and the concept of the completeness defect. We also show that the number of determining nodes is two, that is, the asymptotic behavior of solutions is determined by the values of two sufficiently close points in the interval $[0, 1]$.

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1. INTRODUCTION

The question of the number of parameters (elements) that is necessary for the description of the long-time behavior of solutions to nonlinear partial differential equations was first discussed by Foias and Prodi [9] and Ladyzhenskaya [15] for the two dimensional Navier- Stokes equations (NSE). They proved that the long-time behavior of solutions of NSE is completely determined by dynamics of the first N Fourier modes if N is sufficiently large. More precisely, if the asymptotic behavior of the difference of two solutions at the first N modes tends to zero so does the difference of entire solution as $t \rightarrow \infty$. The corresponding modes are called the

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determining modes. Afterwards, Foias, Manley, Titi and Treve [8] established an upper bound for the number of determining modes of solutions of the NSE in two dimensions, where G , the Grashof number, is the analogue of the Reynolds number. They also introduced the notion of a set of determining nodes (see Definition 3.1).

Concerning determining modes and nodes later similar results have been obtained for other equations (see [5]-[7] and [9]-[14]).

Chueshov [3] presented the approach of a set of determining functionals, a wide collection of finite sets of functionals, containing determining modes and nodes that completely determines the long-time behavior of some first and second order evolution equations. This approach relies on the concept of the completeness defect and it involves some ideas from abstract approximation theory.

In this article we consider an initial-boundary value problem for the strongly damped nonlinear wave equation that describes the purely longitudinal motion of a homogeneous bar. We give a condition (see Theorem 2.1) for a finite set of linear functionals to be a set of determining functionals (see Definition 1.3) for solutions of the given problem.

We note that the approach presented above has been applied to second order in time evolution equations with nonlinear damping (see [2], [3] and [4]). However, none of these studies includes the nonlinear wave equation with strong damping term.

We consider the nonlinear wave equation with strong damping:

$$(1) \quad u_{tt} - (\sigma(u_x))_x - \alpha u_{xxt} + f(u) = g(x), \quad x \in (0, 1), \quad t > 0$$

subject to the initial

$$(2) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1)$$

and boundary conditions

$$(3) \quad u(0, t) = u(1, t) = 0, \quad t > 0,$$

where α is a positive constant and $-\alpha u_{xxt}$ is the strong damping term. Here σ and f are smooth functions satisfying the following properties:

$$(4)$$

$$\sigma(0) = 0, \quad \sigma'(s) \geq \sigma_0 \quad \text{for all } s \in \mathbf{R}$$

where $\sigma_0 > 0$ is a constant;

$$(5) \quad \liminf_{|s| \rightarrow \infty} \frac{F(s)}{s^2} \geq 0$$

and

$$(6) \quad \liminf_{|s| \rightarrow \infty} \frac{sf(s) - wF(s)}{s^2} \geq 0$$

for some $w > 0$ such that $0 < w \leq \frac{1}{2}$ in which $F(s) = \int_0^s f(s)ds$.

Let $I = (0, 1)$ and $L^2(I)$ be the usual Hilbert space with the standard $\|\cdot\|$ and inner product $(\cdot, \cdot)$ and let $Y = H^2(I) \cap H_0^1(I) \times L^2(I)$ be the Hilbert space with

the standard product norm

$$\|(u, v)\|_Y = (\|u_{xx}\|^2 + \|v\|^2)^{1/2}.$$

The local and global existence of solutions of (1)-(3) have been given by the following theorems [1].

Theorem 1.1. *Let σ and f satisfy (4), (5) and (6), respectively, and $g \in L^2(I)$. Then, for $(u_0, u_1) \in Y$ there exists $t_0 = t_0(u_0, u_1) > 0$ such that $(u, u_t) \in C([0, t_0]; Y)$, $u_{tt} \in L^2(I)$ and $u_{xxt} \in L^2(I)$ for a.e. $t \in (0, t_0)$, and u satisfies (1)-(3). Moreover, if t_0 is maximal, then either $t_0 = \infty$ or $\|(u, u_t)\|_Y$ is unbounded on $[0, t_0)$.*

Theorem 1.2. *Let σ , f and g be as above. Then for any $(u_0, u_1) \in Y$, there exists a global solution u of (1)-(3) with $(u, u_t) \in C([0, \infty); Y)$.*

We should also note that the authors [1] have also shown that there exists a compact global attractor in $H^2(I) \cap H_0^1(I) \times L^2(I)$ with finite dimension.

