

PARTIAL LATIN SQUARES HAVING A SANTILLI'S AUTOTOPIISM IN THEIR AUTOTOPIISM GROUPS

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ABSTRACT. An autotopism $\Theta = (\alpha, \beta, \gamma)$ of a partial Latin square $P = (p_{i,j})$ with symbols in $N = \{0, 1, \dots, n-1\}$ is said to be a Santilli's autotopism of P if there exist $j_\alpha, j_\beta, j_\gamma \in N$ such that $\alpha(i) = p_{i,j_\alpha}$, $\beta(i) = p_{i,j_\beta}$ and $\gamma(i) = p_{i,j_\gamma}$, for all $i \in N$. In this paper, we are interested in the set $SPLS(\Theta)$ of partial Latin squares having Θ as a Santilli's autotopism. Specifically, the cardinality of the set of minimal elements of $SPLS(\Theta)$ is obtained and a classification of Latin squares Santilli's isotopisms of order ≤ 5 fixing at least one symbol is given.

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1. INTRODUCTION

A *quasigroup* [1] is a nonempty set G endowed with a product $\cdot$, such that if any two of the three symbols a, b, c in the equation $a \cdot b = c$ are given as elements of G , the third is uniquely determined as an element of G . It is equivalent to say that G is endowed with left $/$ and right $\backslash$ division. Two quasigroups $(G, \cdot)$ and $(H, \circ)$ are *isotopic* [2] if there are three bijections α, β, γ from H to G , such that $\gamma(a \circ b) = \alpha(a) \cdot \beta(b)$, for all $a, b \in H$. The triple $\Theta = (\alpha, \beta, \gamma)$ is called an *isotopism* from $(G, \cdot)$ to $(H, \circ)$.

The multiplication table of a quasigroup is a Latin square. A *Latin square* L of order n is a $n \times n$ array with elements chosen from a set $N = \{x_1, \dots, x_n\}$, such that each symbol occurs precisely once in each row and each column. The set of Latin

squares of order n is denoted by $LS(n)$. An exhaustive study about Latin squares and their applications can be seen, for example, in [6].

In this paper we will consider $N = \{0, 1, \dots, n-1\}$. So, if $L = (l_{i,j})$, the *orthogonal array representation of L* is the set of n^2 triples $\{(i, j, l_{i,j}) : i, j \in N\}$. The previous set is identified with L and then, it is written $(i, j, l_{i,j}) \in L$, for all $i, j \in N$. Keeping in mind that a Latin square is the multiplication table of a quasigroup, an *isotopism* of a Latin square L is therefore a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n = S_n \times S_n \times S_n$, where S_n is the symmetric group on N . It is verified that α, β and γ are then permutations of rows, columns and symbols of L , respectively. The resulting square L^Θ is also a Latin square and it is said to be *isotopic* to L . In particular, if $L = (l_{i,j})$, then $L^\Theta = \{(i, j, \gamma^{-1}(l_{\alpha(i), \beta(j)})) : 0 \leq i, j \leq n-1\}$. If $\gamma = \epsilon$, the identity map on N , Θ is called a *principal isotopism*. An isotopism which maps L to itself is an *autotopism*. $(\epsilon, \epsilon, \epsilon)$ is called the *trivial autotopism*. The stabilizer subgroup of L in $\mathcal{I}_n$ is its *autotopism group*, $\mathcal{U}(L) = \{\Theta \in \mathcal{I}_n : L^\Theta = L\}$. Fixed $\Theta \in \mathcal{I}_n$, the set of all Latin squares L such that $\Theta \in \mathcal{U}(L)$ is denoted by $LS(\Theta)$. Finally, the cardinality of $LS(\Theta)$ is denoted by $\Delta(\Theta)$.

$$\left\{ \begin{array}{l} L_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \\ \Theta = ((0 \ 1)(2 \ 3), (1 \ 2), \epsilon) \end{array} \right\} \Rightarrow L_1^\Theta = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

FIGURE 1. Isotopism permuting 1st with 2nd and 3th with 4th rows and 2nd with 3th columns.

The *cycle structure of a permutation* $\delta \in S_n$ is the sequence $(l_1, l_2, \dots, l_n)$, where l_i is the number of cycles of length i in δ . The following result determines the possible non-trivial autotopisms:

Theorem 1.1 (McKay, Meynert and Myrvold [8]). *Let $L \in LS(n)$. Every non-trivial $\Theta = (\alpha, \beta, \gamma) \in \mathcal{U}(L)$ verifies one of the following assertions:*

- a) α, β, γ have the same cycle structure with at least one and at most $\lfloor \frac{n}{2} \rfloor$ fixed points,
- b) One of α, β, γ has at least one fixed point and the other two have the same cycle structure without fixed points,
- c) None of α, β, γ has fixed points. □

Specifically, the following result is verified:

Proposition 1.2 (Falcón [3]). *Let $\Theta = (\alpha, \beta, \epsilon) \in \mathcal{I}_{n,k}$ be a non-trivial isotopism. Then, $\Delta(\Theta) > 0$ if and only if α and β are both the composition of k cycles of length $\frac{n}{k}$.* □

Fixed $\delta \in S_n$, it is defined the set of its fixed points $Fix(\delta) = \{i \in N : \delta(i) = i\}$. It is said that an isotopism $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$ *fixes at least one symbol* if $Fix(\gamma) \geq 1$.

A *partial Latin square* $P = (p_{i,j})$ of order n is a $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. It is therefore the multiplication table of a partial quasigroup in abstract algebra. The *size* of P , which is denoted by $\#P$, is the number of its *filled cells*, that is, the number of triples $(i, j, k) \in P$ such that $k \neq \emptyset$. The set of partial Latin squares of order n is denoted by $PLS(n)$. *Isotopisms* of partial Latin squares are defined in a similar way as those of Latin squares, although now $\gamma(\emptyset) = \emptyset$. In particular, the sets $\mathcal{U}(P)$ and $PLS(\Theta)$ are similarly defined. It is said that $\Theta = (\alpha, \beta, \gamma) \in \mathcal{U}(P)$ is a *Santilli's autotopism* of P if there exist $j_\alpha, j_\beta, j_\gamma \in N$ such that $\alpha(i) = p_{i,j_\alpha}$, $\beta(i) = p_{i,j_\beta}$ and $\gamma(i) = p_{i,j_\gamma}$, for all $i \in N$. It is indeed a new interpretation [5] of the classical notion of Santilli's isotopisms [9], which has been used, in particular, in the conception of the Hadronic Mechanic [10].

