

APPLICATIONS OF FIXED POINT THEOREMS IN DIVISIBILITY OF POLYNOMIALS

R.P. PANT AND J.S. RAUTELA

DEPARTMENT OF MATHEMATICS, KUMAUN UNIVERSITY, D.S.B. CAMPUS
NAINITAL, 263002, INDIA
E-MAIL: SKY_DREAMZ@REDIFFMAIL.COM

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ABSTRACT. The aim of the present paper is to study the dynamics and fixed points of a class of functions induced by integers and then apply the fixed point theorem to obtain general tests of divisibility of numbers. In the second part, we generalize the theorems on divisibility to obtain conditions of divisibility of polynomials by $(x^4 + q)$ type polynomials. We not only obtain generalizations of the well-known remainder theorem and the factor theorem of algebra but also compute explicit value of the quotient. The results are computationally explicit and can be easily implemented on digital computers.

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1. INTRODUCTION

The study of behavior of points under iteration of a function f is called the dynamics of the function f . A discrete dynamical system consists a function and its iterates. The main objective of dynamics is to study the behavior of the sequence $x, f(x), f^2x, \dots$ for various initial values of x . If f is a function, the set of points $x, f(x), f^2x, \dots$, is also called the orbit of the point x . Orbits can be quite complicated sets even for simple mappings. However, orbits associated with following types of points are especially simple and play a central role in the study of entire system.

Definition 1.1 [1,2]. The point x is called a fixed point of f if $f(x) = x$. A point x is called a periodic point of period n if $f^n(x) = x$. The least positive integer n for which $f^n(x) = x$ is called the prime period of x . The set of all iterates of a periodic

point form a periodic orbit.

Definition 1.2. A point x is defined to be eventually fixed point of a function f , if there exists an integer N such that $f^{n+1}(x) = f^n(x)$ whenever $n \geq N$. A point x is called eventually periodic with period k if there exists $N > 0$ such that $f^{n+k}(x) = f^n(x)$ whenever $n \geq N$.

In this section we study the dynamics of a function defined on the set of non-negative integers and prove a theorem which states that if a number is an integral multiple of a given number then it is an eventually fixed point of a mapping induced by the given number. We also give a number theoretic application of the fixed point theorem to obtain some general tests of divisibility.

Let X denote the set of non-negative integers equipped with the Euclidean metric. In the sequel we shall often write the numbers in X , in the form $10^4p + q$ or in the form $10^4a + 10^3b + 10^2c + 10d + e$ where p, q, a, b, c, d and e are non-negative integers such that $q, b, c, d, e < 10$. We observe that a given integer $10^4p + q$ in X induces a self-mapping f on X under which image of a number $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ in X is given by

$$(1) \quad f(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) = |10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q|.$$

where p, q, a_0, a_1, a_2, a_3 and a_4 are non-negative integers such that $q, a_0, a_1, a_2, a_3 < 10$. From this it can be easily shown that

$$(2) \quad \begin{aligned} f(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) &= |10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q| \\ &< \max \{10^4p + q, 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0\}. \end{aligned}$$

The application of fixed point theorems to the problem of divisibility was initiated by the first author in a recent work [3]. The question of divisibility of one integer by another or of one polynomial by another is an important question. However, the only important known result on the divisibility of one polynomial by another is the well known Factor Theorem which states that a linear polynomial $(x - a)$ divides a polynomial $f(x)$ if and only if $f(a) = 0$. Likewise, the general tests of divisibility given in [3, 4] are perhaps the only results of its type.

In this paper, we generalize the results of [3] to obtain new results that further generalize to yield conditions of divisibility of a polynomial by polynomial of the form $x^4 \pm q$. The results on divisibility by a $x^4 \pm q$ type polynomial are significant since the polynomial $x^4 \pm q$ does not factorize into four real linear factors.

2. RESULTS

Theorem 2.1. Let X be the set of non-negative integers and $p, q \geq 1$ be given integers. Let f be the self-mapping of X which is induced by $10^4p + q$ and defined by (1). Then every integral multiple of $10^4p + q$ is an eventually fixed point of f while every other integer in X is an eventually periodic point of f . Moreover, f

possesses fixed points in X .