Now we give some preliminary materials which will be used in proving existence of determining functionals.

We will start with recalling

Definition 1.3. Let $\mathcal{L} = \{\ell_j : j = 1, \dots, N\}$ be a finite set of linear continuous functionals on $H^2(I) \cap H_0^1(I)$. We will say that $\mathcal{L}$ is a set of determining functionals for (1)-(3) when for any two solutions $u, v \in C([0, \infty), H^2(I) \cap H_0^1(I))$ with u_t and $v_t \in C([0, \infty), L^2(I))$, the conditions

$$(7) \quad \lim_{t \rightarrow \infty} \int_t^{t+1} |\ell_j(u(\tau) - v(\tau))|^2 d\tau = 0 \quad \text{for } j = 1, 2, \dots, N$$

imply

$$(8) \quad \lim_{t \rightarrow \infty} [\|u_{xx}(t) - v_{xx}(t)\|^2 + \|u_t(t) - v_t(t)\|^2] = 0.$$

Definition 1.4. Let V and H be the reflexive Banach spaces and V be continuously and densely embedded into H . Let $\mathcal{L} = \{\ell_j : j = 1, \dots, N\}$ be a set of linear functionals on V . We define the *completeness defect* $\epsilon_{\mathcal{L}} = \epsilon_{\mathcal{L}}(V, H)$ of the set $\mathcal{L}$ with respect to the pair of the spaces V and H by the formula

$$(9) \quad \epsilon_{\mathcal{L}}(V, H) = \sup\{\|u\|_H : u \in V, \ell_j(u) = 0, \ell_j \in \mathcal{L}, \|u\|_V \leq 1\}.$$

The following assertion gives the spectral characterization of the completeness defect in the case when V and H are the Hilbert spaces.

Theorem 1.5. [3] *Let V and H be the separable Hilbert spaces such that V is compactly and densely embedded into H . Let K be the self-adjoint, positive and compact operator in the space V defined by the equality*

$$(Ku, v)_V = (u, v)_H$$

for $u, v \in V$. Then the completeness defect $\epsilon_{\mathcal{L}}(V, H)$ of a set $\mathcal{L}$ of linear functionals on V can be evaluated by the formula

$$\epsilon_{\mathcal{L}}(V, H) = \sqrt{\mu_{\max} P_{\mathcal{L}} K P_{\mathcal{L}}},$$

where $P_{\mathcal{L}}$ is the orthoprojector in the space V on the annihilator

$$\mathcal{L}^\perp = \left\{ v \in V : \ell(v) = 0, \ell \in \mathcal{L} \right\},$$

$\mu_{\max}(S)$ is the maximal eigenvalue of the operator S .

Corollary 1.6. [3] *Let the conditions of Theorem 1.5 be met and let us denote by $\{e_i\}$ the orthonormal basis in the space V that consists of the eigenvectors of the operator K :*

$$(10) \quad Ke_i = \mu_i e_i, \quad (e_i, e_j)_V = \delta_{ij}, \quad \mu_1 \geq \mu_2 \geq \cdots, \quad \lim_{N \rightarrow \infty} \mu_N = 0.$$

Then the completeness defect of the set of functionals

$$\mathcal{L} = \left\{ \ell_j \in V^* : \ell_j(u) = (u, e_j)_V, j = 1, 2, \dots, N \right\}$$

can be evaluated by the formula

$$\epsilon_{\mathcal{L}}(V, H) = \sqrt{\mu_{N+1}}.$$

The following theorem establishes a relation between the completeness defect and the set $\mathcal{L}$.

Theorem 1.7. *Let $\epsilon_{\mathcal{L}} = \epsilon_{\mathcal{L}}(V, H)$ be the completeness defect of a set $\mathcal{L}$ of linear functionals on V with respect to H . Then there exists a positive constant $C_{\mathcal{L}}$ such that*

$$(11) \quad \|u\|_H \leq C_{\mathcal{L}} \sup\{|\ell(u)| : \ell \in \mathcal{L}, \|\ell\|_* \leq 1\} + \epsilon_{\mathcal{L}} \|u\|_V$$

for any $u \in V$, where $\mathcal{L}$ is the closed linear span of the set $\mathcal{L}$ in V^ , the dual space of V and $\|\cdot\|_*$ is the norm in V^* .*

For the proof of this theorem see Chueshov [3].