Properties of those partial Latin squares having in their autotopism groups a Santilli's autotopism have not yet been studied. In this way, we are interested in this paper in studying these properties in case of having a Santilli's autotopism fixing at least one symbol. So, the structure of the paper is the following: in Section 2, some results about Latin square autotopisms fixing at least one symbol are reviewed. In Section 3, by taking into account the canonical construction path method of McKay [7], it is defined the symmetrical structure of any Latin square $L \in LS(\Theta)$, where Θ is an isotopism fixing at least one symbol. In Section 4, the set $SPLS(\Theta)$ of partial Latin squares having an isotopism Θ as a Santilli's autotopism is studied. Specifically, the cardinality of the set of minimal elements of $SPLS(\Theta)$ is determined. Finally, a classification of all Santilli's autotopisms of order $2 \leq n \leq 5$ fixing at least one symbol is given.

2. LATIN SQUARE AUTOTOPISMS FIXING AT LEAST ONE SYMBOL

Every permutation of S_n can be written as the composition of pairwise disjoint cycles. So, from now on, fixed $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$, we will consider that, for all $\delta \in \{\alpha, \beta, \gamma\}$:

$$\delta = C_0^\delta \circ C_1^\delta \circ \dots \circ C_{k_\delta-1}^\delta, \quad (1)$$

where:

- i) For all $i \in \{0, 1, \dots, k_\delta - 1\}$, we have that $C_i^\delta = (c_{i,0}^\delta \ c_{i,1}^\delta \ \dots \ c_{i,\lambda_i^\delta-1}^\delta)$, being $\lambda_i^\delta = \text{length}(C_i^\delta) \leq n$ and $c_{i,0}^\delta = \min_j \{c_{i,j}^\delta\}$.
- ii) $\sum_i \lambda_i^\delta = n$.
- iii) For all $i, j \in \{0, 1, \dots, k_\delta - 1\}$, $\lambda_i^\delta \geq \lambda_j^\delta$, whenever $i \leq j$.
- iv) Given $i, j \in \{0, 1, \dots, k_\delta - 1\}$, being $i < j$ and $\lambda_i^\delta = \lambda_j^\delta$, then $c_{i,0}^\delta < c_{j,0}^\delta$.

Fixed $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in \{0, 1, \dots, k_\delta - 1\}$, we will write $a \in C_i^\delta$ if there exists $j \in \{0, 1, \dots, \lambda_i^\delta - 1\}$ such that $a = c_{i,j}^\delta$. The following result is verified:

Proposition 2.1 (Falcón [4]). *Let $L = (l_{i,j}) \in LS(n)$ and $\Theta = (\alpha, \beta, \gamma) \in \mathcal{U}(L)$ and let us consider $r \in \{0, 1, \dots, k_\alpha - 1\}$ and $s \in \{0, 1, \dots, k_\beta - 1\}$. Let us denote*

$m = l.c.m.(\lambda_r^\alpha, \lambda_s^\beta)$. Now, fixed $a \in C_r^\alpha$ and $b \in C_s^\beta$, let $t \in \{0, 1, \dots, k_\gamma - 1\}$ be such that $l_{a,b} \in C_t^\gamma$. Then, it is verified that:

$$\lambda_t^\gamma = \begin{cases} \text{divides } m, & \text{if } \lambda_r^\alpha = \lambda_s^\beta, \\ = m, & \text{if } \lambda_r^\alpha \neq \lambda_s^\beta. \end{cases}$$

□

Now, let us observe that, from Theorem 1.1, if $\Theta = (\alpha, \beta, \gamma)$ fixes at least one symbol, we can consider in (1), $k = k_\alpha = k_\beta$ ($= k_\gamma = k'$, if we are in case (a)), where:

- i) $k \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{\lceil \frac{n}{2} \rceil}{2} \rfloor$.
- ii) For all $i \in \{0, 1, \dots, k-1\}$, $length(C_i^\alpha) = \lambda_i = length(C_i^\beta)$ ($= length(C_i^\gamma) = \lambda'_i$, if we are in case (a)).

From now on, $\mathcal{I}_{n,k}$ will denote the set of such isotopisms. Fixed $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n,k}$, let us define, for all $m \in \{1, 2, \dots, n\}$:

$$\mathbf{l}_m = \#\{i \in \{0, 1, \dots, k-1\} : \lambda_i = m\}, \quad \mathbf{l}'_m = \#\{i \in \{0, 1, \dots, k'-1\} : \lambda'_i = m\},$$

where $\#$ denotes the cardinality of the corresponding set. Then, let us consider:

$$\mathbf{l} = (\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n), \quad \mathbf{l}' = (\mathbf{l}'_1, \mathbf{l}'_2, \dots, \mathbf{l}'_n).$$

The pair $(\mathbf{l}, \mathbf{l}')$ will be called the *cycle structure of Θ* . Then, the following result is verified:

Lemma 2.2 (Falcón [4]). *The following assertions are verified:*

- i) $\mathbf{l}_1 \leq \lfloor \frac{n}{2} \rfloor$.
- ii) If $\mathbf{l}_1 \neq 0$, then $\mathbf{l}' = \mathbf{l}$.
- iii) $\sum_{m=1}^n \mathbf{l}_m = k \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{\lceil \frac{n}{2} \rceil}{2} \rfloor$ and $\sum_{m=1}^n \mathbf{l}'_m = k'$.
- iv) $\sum_{m=1}^n m \cdot \mathbf{l}_m = \sum_{m=1}^n m \cdot \mathbf{l}'_m = n$.
- v) If $k = 1$, then $\mathbf{l} = (0, \dots, 0, 1)$.
- vi) If there exist $m_1, m_2 \in \{1, 2, \dots, n\}$ such that $l.c.m.(m_1, m_2) > n$, then $\mathbf{l}_{m_1} \cdot \mathbf{l}_{m_2} = 0$. □

From Lemma 2.2, it is possible to obtain a classification of all possible autotopisms fixing at least one symbol. Specifically, we can see in Table 1 this classification and some examples for order $2 \leq n \leq 5$.