Proof. Suppose that $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ is a non-zero integral multiple of $10^4p + q$ and $a_0, a_1, a_2, a_3 < 10$. Then there exists an integer $k \geq 1$ such that

$$10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 = k(10^4p + q).$$

Then,

$$\begin{aligned} f(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) &= |10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q| \\ &= |pk - a_4|(10^4p + q). \end{aligned}$$

That is, $f(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0)$ is an integral multiple of $10^4p + q$ since a_4, p, k are integers.

Further, since $k \geq 1$, $\max\{10^4p + q, 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0\} = 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$. By virtue of (2) we get $f(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) < 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$. Therefore, the f -image of an integral multiple of $10^4p + q$ is a smaller integral multiple of $10^4p + q$. But there are exactly k integral multiples of $10^4p + q$ which are smaller than $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 = k(10^4p + q)$. Moreover, $f(10^4p + q) = |pq - qp| = 0$ and $f(0) = f(10^4 \cdot 0 + 10^3 \cdot 0 + 10^2 \cdot 0 + 10 \cdot 0 + 0) = 0$. Thus, if $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 = k(10^4p + q)$ then $f^n(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) = 0$ for some $n \leq k$, $f^m(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) = 0$ for each $m \geq n$, and $f(0) = 0$. In other words, 0 is a fixed point of f and every other integral multiple of $10^4p + q$ is an eventually fixed point of f . If an integer $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ is a non-multiple of $10^4p + q$ then, by virtue of (2), there exists $n > 0$, such that $0 \leq f^n(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) < 10^4p + q$. Moreover, for each $i > 0$ we get, $0 \leq f^{i+n}(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) < 10^4p + q$. If $f^n(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) = 0$ then $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ is an eventually fixed point of f , otherwise the points $\{f^{i+n}(10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0) : i \geq 0\}$ form a periodic cycle and $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ is an eventually periodic point of f .

This completes the proof.

Now we give a number theoretic application of the above fixed point theorem to obtain a test of divisibility of an integer by another integer. In proving our theorem we shall use the well known elementary result that if the integers a and b are relatively prime and a divides the product of the integers b and c then a divides c . In the sequel, we shall use the notation $x|y$ to show that x divides y .

Theorem 2.2. If $a_0, a_1, a_2, a_3, a_4, p, q$ are non-negative integers, p and q are relatively prime then

- (i) $10^4p + q | 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \Leftrightarrow 10^4p + q | 10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q$,
- (ii) $10^4p - q | 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \Leftrightarrow 10^4p - q | 10^3a_3p + 10^2a_2p + 10a_1p + a_0p + a_4q$.

Proof. Let $10^4p + q \mid 10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q$. Then there exists some integer k such that

$$10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q = k(10^4p + q).$$

Then, $q(a_4 + k) = p(10^3a_3 + 10^2a_2 + 10a_1 + a_0 - 10^4k)$. This implies that $p \mid (a_4 + k)$, as p and q are relatively prime.

We further get $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 = (\frac{a_4+k}{p})(10^4p + q)$.

This means that $10^4p + q$ divides $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ since $p \mid (a_4 + k)$ and $(\frac{a_4+k}{p})$ is an integer.

Conversely, let $10^4p + q \mid 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$.

Then $10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 = k(10^4p + q)$ for some integer k .

This yield $10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q = (pk - a_4)(10^4p + q)$.

That is $10^4p + q \mid 10^3a_3p + 10^2a_2p + 10a_1p + a_0p - a_4q$.

This proves the first part of the theorem. The second part follows on similar lines.

Theorem 2.3. If $a_0, a_1, a_2, a_3, a_4, a_5, p, q$ are non-negative integers, p and q are relatively prime then

- (i) $10^4p + q \mid 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$
 $\Leftrightarrow 10^4p + q \mid 10^3a_3p + 10^2a_2p + 10(a_1p - a_5q) + (a_0p - a_4q),$
- (ii) $10^4p - q \mid 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$
 $\Leftrightarrow 10^4p - q \mid 10^3a_3p + 10^2a_2p + 10(a_1p + a_5q) + (a_0p + a_4q).$