The following version of Gronwall's lemma is also needed to determine behavior of solutions as $t \rightarrow \infty$.

Lemma 1.8. [3] *Let α be a locally integrable real valued function on $(0, \infty)$, satisfying for some $0 < T < \infty$ the following conditions:*

$$(12) \quad \liminf_{t \rightarrow \infty} \int_t^{t+T} \alpha(\tau) d\tau = \gamma > 0,$$

$$(13) \quad \limsup_{t \rightarrow \infty} \int_t^{t+T} \alpha^-(\tau) d\tau = \Gamma < \infty,$$

where $\alpha^- = \max\{-\alpha, 0\}$. Further, let β be a real valued locally integrable function defined on $(0, \infty)$ such that

$$(14) \quad \lim_{t \rightarrow \infty} \int_t^{t+T} \beta^+(\tau) d\tau = 0,$$

where $\beta^+ = \max\{\beta, 0\}$. Suppose that ξ is an absolutely continuous non-negative function on $(0, \infty)$ such that

$$(15) \quad \frac{d\xi}{dt} + \alpha\xi \leq \beta, \quad \text{a.e. on } (0, \infty).$$

Then $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$.

2. EXISTENCE OF DETERMINING FUNCTIONALS

Now we can state and prove the main result concerning existence of a set of determining functionals of solutions to problem (1)-(3)

Theorem 2.1. Let $\mathcal{L} = \{\ell_j : j = 1, \dots, N\}$ be a set of linear continuous functionals on the space $H^2(I) \cap H_0^1(I)$ and let $\epsilon_{\mathcal{L}} = \epsilon_{\mathcal{L}}(H^2(I) \cap H_0^1(I), L^2(I))$ be a positive number satisfying $2A_1\epsilon_{\mathcal{L}}^2 + c_0A_2\epsilon_{\mathcal{L}} < c_4$, where c_4 , A_1 and A_2 are positive constants such that $c_4 = c_4(\sigma_0, \alpha, M, M_1, M_2)$, $A_1 = A_1(M)$, $A_2 = A_2(M_1, M_2, R_2)$ and M , M_1 , M_2 and R_2 are positive constants satisfying

$$\sup_{|s| \leq R_1} |f'(s)| \leq M, \quad \sup_{|s| \leq R} |\sigma'(s)| \leq M_1, \quad \sup_{|s| \leq R} |\sigma''(s)| \leq M_2, \quad \text{and} \quad \|v_{xx}\| \leq R_2.$$

Then, $\mathcal{L}$ is a set of determining functionals for (1)-(3).

Proof. Let u and v be two solutions of problem (1)-(3). Then the difference of these solutions, defined by $w = u - v$, satisfies the initial-boundary value problem

$$(16) \quad w_{tt} - (\sigma(u_x))_x + (\sigma(v_x))_x - \alpha w_{xxt} + f(u) - f(v) = 0, \quad x \in (0, 1), \quad t > 0$$

with the initial

$$(17) \quad w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in (0, 1)$$

and boundary conditions

$$(18) \quad w(0, t) = w(1, t) = 0, \quad t > 0.$$

We start with taking the L^2 inner product of (16) by w_t and integrating over the interval $(0, 1)$ to have

$$(19) \quad \frac{1}{2} \frac{d}{dt} \|w_t\|^2 + \alpha \|w_{xt}\|^2 = I_1 + I_2,$$

where

$$I_1 = \int_0^1 [\sigma(u_x) - \sigma(v_x)] w_{xt} dx, \quad I_2 = \int_0^1 [f(u) - f(v)] w_t dx.$$

First we will give an estimate for I_1 . It follows from the mean-value theorem that

$$\sigma(u_x) - \sigma(v_x) = \sigma'(s)(u_x - v_x)$$

for some s between u_x and v_x . Since $u \in H^2(I) \cap H_0^1(I)$, $u_x \in C[0, 1]$ with $|u_x| \leq R$ for some constant R by the Sobolev embedding theorem. Hence, combining this with the last equality yields

$$|\sigma(u_x) - \sigma(v_x)| \leq M_1 |w_x|,$$

where M_1 is a positive constant such that $\sup_{|s| \leq R} |\sigma'(s)| \leq M_1$.