3. THE SYMMETRICAL STRUCTURE OF Θ .

In this section, fixed $\Theta \in \mathcal{I}_{n,k}$, we are interested in obtaining the symmetrical structure of any $L \in LS(\Theta)$. To obtain it, it can be used the canonical construction path method of McKay [7]. This algorithm allows to decompose L in the subrectangles $R_L^{r,s} = \{(c_{r,u}^\alpha, c_{s,v}^\beta, l_{c_{r,u}^\alpha, c_{s,v}^\beta}) : u \in \{0, 1, \dots, \lambda_r - 1\}, v \in \{0, 1, \dots, \lambda_s - 1\}\}$, for all $r, s \in \{0, 1, \dots, k-1\}$. Let us denote $k_{r,s} = \frac{\lambda_r \cdot \lambda_s}{l.c.m.(\lambda_r, \lambda_s)}$ and let $m = \sum_{r,s} k_{r,s}$. This last number will be called the *structure number of Θ* . It is immediate the following:

n	k	$\mathbf{l}$	$\mathbf{l}'$	$\Theta \in \mathcal{I}_{n,k}$	$L \in LS(\Theta)$
2	1	(0, 1)	(2, 0)	$((01), (01), \epsilon)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	1	(0, 0, 1)	(3, 0, 0)	$((012), (012), \epsilon)$	$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$
	2	(1, 1, 0)	(1, 1, 0)	$((01), (01), (01))$	$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$
4	1	(0, 0, 0, 1)	(2, 1, 0, 0)	$((0123), (0123), (01))$	$\begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{pmatrix}$
			(4, 0, 0, 0)	$((0123), (0123), \epsilon)$	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$
	2	(1, 0, 1, 0)	(1, 0, 1, 0)	$((012), (012), (012))$	$\begin{pmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{pmatrix}$
			(0, 2, 0, 0)	$((01)(23), (01)(23), (01))$	$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 0 \\ 1 & 3 & 0 & 2 \\ 3 & 0 & 2 & 1 \end{pmatrix}$
		(0, 2, 0, 0)	(2, 1, 0, 0)	$((01)(23), (01)(23), \epsilon)$	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$
			(4, 0, 0, 0)	$((01)(23), (01)(23), \epsilon)$	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$
	3	(2, 1, 0, 0)	(2, 1, 0, 0)	$((01), (01), (01))$	$\begin{pmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}$
5	1	(0, 0, 0, 0, 1)	(5, 0, 0, 0, 0)	$((01234), (01234), \epsilon)$	$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$
	2	(1, 0, 0, 1, 0)	(1, 0, 0, 1, 0)	$((0123), (0123), (0123))$	$\begin{pmatrix} 4 & 0 & 3 & 1 & 2 \\ 2 & 4 & 1 & 0 & 3 \\ 1 & 3 & 4 & 2 & 0 \\ 3 & 2 & 0 & 4 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$
	3	(2, 0, 1, 0, 0)	(2, 0, 1, 0, 0)	$((012), (012), (012))$	$\begin{pmatrix} 3 & 4 & 2 & 0 & 1 \\ 0 & 3 & 4 & 1 & 2 \\ 4 & 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$
		(1, 2, 0, 0, 0)	(1, 2, 0, 0, 0)	$((01)(23), (01)(23), (01)(23))$	$\begin{pmatrix} 4 & 0 & 2 & 1 & 3 \\ 1 & 4 & 0 & 3 & 2 \\ 3 & 1 & 4 & 2 & 0 \\ 0 & 2 & 3 & 4 & 1 \\ 2 & 3 & 1 & 0 & 4 \end{pmatrix}$

TABLE 1. Classification of Latin square autotopisms of order $2 \leq n \leq 5$, fixing at least one symbol.

Lemma 3.1. *Let $\Theta_1, \Theta_2 \in \mathcal{I}_{n,k}$ be such that they have the same cycle structure. Then, their structure numbers also coincide.*

Now, to obtain the symmetrical structure of $LS(\Theta)$, let us consider an ordered set of m different symbols $X = \{x_1, x_2, \dots, x_m\}$, none of them in N , and let us follow the next:

Algorithm 3.2. Let us fixe a subrectangle $R_L^{r_0, s_0}$ and let $m_{r_0, s_0} = \sum_{r < r_0, s} k_{r, s} + \sum_{s < s_0} k_{r_0, s}$. Then:

1) Let us impose that, for all $i \in \{0, 1, \dots, l.c.m.(\lambda_{r_0}, \lambda_{s_0}) - 1\}$:

$$l_{c_{r_0, i}^\alpha \text{ (mod } \lambda_{r_0}), c_{s_0, i}^\beta \text{ (mod } \lambda_{s_0})} = x_{m_{r_0, s_0} + 1}.$$

2) Let n_1 be the smallest positive integer such that $l_{c_{r_0, 0}^\alpha, c_{s_0, n_1}^\beta}$ has not been obtained from the previous step. Then, let us impose that, for all $i \in \{0, 1, \dots, l.c.m.(\lambda_{r_0}, \lambda_{s_0}) - 1\}$:

$$l_{c_{r_0, i}^\alpha \text{ (mod } \lambda_{r_0}), c_{s_0, n_1 + i}^\beta \text{ (mod } \lambda_{s_0})} = x_{m_{r_0, s_0} + 2}.$$

Now, for j from 3 to k_{r_0, s_0} we do:

j) Let n_{j-1} be the smallest positive integer such that $l_{c_{r_0, 0}^\alpha, c_{s_0, n_{j-1}}^\beta}$ has not been obtained from the previous steps. Then, let us impose that, for all $i \in \{0, 1, \dots, l.c.m.(\lambda_{r_0}, \lambda_{s_0}) - 1\}$:

$$l_{c_{r_0, i}^\alpha \text{ (mod } \lambda_{r_0}), c_{s_0, n_{j-1} + i}^\beta \text{ (mod } \lambda_{s_0})} = x_{m_{r_0, s_0} + j}.$$

By following the previous algorithm, we determine completely $R_L^{r_0, s_0}$ by using symbols of X . Analogously, we can repeat the process with all subrectangles $R_L^{r, s}$ and therefore, we can obtain an $n \times n$ array L_Θ^X with elements chosen from X . Let us observe that if we choose a distinct set of symbols $X' = \{x'_1, x'_2, \dots, x'_m\}$, then the array $L_\Theta^{X'}$ coincides with L_Θ^X except for the bijection on the symbols $\phi(x_i) = x'_i$, for all $i \in \{1, 2, \dots, m\}$. So, from now on, once we have chosen a set of symbols to follow the previous algorithm, we will denote the resulting array by L_Θ and we will say that L_Θ determines the symmetrical structure of Θ . The following result is verified:

Proposition 3.3. Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n, k}$ and $P \in PLS(\Theta)$. Let $L_\Theta = (l_{i, j})$ be the symmetrical structure of Θ . Finally, let $i_0, j_0 \in N$ be such that $p_{i_0, j_0} \neq \emptyset$. Then, $p_{i, j} \neq \emptyset$, for all $i, j \in N$ such that $l_{i, j} = l_{i_0, j_0}$. Moreover, if $r_0, s_0 \in \{0, 1, \dots, k - 1\}$, $u_0 \in \{0, 1, \dots, \lambda_{r_0} - 1\}$ and $v_0 \in \{0, 1, \dots, \lambda_{s_0} - 1\}$ are such that $c_{r_0, u_0}^\alpha = i_0$ and $c_{s_0, v_0}^\beta = j_0$, respectively, then, the symbols of all these filled cells of P must be in a same cycle $C_{k_0}^\gamma$ of γ , such that λ'_{k_0} divides $l.m.c.(\lambda_{r_0}, \lambda_{s_0})$.