Proof. Since p and q are relatively prime, using Theorem 2.2, we get

$$\begin{aligned} 10^4p + q \mid 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\ \Leftrightarrow 10^4p + q \mid 10^4(10a_5 + a_4) + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\ \Leftrightarrow 10^4p + q \mid 10^3a_3p + 10^2a_2p + 10a_1p + a_0p - q(10a_5 + a_4) \\ \Leftrightarrow 10^4p + q \mid 10^3a_3p + 10^2a_2p + 10(a_1p - a_5q) + (a_0p - a_4q). \end{aligned}$$

For the prove of part (ii) replacing q by $-q$ in the above equation, we get

$$\begin{aligned} 10^4p - q \mid 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\ \Leftrightarrow 10^4p - q \mid 10^3a_3p + 10^2a_2p + 10(a_1p + a_5q) + (a_0p + a_4q). \end{aligned}$$

This completes the proof.

The proofs of the next two theorems follow along the lines of Theorem 2.2, 2.3; therefore we state the result but omit the proof.

Theorem 2.4. If $a_0, a_1, a_2, a_3, a_4, a_5, a_6, p, q$ are non-negative integers and p, q are relatively prime then

- (i) $10^4p + q \mid 10^6a_6 + 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0$
 $\Leftrightarrow 10^4p + q \mid 10^3a_3p + 10^2(a_2p - a_6q) + 10(a_1p -$

$$\begin{aligned}
& a_5q) + (a_0p - a_4q), \\
(ii) \quad & 10^4p - q | 10^6a_6 + 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\
& \Leftrightarrow 10^4p - q | 10^3a_3p + 10^2(a_2p + a_6q) + 10(a_1p + \\
& a_5q) + (a_0p + a_4q).
\end{aligned}$$

Theorem 2.5. If $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, p, q$ are non-negative integers and p, q are relatively prime then

$$\begin{aligned}
(i) \quad & 10^4p + q | 10^7a_7 + 10^6a_6 + 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\
& \Leftrightarrow 10^4p + q | 10^3(a_3p - a_7q) + 10^2(a_2p - a_6q) + 10(a_1p - \\
& a_5q) + (a_0p - a_4q), \\
(ii) \quad & 10^4p - q | 10^7a_7 + 10^6a_6 + 10^5a_5 + 10^4a_4 + 10^3a_3 + 10^2a_2 + 10a_1 + a_0 \\
& \Leftrightarrow 10^4p - q | 10^3(a_3p + a_7q) + 10^2(a_2p + a_6q) + 10(a_1p + \\
& a_5q) + (a_0p + a_4q).
\end{aligned}$$

Similarly, using mathematical induction, the above theorems can be generalized to obtain the following theorems:

Theorem 2.6. If $p > 0, q > 0, a_0, a_1, \dots, a_{4n}$ are non-negative integers and p, q are relatively prime then

$$\begin{aligned}
(i) \quad & 10^4p + q | 10^{4n}a_{4n} + 10^{(4n-1)}a_{(4n-1)} + 10^{(4n-2)}a_{(4n-2)} + \dots + 10^2a_2 + 10a_1 + a_0 \\
& \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
& 10^4p + q | 10^3p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+2)} p^{(n-i-1)} q^i \right] \\
& + 10p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+1)} p^{(n-i-1)} q^i \right] + \left[\sum_{i=0}^n (-1)^i a_{4i} p^{(n-i)} q^i \right],
\end{aligned}$$

$$(ii) \quad 10^4p - q | 10^{4n}a_{4n} + 10^{(4n-1)}a_{(4n-1)} + 10^{(4n-2)}a_{(4n-2)} + \dots + 10^2a_2 + 10a_1 + a_0$$

$$\begin{aligned}
& \Leftrightarrow 10^4p - q | 10^3p \left[\sum_{i=0}^{(n-1)} a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2p \left[\sum_{i=0}^{(n-1)} a_{(4i+2)} p^{(n-i-1)} q^i \right] \\
& + 10p \left[\sum_{i=0}^{(n-1)} a_{(4i+1)} p^{(n-i-1)} q^i \right] + \left[\sum_{i=0}^n a_{4i} p^{(n-i)} q^i \right].
\end{aligned}$$