Thus,

$$(20) \quad |I_1| \leq M_1 \int_0^1 |w_x| |w_{xt}| dx.$$

Using Cauchy's inequality (with ε), we get from (20)

$$(21) \quad |I_1| \leq \frac{\varepsilon}{2} \|w_{xt}\|^2 + \frac{M_1^2}{2\varepsilon} \|w_x\|^2.$$

Employing the same techniques along with the Poincaré inequality

$$\lambda_1 \|w_t\|^2 \leq \|w_{xt}\|^2$$

leads to

$$(22) \quad |I_2| \leq \frac{\varepsilon_1}{2\lambda_1} \|w_{xt}\|^2 + \frac{M^2}{2\varepsilon_1} \|w\|^2,$$

where M is a positive constant such that $\sup_{|s| \leq R_1} |f'(s)| \leq M$. Now, inserting (21) and (22) in (19) gives

$$(23) \quad \frac{1}{2} \frac{d}{dt} \|w_t\|^2 + \frac{\alpha}{2} \|w_{xt}\|^2 \leq \frac{M^2}{2\varepsilon_1} \|w\|^2 + \frac{M_1^2}{2\varepsilon} \|w_x\|^2,$$

where ε and ε_1 have been chosen so that $\alpha = \varepsilon + \frac{\varepsilon_1}{\lambda_1} > 0$.

On the other hand, the L^2 inner product of (16) with $-w_{xx}$ and integration by parts over $(0, 1)$ yields

$$(24) \quad -(w_{tt}, w_{xx}) + \frac{\alpha}{2} \frac{d}{dt} \|w_{xx}\|^2 + \int_0^1 \left[(\sigma(u_x))_x - (\sigma(v_x))_x \right] w_{xx} dx = J_1,$$

where

$$J_1 = \int_0^1 [f(u) - f(v)] w_{xx} dx.$$

By making use of the identity

$$(w_{tt}, w_{xx}) = \frac{d}{dt} (w_t, w_{xx}) - \|w_{xt}\|^2$$

along with some rearrangement, (24) can be written as

$$\frac{d}{dt} \left[\frac{\alpha}{2} \|w_{xx}\|^2 - (w_t, w_{xx}) \right] + \int_0^1 \sigma'(u_x) |w_{xx}|^2 dx + \|w_{xt}\|^2 = J_1 + J_2,$$

where

$$J_2 = - \int_0^1 [\sigma'(u_x) - \sigma'(v_x)] v_{xx} w_{xx} dx.$$

Furthermore, using (4) in the last expression gives

$$(25) \quad \frac{d}{dt} \left[\frac{\alpha}{2} \|w_{xx}\|^2 - (w_t, w_{xx}) \right] + \sigma_0 \|w_{xx}\|^2 + \|w_{xt}\|^2 \leq J_1 + J_2.$$

Utilizing a similar procedure we have used for I_2 , except for the last step, we derive an estimate for J_1

$$(26) \quad |J_1| \leq \frac{\delta}{2} \|w_{xx}\|^2 + \frac{M^2}{2\delta} \|w\|^2,$$

where M is a constant as before.

An estimate for J_2 is found as follows. First, it follows, from the mean-value theorem, that

$$(27) \quad |J_2| \leq M_2 \int_0^1 |w_x| |v_{xx}| |w_{xx}| dx,$$

where M_2 is a constant such that $\sup_{|s| \leq R} |\sigma''(s)| \leq M_2$. Next, we have

$$(28) \quad |J_2| \leq M_2 \|w_x\|_\infty \|v_{xx}\| \|w_{xx}\|,$$

where the Cauchy-Schwarz inequality has been used. Since $\|v_{xx}\| \leq R_2$,

$$(29) \quad |J_2| \leq R_2 M_2 \|w_x\|_\infty \|w_{xx}\|$$

for some constant $R_2 > 0$. Using Agmon's inequality

$$\|w_x\|_\infty \leq C \|w_x\|^{1/2} \|w_{xx}\|^{1/2}$$

in (29) we have

$$(30) \quad |J_2| \leq K_0 \|w_x\|^{1/2} \|w_{xx}\|^{3/2},$$

where $K_0 = CR_2 M_2$. Then, applying Young's inequality (with $p = \frac{4}{3}, q = 4$) in (30) we deduce

$$(31) \quad |J_2| \leq \frac{K_0^2}{4\mu^4} \|w_x\|^2 + \frac{3\mu^{3/4}}{4} \|w_{xx}\|^2.$$