Proof. From Algorithm 3.2, we know that, for all $i \in \{0, 1, \dots, l.c.m.(\lambda_{r_0}, \lambda_{s_0}) - 1\}$:

$$l_{c_{r_0, u_0 + i}^\alpha \text{ (mod } \lambda_{r_0}), c_{s_0, v_0 + i}^\beta \text{ (mod } \lambda_{s_0})} = l_{c_{r_0, u_0}^\alpha, c_{s_0, v_0}^\beta} = l_{i_0, j_0}.$$

Indeed, the previous cells of L_Θ are the only ones filled with l_{i_0, j_0} . Now, let $t_0 \in \{0, 1, \dots, k' - 1\}$ and $w_0 \in \{0, 1, \dots, \lambda'_{t_0} - 1\}$ be such that $c_{t_0, w_0}^\gamma = p_{i_0, j_0}$. Then, as $P \in PLS(\Theta)$, it must be, for all $i \in \{0, 1, \dots, l.c.m.(\lambda_{r_0}, \lambda_{s_0}) - 1\}$:

$$p_{c_{r_0, u_0 + i}^\alpha \text{ (mod } \lambda_{r_0}), c_{s_0, v_0 + i}^\beta \text{ (mod } \lambda_{s_0})} = c_{t_0, w_0 + i}^\gamma \text{ (mod } \lambda_{t_0}).$$

The first assertion of the result is therefore proven. Then, the second one is immediate, by keeping in mind Proposition 2.1. $\square$

The symmetrical structure has been used by Falcón [3] to determine the group $LS(\Theta)$ when Θ is a principal isotopism. By the other way, let us observe that, in general, the symmetrical structure is not useful to identify a cycle structure. So, for example, $\Theta_1 = ((0123), (0123), (01))$ and $\Theta_2 = ((0123), (0123), \epsilon)$ have different cycle structures, but their symmetrical structures are both equal to the following array:

$$L_{\Theta_1} = L_{\Theta_2} = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}.$$

Nevertheless, if we extend, in a natural way, the concept of two isotopic Latin squares to any two $n \times n$ arrays with symbols belonging to a set S , in such a way that the third component of each isotopism corresponds to a permutation of the elements of S , then, by keeping in mind the previous algorithm, the following result is verified:

Proposition 3.4. *Let $\Theta_1 = (\alpha_1, \beta_1, \gamma_1), \Theta_2 = (\alpha_2, \beta_2, \gamma_2) \in \mathcal{I}_{n,k}$ be such that they have the same cycle structure. Then, L_{Θ_1} and L_{Θ_2} are isotopic.*

Proof. From Lemma 3.1, Θ_1 and Θ_2 have the same structure number m . So, we can suppose that L_{Θ_1} and L_{Θ_2} have the same set of m symbols $X = \{x_1, \dots, x_m\}$. Now, let us consider the permutations $\alpha, \beta : N \rightarrow N$, such that $\alpha(c_{i,j}^{\alpha_1}) = c_{i,j}^{\alpha_2}$ and $\beta(c_{i,j}^{\beta_1}) = c_{i,j}^{\beta_2}$, for all $i \in \{0, 1, \dots, k-1\}$ and $j \in \{0, 1, \dots, \lambda_i-1\}$. By taking into account Algorithm 3.2, it is verified that if we apply $\Theta = (\alpha, \beta, \epsilon)$ to L_{Θ_1} , we obtain L_{Θ_2} . So, L_{Θ_1} and L_{Θ_2} are isotopic. $\square$

Θ	$((01), (01), (01))$	$((02), (12), (01))$
L_{Θ}	$\begin{pmatrix} a & b & c \\ b & a & c \\ d & d & e \end{pmatrix}$	$\begin{pmatrix} c & a & b \\ e & d & d \\ c & b & a \end{pmatrix}$

FIGURE 2. Isotopic symmetrical structures of two autotopisms of order 3 with the same cycle structure. In this example we have chosen the set of symbols $X = \{a, b, c, d, e\}$.

4. MINIMAL PARTIAL LATIN SQUARES RELATED TO A SANTILLI'S AUTOTOPISM

Let $\Theta \in \mathcal{I}_{n,k}$. From now on, we will denote by $SPLS(\Theta)$ the set of all partial Latin squares having Θ as a Santilli's autotopism. So, if $SPLS(\Theta) = \emptyset$, then Θ cannot be a Santilli's autotopism. In this way, let us define the set $\mathcal{ST}_{n,k} = \{\Theta \in \mathcal{I}_{n,k} : SPLS(\Theta) \neq \emptyset\}$. By the other way, if $P = (p_{i,j}) \in SPLS(\Theta)$, we will denote

Θ	L_Θ	Θ	L_Θ
$((01), (01), \epsilon)$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$((01)(23), (01)(23), \epsilon)$	$\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ e & f & g & h \\ f & e & h & g \end{pmatrix}$
$((012), (012), \epsilon)$	$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$	$((01), (01), (01))$	$\begin{pmatrix} a & b & c & d \\ b & a & c & d \\ e & e & g & h \\ f & f & i & j \end{pmatrix}$
$((01), (01), (01))$	$\begin{pmatrix} a & b & c \\ b & a & c \\ d & d & e \end{pmatrix}$	$((01234), (01234), \epsilon)$	$\begin{pmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{pmatrix}$
$((0123), (0123), (01))$	$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$	$((0123), (0123), (0123))$	$\begin{pmatrix} a & b & c & d & e \\ d & a & b & c & e \\ c & d & a & b & e \\ b & c & d & a & e \\ f & f & f & f & g \end{pmatrix}$
$((0123), (0123), \epsilon)$	$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$	$((012), (012), (012))$	$\begin{pmatrix} a & b & c & d & e \\ c & a & b & d & e \\ b & c & a & d & e \\ f & f & f & h & i \\ g & g & g & j & k \end{pmatrix}$
$((012), (012), (012))$	$\begin{pmatrix} a & b & c & d \\ c & a & b & d \\ b & c & a & d \\ e & e & e & f \end{pmatrix}$	$((01)(23), (01)(23), (01)(23))$	$\begin{pmatrix} a & b & c & d & e \\ b & a & d & c & e \\ f & g & h & i & j \\ g & f & i & h & j \\ k & k & l & l & m \end{pmatrix}$
$((01)(23), (01)(23), (01))$	$\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ e & f & g & h \\ f & e & h & g \end{pmatrix}$		