Theorem 2.7. If $p > 0, q > 0, a_0, a_1, \dots, a_{(4n+1)}$ are non-negative integers and p, q are relatively prime then

$$(i) \quad 10^4p + q | 10^{(4n+1)}a_{(4n+1)} + 10^{4n}a_{4n} + 10^{(4n-1)}a_{(4n-1)} + \dots + 10^2a_2 + 10a_1 + a_0$$

$\Leftrightarrow$

$$10^4 p + q | 10^3 p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2 p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+2)} p^{(n-i-1)} q^i \right] \\ + 10 \left[\sum_{i=0}^n (-1)^i a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n (-1)^i a_{4i} p^{(n-i)} q^i \right],$$

$$(ii) \quad 10^4 p - q | 10^{(4n+1)} a_{(4n+1)} + 10^{4n} a_{4n} + 10^{(4n-1)} a_{(4n-1)} + \dots + 10^2 a_2 + 10 a_1 + a_0$$

$$\Leftrightarrow 10^4 p - q | 10^3 p \left[\sum_{i=0}^{(n-1)} a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2 p \left[\sum_{i=0}^{(n-1)} a_{(4i+2)} p^{(n-i-1)} q^i \right] \\ + 10 \left[\sum_{i=0}^n a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n a_{4i} p^{(n-i)} q^i \right].$$

Theorem 2.8. If $p > 0$, $q > 0$, $a_0, a_1, \dots, a_{(4n+2)}$ are non-negative integers and p, q are relatively prime then

$$(i) \quad 10^4 p + q | 10^{(4n+2)} a_{(4n+2)} + 10^{(4n+1)} a_{(4n+1)} + 10^{4n} a_{4n} + \dots + 10^2 a_2 + 10 a_1 + a_0$$

$$\Leftrightarrow 10^4 p + q | 10^3 p \left[\sum_{i=0}^{(n-1)} (-1)^i a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2 \left[\sum_{i=0}^n (-1)^i a_{(4i+2)} p^{(n-i)} q^i \right] \\ + 10 \left[\sum_{i=0}^n (-1)^i a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n (-1)^i a_{4i} p^{(n-i)} q^i \right],$$

$$(ii) \quad 10^4 p - q | 10^{(4n+2)} a_{(4n+2)} + 10^{(4n+1)} a_{(4n+1)} + 10^{4n} a_{4n} + \dots + 10^2 a_2 + 10 a_1 + a_0$$

$$\Leftrightarrow 10^4 p - q | 10^3 p \left[\sum_{i=0}^{(n-1)} a_{(4i+3)} p^{(n-i-1)} q^i \right] + 10^2 \left[\sum_{i=0}^n a_{(4i+2)} p^{(n-i)} q^i \right] \\ + 10 \left[\sum_{i=0}^n a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n a_{4i} p^{(n-i)} q^i \right].$$

Theorem 2.9. If $p > 0$, $q > 0$, $a_0, a_1, \dots, a_{(4n+3)}$ are non-negative integers and p, q are relatively prime then

$$(i) \quad 10^4 p + q | 10^{(4n+3)} a_{(4n+3)} + 10^{(4n+2)} a_{(4n+2)} + 10^{(4n+1)} a_{(4n+1)} + \dots + 10^2 a_2 + 10 a_1 + a_0$$

$$\Leftrightarrow 10^4 p + q | 10^3 \left[\sum_{i=0}^n (-1)^i a_{(4i+3)} p^{(n-i)} q^i \right] + 10^2 \left[\sum_{i=0}^n (-1)^i a_{(4i+2)} p^{(n-i)} q^i \right] \\ + 10 \left[\sum_{i=0}^n (-1)^i a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n (-1)^i a_{4i} p^{(n-i)} q^i \right],$$

$$(ii) \quad 10^4 p - q | 10^{(4n+3)} a_{(4n+3)} + 10^{(4n+2)} a_{(4n+2)} + 10^{(4n+1)} a_{(4n+1)} + \dots + 10^2 a_2 + 10 a_1 + a_0$$

$$\Leftrightarrow 10^4 p - q | 10^3 \left[\sum_{i=0}^n a_{(4i+3)} p^{(n-i)} q^i \right] + 10^2 \left[\sum_{i=0}^n a_{(4i+2)} p^{(n-i)} q^i \right] + 10 \left[\sum_{i=0}^n a_{(4i+1)} p^{(n-i)} q^i \right] + \left[\sum_{i=0}^n a_{4i} p^{(n-i)} q^i \right].$$