Thus, substituting (26) and (31) into (25) we find

$$(32) \quad \begin{aligned} & \frac{d}{dt} \left[\frac{\alpha}{2} \|w_{xx}\|^2 - (w_t, w_{xx}) \right] + \sigma_0 \|w_{xx}\|^2 + \|w_{xt}\|^2 \\ & \leq A_0 \|w_{xx}\|^2 + \frac{M^2}{2\delta} \|w\|^2 + \frac{K_0^2}{4\mu^4} \|w_x\|^2, \end{aligned}$$

where $A_0 = \frac{\delta}{2} + \frac{3\mu^{3/4}}{4}$. We can choose δ and μ so that $c_2 = \sigma_0 - A_0 > 0$, and hence from (32) we have

$$(33) \quad \begin{aligned} & \frac{d}{dt} \left[\frac{\alpha}{2} \|w_{xx}\|^2 - (w_t, w_{xx}) \right] + c_2 \|w_{xx}\|^2 + \|w_{xt}\|^2 \\ & \leq \frac{M^2}{2\delta} \|w\|^2 + \frac{K_0^2}{4\mu^4} \|w_x\|^2. \end{aligned}$$

Multiplying (33) by $\frac{\alpha}{4}$ and adding this to (23), we obtain the following

$$(34) \quad \frac{d}{dt} E(t) + \frac{c_2 \alpha}{4} \|w_{xx}\|^2 + \frac{3\alpha}{4} \|w_{xt}\|^2 \leq A_1 \|w\|^2 + A_2 \|w_x\|^2,$$

where

$$E(t) = \frac{1}{2} \|w_t\|^2 + \frac{\alpha}{4} \left[\frac{\alpha}{2} \|w_{xx}\|^2 - (w_t, w_{xx}) \right]$$

and $A_1 = \frac{M^2}{2\varepsilon_1} + \frac{\alpha M^2}{8\delta}$ and $A_2 = \frac{M_1^2}{2\varepsilon} + \frac{\alpha K_0^2}{16\mu^4}$. In (34), revisiting the Poincaré inequality for w_{xt} , we get

$$(35) \quad \frac{d}{dt} E(t) + \frac{c_2 \alpha}{4} \|w_{xx}\|^2 + \frac{3\alpha \lambda_1}{4} \|w_t\|^2 \leq A_1 \|w\|^2 + A_2 \|w_x\|^2.$$

Let $\varepsilon_{\mathcal{L}}$ denote the completeness defect between $L^2(I)$ and $X = H^2(I) \cap H_0^1(I)$. That is,

$$\varepsilon_{\mathcal{L}}(X, L^2(I)) = \sup\{\|w\| : w \in X, \ell_j(w) = 0, \ell_j \in \mathcal{L}, \|w\|_X \leq 1\}.$$

Then by Theorem 1.7 we have,

$$(36) \quad \|w\| \leq \varepsilon_{\mathcal{L}} \|w\|_{H^2(I) \cap H_0^1(I)} + C_{\mathcal{L}} \max_{j=1, \dots, N} |\ell_j(w)|,$$

or equivalently,

$$(37) \quad \|w\| \leq \varepsilon_{\mathcal{L}} \|w_{xx}\| + C_{\mathcal{L}} \max_{j=1, \dots, N} |\ell_j(w)|$$

for all $w \in H^2(I) \cap H_0^1(I)$. Squaring both sides of (37) and using Cauchy's inequality we get

$$(38) \quad \|w\|^2 \leq 2\varepsilon_{\mathcal{L}}^2 \|w_{xx}\|^2 + 2C_{\mathcal{L}}^2 \max_{j=1, \dots, N} |\ell_j(w)|^2.$$