TABLE 2. Symmetrical structures of the autotopisms appearing in Table 1.

by $S(\Theta, P)$, the triple $(j_\alpha, j_\beta, j_\gamma) \in N^3$, such that $\alpha(i) = p_{i,j_\alpha}$, $\beta(i) = p_{i,j_\beta}$ and $\gamma(i) = p_{i,j_\gamma}$, for all $i \in N$. In this section, we are interested in studying several properties of $SPLS(\Theta)$. In this way, the following results are verified:

Lemma 4.1. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_{n,k}$. If there exist $\delta, \delta' \in \{\alpha, \beta, \gamma\}$ and $i_0 \in N$ such that $\delta(i_0) = \delta'(i_0)$, then, it must be $\delta = \delta'$.*

Proof. Let $P = (p_{i,j}) \in SPLS(\Theta)$. There must exist $j_\delta, j_{\delta'} \in N$ such that $p_{i,j_\delta} = \delta(i)$ and $p_{i,j_{\delta'}} = \delta'(i)$, for all $i \in N$. Then, if we consider $i = i_0$, we have that $p_{i_0,j_\delta} = \delta(i_0) = \delta'(i_0) = p_{i_0,j_{\delta'}}$. So, keeping in mind that P is a partial Latin square, it must be $j_\delta = j_{\delta'}$. But then, $\delta(i) = p_{i,j_\delta} = p_{i,j_{\delta'}} = \delta'(i)$, for all $i \in N$. Therefore, $\delta = \delta'$. $\square$

Lemma 4.2. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_{n,k}$ be such that $\mathbf{l}_1 \geq 1$ in its cycle structure $((\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n), (\mathbf{l}'_1, \mathbf{l}'_2, \dots, \mathbf{l}'_n))$. Let $P = (p_{i,j}) \in SPLS(\Theta)$. If there exist $\delta \in \{\alpha, \beta, \gamma\}$, $i_0 \in \text{Fix}(\alpha)$ and $k_0 \in \text{Fix}(\gamma)$ such that $\delta(i_0) = k_0$, then $j_\delta \in \text{Fix}(\beta)$. Moreover, if $j_\beta \in \text{Fix}(\beta)$, then $\beta(i) \in \text{Fix}(\gamma)$, for all $i \in \text{Fix}(\alpha)$.*

Proof. Let us consider $L_\Theta = (l_{i,j})$, the symmetrical structure of Θ . As $i_0 \in \text{Fix}(\alpha)$, Algorithm 3.2 implies that there do not exist $i \in N \setminus \{i_0\}$ and $j \in N$

such that $l_{i,j} \in \{l_{i_0,0}, l_{i_0,1}, \dots, l_{i_0,n-1}\}$. Now, as $p_{i_0,j_\delta} = \delta(i_0) = k_0 \in \text{Fix}(\gamma)$, then, Proposition 3.3 implies that there does not exist $j \in N \setminus \{j_\delta\}$ such that $l_{i_0,j} = l_{i_0,j_\delta}$. Therefore, j_δ must be a fixed point of β .

Now, let us suppose that $j_\beta \in \text{Fix}(\beta)$ and let us consider $i_0 \in \text{Fix}(\alpha)$. Let us see that $\beta(i_0) \in \text{Fix}(\gamma)$. To prove it, let us consider $r \in \{0, 1, \dots, k-1\}$ such that $i_0 \in C_r^\alpha$ and let $s \in \{0, 1, \dots, \lambda_r - 1\}$ be such that $i_0 = c_{r,s}^\alpha$. Then, it is verified that:

$$\beta(i_0) = p_{i_0,j_\beta} = p_{\alpha(i_0),j_\beta} = p_{c_{r,s+1 \pmod{\lambda_r}}^\alpha, j_\beta} = \gamma(p_{c_{r,s}^\alpha, j_\beta}) = \gamma(p_{i_0,j_\beta}) = \gamma(\beta(i_0)).$$

Therefore, $\beta(i_0) \in \text{Fix}(\gamma)$. $\square$

Lemma 4.3. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{ST}_{n,k}$ and let $P = (p_{i,j}) \in \text{SPLS}(\Theta)$ be such that $S(\Theta, P) = (j_\alpha, j_\beta, j_\gamma)$. If there exist $m \in \mathbb{N}$ and $i_0 \in N$ such that $\gamma^m(\beta(i_0)) = \beta(\alpha^m(i_0))$, then $j_\beta \in \text{Fix}(\beta^m)$.*

Proof. Let $r, s \in \{0, 1, \dots, k-1\}$ be such that $i_0 \in C_r^\alpha$ and $j_\beta \in C_s^\beta$. Now, let $u \in \{0, 1, \dots, \lambda_r - 1\}$ and $v \in \{0, 1, \dots, \lambda_s - 1\}$ be such that $i_0 = c_{r,u}^\alpha$ and $j_\beta = c_{s,v}^\beta$. Then:

$$\begin{aligned} p_{\alpha^m(i_0), \beta^m(j_\beta)} &= p_{c_{r,u+m \pmod{\lambda_r}}^\alpha, c_{s,v+m \pmod{\lambda_s}}^\beta} = \gamma^m(p_{c_{r,u}^\alpha, c_{s,v}^\beta}) = \gamma^m(p_{i_0, j_\beta}) = \\ &= \gamma^m(\beta(i_0)) = \beta(\alpha^m(i_0)) = p_{\alpha^m(i_0), j_\beta}. \end{aligned}$$

Therefore, it must be $j_\beta \in \text{Fix}(\beta^m)$, because $P \in \text{PLS}(n)$. $\square$

Lemma 4.1 can be used, for example, to determine that $((012), (013), (012)) \in \mathcal{I}_{4,2}$ is not a Santilli's autotopism. Lemma 4.2 is useful, for example, to determine that if $\Theta = (\alpha, \beta, \gamma) = ((012), (134), (012)) \in \mathcal{I}_{5,3}$ is a Santilli's autotopism, then, fixed $P \in \text{SPLS}(\Theta)$ such that $S(\Theta, P) = (j_\alpha, j_\beta, j_\gamma)$, it must be $j_\alpha \in \{0, 2\}$. By the other way, Θ verifies the hypothesis of Lemma 4.3, by considering $m = 2$ and $i_0 = 0$. So, it must be $j_\beta \in \text{Fix}(\beta^2) = \{0, 2\} = \text{Fix}(\beta)$. So, from Lemma 4.2, it must be $\beta(i) \in \text{Fix}(\gamma)$, for all $i \in \text{Fix}(\alpha) = \{3, 4\}$. However, $\beta(4) = 1 \notin \text{Fix}(\gamma)$. Therefore, it must be $\text{SPLS}(\Theta) = \emptyset$.