It is easy to see that Theorem 2.6 can be obtained from Theorem 2.9 by taking $a_{(4n+3)} = a_{(4n+2)} = a_{(4n+1)} = 0$. Similarly, Theorem 2.7 and Theorem 2.8 can be obtained from Theorem 2.9 by taking $a_{(4n+3)} = a_{(4n+2)} = 0$, $a_{(4n+3)} = 0$ respectively. Therefore, Theorem 2.9 can be considered as the generalized divisibility theorem.

3. DIVISIBILITY OF POLYNOMIALS

We now generalize the above results on divisibility to obtain conditions of divisibility of polynomials by polynomial of the form $x^4 \pm q$. In the process of generalization, we can drop the condition of p and q being relatively prime. Without loss of generality, we can take $p = 1$. In the sequel R denotes the field of real numbers.

Theorem 3.1. If $x^4 \pm q$ and $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ are polynomials over R , where a_4 is a polynomial, then

- (i) $x^4 + q | a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \Leftrightarrow x^4 + q | a_3 x^3 + a_2 x^2 + a_1 x + a_0 - a_4 q$,
- (ii) $x^4 - q | a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \Leftrightarrow x^4 - q | a_3 x^3 + a_2 x^2 + a_1 x + a_0 + a_4 q$.

Proof. Let $x^4 + q | a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. Then there exists some polynomial $k(x)$ over R such that

$$(3) \quad a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = k(x)(x^4 + q).$$

Then, $a_0 = k(x)(x^4 + q) - (a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x)$ and $a_3 x^3 + a_2 x^2 + a_1 x + a_0 - a_4 q = (k(x) - a_4)(x^4 + q)$. This implies that $(x^4 + q)$ divides $a_3 x^3 + a_2 x^2 + a_1 x + a_0 - a_4 q$. Conversely, suppose that $x^4 + q | a_3 x^3 + a_2 x^2 + a_1 x + a_0 - a_4 q$. Then there exists some polynomial $h(x)$ over R such that $a_3 x^3 + a_2 x^2 + a_1 x + a_0 - a_4 q = h(x)(x^4 + q)$. Then $a_0 = h(x)(x^4 + q) - (a_3 x^3 + a_2 x^2 + a_1 x - a_4 q)$ and

$$(4) \quad a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = (k(x) + a_4)(x^4 + q).$$

This implies that $x^4 + q$ divides $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. This proves the first part of the theorem. The second part follows on similar lines.

As a corollary of the above theorem, we obtain the following analogue of Theorem 2.2.

Theorem 3.2. If $x^4 \pm q$ and $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ are polynomials over R , then

$$\begin{aligned} (i) \quad x^4 + q | a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 &\Leftrightarrow x^4 + q | a_3x^3 + a_2x^2 + a_1x + a_0 - a_4q \\ &\Leftrightarrow a_1 = a_2 = a_3 = 0 \text{ and } a_0 - a_4q = 0, \\ (ii) \quad x^4 - q | a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 &\Leftrightarrow x^4 - q | a_3x^3 + a_2x^2 + a_1x + a_0 + a_4q \\ &\Leftrightarrow a_1 = a_2 = a_3 = 0 \text{ and } a_0 + a_4q = 0. \end{aligned}$$

Proof. The proof follows from Theorem 3.1 by virtue of the fact that the constant a_4 is a polynomial of degree zero. In this case, the polynomials appearing in (3) and (4) will be constants. Moreover, the cubic polynomial $a_3x^3 + a_2x^2 + a_1x + a_0 - a_4q$ will be divisible by $x^4 + q$ if and only if $a_3x^3 + a_2x^2 + a_1x + a_0 - a_4q$ is identically zero, that is, if and only if $a_1 = a_2 = a_3 = 0$ and $a_0 - a_4q = 0$. This proves the first part of the theorem. The second part follows on similar lines.

Theorems 3.1 and 3.2 can be generalized by using an argument similar to that used in proving Theorem 2.9 to obtain the following analogues of Theorem 2.9.