Moreover, from (37) and the multiplicative inequality

$$(39) \quad \|w_x\|^2 \leq c_0 \|w_{xx}\| \|w\|$$

we find

$$(40) \quad \|w_x\|^2 \leq c_0 \varepsilon_{\mathcal{L}} \|w_{xx}\|^2 + \frac{\varepsilon_4}{2} \|w_{xx}\|^2 + \frac{c_0^2 C_{\mathcal{L}}^2}{2\varepsilon_4} \max_{j=1, \dots, N} |\ell_j(w)|^2,$$

where the Cauchy inequality has also been used. Combining (38) and (40) in (35) leads to

$$\frac{d}{dt} E(t) + c_3 \|w_{xx}\|^2 + \frac{3\alpha \lambda_1}{4} \|w_t\|^2 \leq A_3 \|w_{xx}\|^2 + \frac{c_0^2 C_{\mathcal{L}}^2}{2\varepsilon_4} \max_{j=1, \dots, N} |\ell_j(w)|^2,$$

where ε_4 is chosen such that $c_3 = \frac{c_2 \alpha}{4} - \frac{\varepsilon_4}{2} > 0$ and $A_3 = 2A_1 \varepsilon_{\mathcal{L}}^2 + c_0 A_2 \varepsilon_{\mathcal{L}}$. Now, we choose $\varepsilon_{\mathcal{L}} > 0$ as small as possible so that $c_4 = c_3 - A_3 > 0$. Thus, from the above inequality we have

$$(41) \quad \frac{d}{dt} E(t) + c_4 \|w_{xx}\|^2 + \frac{\alpha \lambda_1}{4} \|w_t\|^2 \leq 2C_{\mathcal{L}}^2 \max_{j=1, \dots, N} |\ell_j(w)|^2.$$

We can find an upper and lower bounds for the functional $E(t)$. Due to the Cauchy-Schwarz and the Cauchy inequalities, we see that

$$(42) \quad |(w_t, w_{xx})| \leq \frac{1}{\alpha} \|w_t\|^2 + \frac{\alpha}{4} \|w_{xx}\|^2.$$

Hence, using (42) we can show that

$$(43) \quad \frac{1}{4}\|w_t\|^2 + \frac{\alpha^2}{16}\|w_{xx}\|^2 \leq E(t) \leq \frac{3}{4}\|w_t\|^2 + \frac{3\alpha^2}{16}\|w_{xx}\|^2.$$

Thus, from (43) we observe that there exists a positive constant

$$\eta = \min\left\{\frac{16c_4}{3\alpha^2}, \alpha\lambda_1\right\}$$

such that

$$(44) \quad \frac{d}{dt}E(t) + \eta E(t) \leq 2C_{\mathcal{L}}^2 \max_{j=1,\dots,N} |\ell_j(w)|^2.$$

Now, we apply Lemma 1.8 to (44) with

$$\beta^+(t) = 2C_{\mathcal{L}}^2 \max_{j=1,\dots,N} |\ell_j(w(t))|^2, \quad \alpha(t) = \eta \quad \text{and} \quad \xi(t) = E(t).$$

Thus, as a result of Lemma 1.8, we see that if

$$\int_t^{t+1} \beta^+(s) ds$$

tends to zero as $t \rightarrow \infty$, then $E(t) \rightarrow 0$. Therefore, it is clear that

$$\lim_{t \rightarrow \infty} [\|w_{xx}(t)\|^2 + \|w_t(t)\|^2] = 0$$

or

$$\lim_{t \rightarrow \infty} [\|u_{xx}(t) - v_{xx}(t)\|^2 + \|u_t(t) - v_t(t)\|^2] = 0.$$

Thus, according to Definition 1.3, the set $\mathcal{L}$ defined on $H^2(I) \cap H_0^1(I)$ is a set of determining functionals for (1)-(3). This completes the proof of Theorem 2.1.

Remark 2.2. Theorem 2.1 shows that for $\mathcal{L}$ to be a set of determining elements, it is sufficient that the value of the completeness defect $\epsilon_{\mathcal{L}}$ be small.

3. DETERMINING NODES

In this section we will show that there are two points in the interval $[0, 1]$, called determining nodes, which completely determines the long time behavior of solutions of the strongly damped nonlinear wave equation. There are many studies about the number of determining nodes for some nonlinear partial differential equations. Among those we may mention the studies for the Navier-Stokes equations [10], [11] and [12], for the generalized Ginzburg-Landau equation [6] and [13], for the Kuramoto-Sivashinsky equation [7], and for the Ginzburg-Landau equation [14].