All the previous results can be used to obtain the set of partial Latin squares of $\text{SPLS}(\Theta)$ of the smallest size. In this way, we will say that $P \in \text{SPLS}(\Theta)$ is *minimal* if there does not exist $P' \in \text{SPLS}(\Theta)$ such that P' is a proper subset of P . We will denote by $\text{SPLS}^-(\Theta)$ the set of all these minimal partial Latin squares. The following result is the most remarkable one of this section:

Theorem 4.4. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{ST}_{n,k}$ be an isotopism with symmetrical structure $L_\Theta = (l_{i,j})$ and let us consider $P = (p_{i,j}) \in \text{SPLS}^-(\Theta)$. If $S(\Theta, P) = (j_\alpha, j_\beta, j_\gamma)$, let $r_\alpha, r_\beta \in \{0, 1, \dots, k-1\}$ be such that $j_\alpha \in C_{r_\alpha}^\beta$ and $j_\beta \in C_{r_\beta}^\beta$. Then:*

$$\#P = \begin{cases} n \cdot \lambda_{r_\alpha}, & \text{if there exists } m \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\} \text{ such that } \beta = \gamma^m \circ \alpha^{1-m}, \\ n \cdot (\lambda_{r_\alpha} + \lambda_{r_\beta}), & \text{otherwise.} \end{cases}$$

Proof. Let $r_\gamma \in \{0, 1, \dots, k-1\}$ be such that $j_\gamma \in C_{r_\gamma}^\beta$. As $P \in \text{SPLS}(\Theta)$, it must be $p_{i,j_\delta} = \delta(i) \neq \emptyset$, for all $i \in N$ and $\delta \in \{\alpha, \beta, \gamma\}$. Now, from Proposition 3.3,

it must be $p_{i,j} \neq \emptyset$, for all $i, j \in N$, such that there exist $u \in N$ and $v \in \{j_\alpha, j_\beta, j_\gamma\}$ verifying $l_{i,j} = l_{u,v}$. From Algorithm 3.2, it is equivalent to say that all the cells of the subsquares $R_P^{a,b} = \{(c_{a,u}^\alpha, c_{b,v}^\beta, p_{c_{a,u}^\alpha, c_{b,v}^\beta}) : u \in \{0, 1, \dots, \lambda_a - 1\}, v \in \{0, 1, \dots, \lambda_b - 1\}\}$, where $a \in \{0, 1, \dots, k-1\}$ and $b \in \{r_\alpha, r_\beta, r_\gamma\}$, are filled. Therefore, $p_{i,j} \neq \emptyset$, whenever $j \in C_{r_\alpha}^\beta \cup C_{r_\beta}^\beta \cup C_{r_\gamma}^\beta$. Besides, as P is minimal, these cells are the only ones being filled.

Now, let $h \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\}$ and $i_0 \in N$. Let $r_{i_0} \in \{0, 1, \dots, k-1\}$ be such that $i_0 \in C_{r_{i_0}}^\alpha$. By the other way, let $u \in \{0, 1, \dots, \lambda_{r_{i_0}} - 1\}$ and $v \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\}$ be such that $i_0 = c_{r_{i_0},u}^\alpha$ and $j_\alpha = c_{r_\alpha,v}^\beta$. Finally, let $s \in \{0, 1, \dots, k'-1\}$ be such that $\alpha^{1-h}(i_0) \in C_s^\gamma$ and let $w \in \{0, 1, \dots, \lambda'_s - 1\}$ be such that $\alpha^{1-h}(i_0) = c_{s,w}^\gamma$. Then, it is verified that $c_{s,w}^\gamma = \alpha^{1-h}(i_0) = \alpha(\alpha^{-h}(i_0)) = p_{\alpha^{-h}(i_0), j_\alpha} = p_{c_{r_{i_0},u-h \pmod{\lambda_{r_{i_0}}}}^\alpha, c_{r_\alpha,v}^\beta}$ and, therefore:

$$\gamma^h(\alpha^{1-h}(i_0)) = c_{s,w+h \pmod{\lambda'_s}}^\gamma = p_{c_{r_{i_0},u}, c_{r_\alpha,v+h \pmod{\lambda_{r_\alpha}}}^\beta} = p_{i_0, c_{r_\alpha,v+h \pmod{\lambda_{r_\alpha}}}^\beta}.$$

So, we have that $\gamma^h(\alpha^{1-h}(i)) = p_{i, c_{r_\alpha,v+h \pmod{\lambda_{r_\alpha}}}^\beta}$, for all $i \in N$ and $h \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\}$. Specifically, if $h = 1$, then $\gamma(i) = p_{i, c_{r_\alpha,v+1 \pmod{\lambda_{r_\alpha}}}^\beta}$, for all $i \in N$ and, therefore, $j_\gamma = c_{r_\alpha,v+1 \pmod{\lambda_{r_\alpha}}}^\beta$. So, $j_\gamma \in C_{r_\alpha}^\beta$ and $r_\gamma = r_\alpha$. By the other way, if there exists $m \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\}$ such that $\beta = \gamma^m \circ \alpha^{1-m}$, it must be $j_\beta = c_{r_\alpha,v+m \pmod{\lambda_{r_\alpha}}}^\beta$ and so, $r_\beta = r_\alpha$. Therefore, in this case, $\sharp P = n \cdot \lambda_{r_\alpha}$.