Theorem 3.3. If $x^4 \pm q$ and $a_{(4n+3)}x^{(4n+3)} + \dots + a_1x + a_0$ are polynomials over R , then

$$\begin{aligned} (i) \quad x^4 + q | a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0 &\Leftrightarrow \\ x^4 + q | \left[\sum_{i=0}^n (-1)^i a_{(4i+3)} q^i \right] x^3 + \left[\sum_{i=0}^n (-1)^i a_{(4i+2)} q^i \right] x^2 + \left[\sum_{i=0}^n (-1)^i a_{(4i+1)} q^i \right] x + & \\ \left[\sum_{i=0}^n (-1)^i a_{4i} q^i \right] & \\ \Leftrightarrow \sum_{i=0}^n (-1)^i a_{(4i+3)} q^i = & \\ 0, \sum_{i=0}^n (-1)^i a_{(4i+2)} q^i = 0, \sum_{i=0}^n (-1)^i a_{(4i+1)} q^i \text{ and } \sum_{i=0}^n (-1)^i a_{4i} q^i = 0, & \\ (ii) \quad x^4 - q | a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0 & \\ \Leftrightarrow x^4 - q | \left[\sum_{i=0}^n a_{(4i+3)} q^i \right] x^3 + \left[\sum_{i=0}^n a_{(4i+2)} q^i \right] x^2 + \left[\sum_{i=0}^n a_{(4i+1)} q^i \right] x + \left[\sum_{i=0}^n a_{4i} q^i \right] & \\ \Leftrightarrow \sum_{i=0}^n a_{(4i+3)} q^i = 0, \sum_{i=0}^n a_{(4i+2)} q^i = 0, \sum_{i=0}^n a_{(4i+1)} q^i \text{ and } \sum_{i=0}^n a_{4i} q^i = 0. & \end{aligned}$$

In Theorems 3.3 if we take $a_1 = a_5 = \dots = a_{(4n+1)} = 0$, $a_2 = a_6 = \dots = a_{(4n+2)} = 0$ and $a_3 = a_7 = \dots = a_{(4n+3)} = 0$, write b_n for a_{4n} and write y for x^4 then we obtain the well-known Factor Theorem of algebra as a corollary of Theorem 3.3.

Corollary 3.1. If $y - q$ and $b_n y^n + b_{(n-1)} y^{(n-1)} + \dots + b_1 y + b_0$ are polynomials over R , then $y - q | b_n y^n + b_{(n-1)} y^{(n-1)} + \dots + b_1 y + b_0 \Leftrightarrow b_n q^n + b_{(n-1)} q^{(n-1)} + \dots + b_1 q + b_0 = f(q) = 0$.

We thus see that Theorem 3.3 is generalization of the Factor Theorem. Moreover, Theorem 3.3 can be rewritten in the following form to obtain the generalization of the Remainder Theorem of algebra.

Theorem 3.4. The remainders on dividing the polynomial $f(x) = a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ by $x^4 \pm q$ are respectively given by

$$\left[\sum_{i=0}^n (-1)^i a_{(4i+3)} q^i \right] x^3 + \left[\sum_{i=0}^n (-1)^i a_{(4i+2)} q^i \right] x^2 + \left[\sum_{i=0}^n (-1)^i a_{(4i+1)} q^i \right] x + \left[\sum_{i=0}^n (-1)^i a_{4i} q^i \right]$$

$$\text{and } \left[\sum_{i=0}^n a_{(4i+3)} q^i \right] x^3 + \left[\sum_{i=0}^n a_{(4i+2)} q^i \right] x^2 + \left[\sum_{i=0}^n a_{(4i+1)} q^i \right] x + \left[\sum_{i=0}^n a_{4i} q^i \right].$$

Remark 3.1. The polynomial $f(x) = a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ in Theorem 3.4 can be written as

$$\begin{aligned} f(x) &= x^3(a_{(4n+3)}x^{4n} + a_{(4n-1)}x^{(4n-4)} + \dots + a_3) \\ &\quad + x^2(a_{(4n+2)}x^{4n} + a_{(4n-2)}x^{(4n-4)} + \dots + a_2) \\ &\quad + x(a_{(4n+1)}x^{4n} + a_{(4n-3)}x^{(4n-4)} + \dots + a_1) \\ &\quad + (a_{4n}x^{4n} + a_{(4n-4)}x^{(4n-4)} + \dots + a_0) \\ &= x^3f_1(x^4) + x^2f_2(x^4) + xf_3(x^4) + f_4(x^4). \end{aligned}$$