Definition 3.1. Let $E = \{x_1, x_2, \dots, x_N\}$ be a set of points in $[0, 1]$. The set E is called a set of determining nodes if for any two solutions $u(x, t)$ and $v(x, t)$ of problem (1)-(3) satisfying

$$\lim_{t \rightarrow \infty} |u(x_i, t) - v(x_i, t)| = 0, \quad i = 1, 2, \dots, N$$

we have

$$\lim_{t \rightarrow \infty} |u(x, t) - v(x, t)| = 0,$$

for all $x \in [0, 1]$.

The main result we have obtained in this direction is

Theorem 3.2. *There is a positive number $d_1 = d_1(\alpha, \sigma, f)$ with the following property: if $x_1 < x_2$ are real numbers such that $d = x_2 - x_1 \leq d_1$, and if two solutions u, v of the strongly damped nonlinear wave equation satisfy*

$$\lim_{t \rightarrow \infty} |u(x_i, t) - v(x_i, t)| = 0, \quad i = 1, 2$$

then

$$\lim_{t \rightarrow \infty} |u(x, t) - v(x, t)| = 0,$$

for all $x \in [0, 1]$.

Proof. We will recall the differential inequality (34):

$$\frac{d}{dt}E(t) + \frac{c_2\alpha}{4}\|w_{xx}\|^2 + \frac{3\alpha\lambda_1}{4}\|w_t\|^2 \leq A_1\|w\|^2 + A_2\|w_x\|^2,$$

where

$$E(t) = \frac{1}{2}\|w_t\|^2 + \frac{\alpha}{4}\left[\frac{\alpha}{2}\|w_{xx}\|^2 - (w_t, w_{xx})\right],$$

and $A_1 = \frac{M^2}{2\varepsilon_1} + \frac{\alpha M^2}{8\delta}$ and $A_2 = \frac{M_1^2}{2\varepsilon} + \frac{\alpha K_0^2}{16\mu^4}$. Here $\|\cdot\|$ and $(\cdot, \cdot)$ denote the L^2 -norm and inner product over the interval $[x_1, x_2] \subset [0, 1]$, respectively.

Now we want to find an upper bound for the terms in the right-side of (34). In order to do this, we first have the elementary inequality [14],

$$(45) \quad \|w\|^2 \leq 2d|w(x_1)|^2 + 2d^2\|w_x\|^2,$$

where d is a positive constant such that $x_2 - x_1 \leq d$ and to be determined. Moreover, using the Sobolev inequality, $\|w_x\| \leq c_0\|w_{xx}\|$, we have

$$(46) \quad \|w\|^2 \leq 2d|w(x_1)|^2 + 2d^2c_0^2\|w_{xx}\|^2.$$

To give an estimate for the second term of (34) we use the interpolation inequality,

$$\|w_x\|^2 \leq c\|w_{xx}\|\|w\|,$$

and the Cauchy inequality,

$$(47) \quad \|w_x\|^2 \leq \frac{c^2\varepsilon}{2}\|w_{xx}\|^2 + \frac{1}{2\varepsilon}\|w\|^2.$$

Substituting (46) and (47) in (34) we get

$$(48) \quad \frac{d}{dt}E(t) + A_3\|w_{xx}\|^2 + \frac{3\alpha\lambda_1}{4}\|w_t\|^2 \leq A_4|w(x_1)|^2,$$

where

$$A_3 = \frac{c_2\alpha}{4} - \frac{A_2c^2\varepsilon^2}{2} - (2A_1c_0^2 + \frac{2A_2c_0^2}{\varepsilon})d^2 \text{ and } A_4 = 2A_1 + \frac{A_2d}{\varepsilon} > 0.$$

Since ε is an arbitrary positive constant, we choose ε so that

$$\frac{c_2\alpha}{4} - \frac{A_2c^2\varepsilon}{2} > 0.$$

Thus, for a sufficiently small d , A_3 will be positive and hence, from equation (48), we get

$$(49) \quad \frac{d}{dt}E(t) + \beta E(t) \leq A_4|w(x_1)|^2$$

for some $\beta > 0$. Using Lemma 1.8 we conclude that

$$E(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

whenever

$$\max\{|w(x_i)|\} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad i = 1, 2$$

and hence by the Sobolev embedding theorem we deduce

$$\lim_{t \rightarrow \infty} |u(x, t) - v(x, t)| = 0,$$

for all $x \in [0, 1]$.

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