Otherwise, if there does not exist $m \in \{0, 1, \dots, \lambda_{r_\alpha} - 1\}$ such that $\beta = \gamma^m \circ \alpha^{1-m}$, then, it must be $r_\beta \neq r_\alpha$ and so, $\sharp P = n \cdot (\lambda_{r_\alpha} + \lambda_{r_\beta})$. $\square$

The following results are immediate consequences of the previous theorem:

Corollary 4.5. *Let $\Theta = (\alpha, \alpha, \alpha) \in \mathcal{I}_{n,k}$ an isomorphism. Then, $SPLS(\Theta) \neq \emptyset$. Moreover, $\sharp P = n$, for all $P \in SPLS^-(\Theta)$, and all the filled cells of P are in the same column. Besides, it is verified that $\sharp SPLS^-(\Theta) = \sharp Fix(\alpha)$.*

Proof. Let us observe that $\sharp Fix(\alpha) \geq 1$, because $\Theta \in \mathcal{I}_{n,k}$. Let $i_0 \in Fix(\alpha)$ and let us define the partial Latin square $P = (p_{i,j}) \in PLS(n)$, such that $p_{i,j} = \begin{cases} \alpha(j), & \text{if } j = i_0, \\ \emptyset, & \text{otherwise.} \end{cases}$, for all $i, j \in N$. Then, it is verified that $P \in SPLS^-(\Theta)$, being $S(\Theta, P) = (i_0, i_0, i_0)$. So, $SPLS(\Theta) \neq \emptyset$. Moreover, as i_0 has been arbitrarily taken from $Fix(\alpha)$, it must be $\sharp SPLS^-(\Theta) \geq \sharp Fix(\alpha)$.

Now, if we fixe any $Q = (q_{i,j}) \in SPLS^-(\Theta)$, then, from Lemma 4.2, $j_\alpha \in Fix(\alpha)$. Indeed, $S(\Theta, Q) = (j_\alpha, j_\alpha, j_\alpha)$. Let $r \in \{0, 1, \dots, k-1\}$ be such that $j_\alpha \in C_r^\alpha$. Keeping in mind that $j_\alpha \in Fix(\alpha)$, it must be $\lambda_r = 1$. So, from Theorem 4.4, it must be $\sharp Q = n$ and, then, all the filled cells are in the $(j_\alpha + 1)^{th}$ column of Q . Therefore, $\sharp SPLS(\Theta) = \sharp Fix(\alpha)$. $\square$

Corollary 4.6. *Let $\Theta = (\alpha, \beta, \epsilon) \in \mathcal{SI}_{n,k}$ a principal isotopism. If there exists $m \in \{1, 2, \dots, \frac{n}{k}\}$ such that $\beta = \alpha^m$, then $\sharp SPLS^-(\Theta) = n$ and $\sharp P = \frac{n^2}{k}$, for all*

$P \in SPLS^-(\Theta)$. Otherwise, $\sharp SPLS^-(\Theta) = n^2 \cdot (1 - \frac{1}{k})$ and $\sharp P = 2\frac{n^2}{k}$, for all $P \in SPLS^-(\Theta)$.

Proof. Let $P = (p_{i,j}) \in SPLS(\Theta)$ and $S(\Theta, P) = (j_\alpha, j_\beta, j_\epsilon)$. Let $r_\alpha \in \{0, 1, \dots, k-1\}$ be such that $j_\alpha \in C_{r_\alpha}^\beta$. From Proposition 1.2, it must be $\lambda_{r_\alpha} = \frac{n}{k}$. Now, if there exists $m_0 \in \{1, 2, \dots, \frac{n}{k}\}$ such that $\beta = \alpha^{m_0}$, then, $\beta = \epsilon^{1-m_0} \circ \alpha^{1-(1-m_0)}$. Therefore, from Theorem 4.4, it must be $\sharp P = n \cdot \frac{n}{k} = \frac{n^2}{k}$. By the other way, let us observe that there exist so many minimal partial Latin squares in $SPLS(\Theta)$ as different possible values of j_α . Specifically, fixed a cycle of β , it is verified that j_α can take $\frac{n}{k}$ different values. Therefore, as there exist k cycles in β , we obtain that $\sharp SPLS^-(\Theta) = n$.

Otherwise, let us suppose that there does not exist $m \in \{1, 2, \dots, \frac{n}{k}\}$ such that $\beta = \alpha^m$. Then, it must exist $r_\beta \in \{0, 1, \dots, k-1\} \setminus \{r_\alpha\}$ such that $j_\beta \in C_{r_\beta}^\beta$. So, j_β can take $n - \frac{n}{k}$ different values and, therefore, $\sharp SPLS^-(\Theta) = n \cdot (n - \frac{n}{k}) = n^2 \cdot (1 - \frac{1}{k})$. Besides, from Theorem 4.4, it must be $\sharp P = n \cdot (\frac{n}{k} + \frac{n}{k}) = 2\frac{n^2}{k}$. $\square$

Corollary 4.7. Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n,1}$. If there does not exist $m \in \{1, 2, \dots, n\}$ such that $\beta = \gamma^m \circ \alpha^{1-m}$, then $SPLS(\Theta) = \emptyset$.

Proof. Let us suppose that $SPLS(\Theta) \neq \emptyset$ and let $P \in SPLS(\Theta)$ be such that $S(\Theta, P) = (j_\alpha, j_\beta, j_\gamma)$. As $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n,1}$, it is verified that $\alpha = C_0^\alpha$ and $\beta = C_0^\beta$ are both an only cycle of length n . So, $P \in LS(n)$ and the $(j_\alpha + 1)^{th}$ column of P determines all the cells of this Latin square. Therefore, as there does not exist $m \in \{1, 2, \dots, n\}$ such that $\beta = \gamma^m \circ \alpha^{1-m}$, we know, from the proof of Theorem 4.4, that it is not possible to find $j \in \{0, 1, \dots, n-1\}$ such that $p_{i,j} = \beta(i)$ for all $i \in N$, which is a contradiction because $P \in SPLS(\Theta)$. So, it must be $SPLS(\Theta) = \emptyset$. $\square$

By keeping in mind Theorem 4.4, there are three classes of isotopisms in the set $\mathcal{I}_{n,k}$:

- I) Those isotopisms $\Theta \in \mathcal{I}_{n,k}$ such that $SPLS(\Theta) = \emptyset$. That is to say, those ones not being a Santilli's autotopism.
- II) Those isotopisms $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n,k}$ such that $SPLS(\Theta) \neq \emptyset$ and there exists $m \in \mathbb{N}$ verifying that $\beta = \gamma^m \circ \alpha^{1-m}$.
- III) Those isotopisms $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_{n,k}$ such that $SPLS(\Theta) \neq \emptyset$ and there does not exist $m \in \mathbb{N}$ verifying that $\beta = \gamma^m \circ \alpha^{1-m}$.