Using this notation, Theorem 3.4 can be restated as:

The remainder on dividing the polynomial $f(x) = a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0 = x^3f_1(x^4) + x^2f_2(x^4) + xf_3(x^4) + f_4(x^4)$ by $x^4 \pm q$ is given by $x^3f_1(-q) + x^2f_2(-q) + xf_3(-q) + f_4(-q)$ and $x^3f_1(q) + x^2f_2(q) + xf_3(q) + f_4(q)$ respectively.

Moreover, Theorem 3.4 can be restated in a compact form as follows:

If $x^4 + q$ and $a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ are polynomials over R , then

$$\begin{aligned} x^4 + q | f(x) &= x^3f_1(x^4) + x^2f_2(x^4) + xf_3(x^4) + f_4(x^4) \\ \Leftrightarrow x^4 + q | x^3f_1(-q) + x^2f_2(-q) + xf_3(-q) + f_4(-q) \\ \Leftrightarrow f_1(-q) &= f_2(-q) = f_3(-q) = f_4(-q) = f(-\alpha q^{1/4}) = 0. \end{aligned}$$

If $x^4 - q$ and $a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ are polynomials over R , then

$$\begin{aligned} x^4 - q | f(x) &= x^3f_1(x^4) + x^2f_2(x^4) + xf_3(x^4) + f_4(x^4) \\ \Leftrightarrow x^4 - q | x^3f_1(q) + x^2f_2(q) + xf_3(q) + f_4(q) \\ \Leftrightarrow f_1(q) &= f_2(q) = f_3(q) = f_4(q) = f(\alpha q^{1/4}) = 0. \end{aligned}$$

where $\alpha = \cos(\pi/2) + i \sin(\pi/2)$ is a forth root of unity.

With the help of Theorem 3.4 we can compute the quotient obtained on dividing a polynomial $f(x)$ by $x^4 \pm q$. We do this in the next theorem, which may be called as the Quotient Theorem.

Theorem 3.5 (Quotient Theorem). The quotients on dividing the polynomial $f(x) = a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ by $x^4 \pm q$ are respectively given by

$$\sum_{j=4}^{(4n+3)} a_j \left(x^{j-4} - qx^{j-8} + q^2x^{j-12} - \dots + (-1)^{\left[\frac{r}{4}-1\right]} q^{\left[\frac{r}{4}-1\right]} x^{\left[\frac{r}{4}\right](r-4)} \right)$$

and

$$\sum_{j=4}^{(4n+3)} a_j \left(x^{j-4} + qx^{j-8} + q^2x^{j-12} + \dots + q^{\left[\frac{r}{4}-1\right]} x^{\left[\frac{r}{4}\right](r-4)} \right).$$

Proof. Consider the division of $f(x) = a_{(4n+3)}x^{(4n+3)} + a_{(4n+2)}x^{(4n+2)} + a_{(4n+1)}x^{(4n+1)} + \dots + 10^2a_2 + 10a_1 + a_0$ by $x^4 + q$. Then, by virtue of Theorem 3.4, we get $f(x) - \text{remainder} = x^4 + q(a_4 + a_5x + a_6x^2 + a_7x^3 + a_8(x^4 - q) + a_9(x^5 - qx) + a_{10}(x^6 - qx^2) + a_{11}(x^7 - qx^3) + a_{12}(x^8 - qx^4 + q^2) + \dots)$. This yields the first part of the theorem. The second part follows similarly. In a similar manner, Corollary 3.1 leads to the following

Corollary 3.2. The quotient on dividing the polynomial $f(y) = b_ny^n + b_{n-1}y^{n-1} + \dots + b_1y + b_0$ by $y - q$ is given by

$$\sum_{k=1}^n b_k (y^{k-1} + qy^{k-2} + q^2y^{k-3} + \dots + q^{k-1}).$$

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