To finish our study about Santilli's autotopisms fixing at least one symbol, we give in Tables 3, 4 and 5 the classification and some examples of all the autotopisms of $\mathcal{I}_{n,k}$ when $2 \leq n \leq 5$. In this classification, we keep in mind the structure cycle of each autotopism and whether it is of type (I), (II) or (III), according to the above classes.

n	k	$\mathbf{I}$	$\mathbf{I}'$	Type	$\Theta \in \mathcal{I}_{n,k}$	$P \in SPLS^-(\Theta)$	$(j_\alpha, j_\beta, j_\gamma)$
2	1	(0, 1)	(2, 0)	<i>II</i>	$((01), (01), \epsilon)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	(1, 1, 0)
3	1	(0, 0, 1)	(3, 0, 0)	<i>II</i>	$((012), (012), \epsilon)$	$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$	(2, 2, 0)
	2	(1, 1, 0)	(1, 1, 0)	<i>II</i>	$((01), (01), (01))$	$\begin{pmatrix} * & * & 1 \\ * & * & 0 \\ * & * & 2 \end{pmatrix}$	(2, 2, 2)
				<i>III</i>	$((01), (02), (01))$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$	(1, 0, 1)

TABLE 3. Classification of isotopisms of order $n \in \{2, 3\}$ fixing at least one symbol.

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k	$\mathbf{l}$	$\mathbf{l}'$	Type	$\Theta \in \mathcal{I}_{n,k}$	$P \in SPLS^-(\Theta)$	$(j_\alpha, j_\beta, j_\gamma)$
1	$(0, 0, 0, 1)$	$(2, 1, 0, 0)$	I	$((0123), (0123), (01))$	$\emptyset$	$-$
			II	$((0123), (0123), (02))$	$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 0 & 3 & 2 & 1 \end{pmatrix}$	$(0, 0, 1)$
		$(4, 0, 0, 0)$	I	$((0123), (0132), \epsilon)$	$\emptyset$	$-$
			II	$((0123), (0123), \epsilon)$	$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{pmatrix}$	$(0, 0, 1)$
2	$(1, 0, 1, 0)$	$(1, 0, 1, 0)$	I	$((012), (013), (012))$	$\emptyset$	$-$
			II	$((012), (012), (012))$	$\begin{pmatrix} * & * & * & 1 \\ * & * & * & 2 \\ * & * & * & 0 \\ * & * & * & 3 \end{pmatrix}$	$(3, 3, 3)$
			III	$((012), (023), (012))$	$\begin{pmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix}$	$(1, 0, 1)$
	$(0, 2, 0, 0)$	$(2, 1, 0, 0)$	I	$((01)(23), (01)(23), (01))$	$\emptyset$	$-$
			II	$((01)(23), (01)(23), (02))$	$\begin{pmatrix} 1 & 2 & * & * \\ 0 & 1 & * & * \\ 3 & 0 & * & * \\ 2 & 3 & * & * \end{pmatrix}$	$(0, 0, 1)$
			III	$((01)(23), (03)(12), (02))$	$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 0 \\ 2 & 0 & 1 & 3 \end{pmatrix}$	$(0, 1, 3)$
		$(4, 0, 0, 0)$	II	$((01)(23), (01)(23), \epsilon)$	$\begin{pmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \\ 2 & 3 & * & * \\ 3 & 2 & * & * \end{pmatrix}$	$(1, 1, 0)$
			III	$((01)(23), (02)(13), \epsilon)$	$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix}$	$(2, 1, 0)$
3	$(2, 1, 0, 0)$	$(2, 1, 0, 0)$	I	$((01), (02), (01))$	$\emptyset$	$-$
			II	$((01), (01), (01))$	$\begin{pmatrix} * & * & * & 1 \\ * & * & * & 0 \\ * & * & * & 2 \\ * & * & * & 3 \end{pmatrix}$	$(3, 3, 3)$
			III	$((01), (23), (01))$	$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 2 & 3 & * & * \\ 3 & 2 & * & * \end{pmatrix}$	$(0, 1, 0)$

TABLE 4. Classification of isotopisms of order 4 fixing at least one symbol.

k	$\mathbf{l}$	$\mathbf{l}'$	Type	$\Theta \in \mathcal{I}_{n,k}$	$P \in SPLS^-(\Theta)$	$(j_\alpha, j_\beta, j_\gamma)$
1	(0,0,0,0,1)	(5,0,0,0,0)	I	$((01234),$ $(02143),$ $\epsilon)$	$\emptyset$	—
			II	$((01234),$ $(01234),$ $\epsilon)$	$\begin{pmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$	$(4, 4, 0)$
2	(1,0,0,1,0)	(1,0,0,1,0)	I	$((0123),$ $(0132),$ $(0123))$	$\emptyset$	—
			II	$((0123),$ $(0123),$ $(0123))$	$\begin{pmatrix} * & * & * & * & 1 \\ * & * & * & * & 2 \\ * & * & * & * & 3 \\ * & * & * & * & 0 \\ * & * & * & * & 4 \end{pmatrix}$	$(4, 4, 4)$
			III	$((0123),$ $(0241),$ $(0123))$	$\begin{pmatrix} 2 & 3 & 0 & 1 & 4 \\ 0 & 4 & 3 & 2 & 1 \\ 4 & 2 & 1 & 3 & 0 \\ 3 & 1 & 4 & 0 & 2 \\ 1 & 0 & 2 & 4 & 3 \end{pmatrix}$	$(3, 0, 3)$
3	(2,0,1,0,0)	(2,0,1,0,0)	I	$((012),$ $(034),$ $(012))$	$\emptyset$	—
			II	$((012),$ $(012),$ $(012))$	$\begin{pmatrix} * & * & * & 1 & * \\ * & * & * & 2 & * \\ * & * & * & 0 & * \\ * & * & * & 3 & * \\ * & * & * & 4 & * \end{pmatrix}$	$(3, 3, 3)$
	(1,2,0,0,0)	(1,2,0,0,0)	I	$((01)(23),$ $(01)(24),$ $(01)(23))$	$\emptyset$	—
			II	$((01)(23),$ $(01)(23),$ $(01)(23))$	$\begin{pmatrix} * & * & * & * & 1 \\ * & * & * & * & 0 \\ * & * & * & * & 3 \\ * & * & * & * & 2 \\ * & * & * & * & 4 \end{pmatrix}$	$(4, 4, 4)$
			III	$((01)(23),$ $(02)(14),$ $(01)(23))$	$\begin{pmatrix} 2 & * & 4 & * & 1 \\ 4 & * & 3 & * & 0 \\ 0 & * & 2 & * & 3 \\ 3 & * & 1 & * & 2 \\ 1 & * & 0 & * & 4 \end{pmatrix}$	$(4, 0, 4)$

TABLE 5. Classification of isotopisms of order 5 fixing at least one symbol.