ON STRONG CHAIN CLASSES OF HOMEOMORPHISMS OF COMPACT METRIC SPACES

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Abstract. The strong chain recurrent points and strong chain transitive sets of a homeomorphism f on a compact metric space X, first introduced by Easton [4]. He note that, if chain recurrent set of f is all of X, then strong chain recurrent set and strong chain transitive set of f are useful. Here, we obtain some results about strong chain recurrency. In particular, we show that an isolated strong chain class S of generic homeomorphism f has a generic continuation S_g, where g close to f, in C°-topology. Moreover, we have the generic persistence of Lipschitz ergodicity at f near S.

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1. Introduction

One of the main problems in dynamics is to describe the phase portrait or orbit structure of a system from a topological point of view. One of the most important topological view is recurrence behaviour. In fact, interesting dynamics is always associated with some sort of orbit recurrence. So, in trying to understand the dynamical behaviour one focuses on the recurrent part of the ambient space. Various notions of recurrence have been considered in dynamics such as recurrent points and non-wandering points. In 70^{th}, Conley and Bowen considered a weaker notion of recurrence. They introduced the notion of chains or pseudo orbits (see [2]

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and [3]). Moreover, the notion of chain recurrence introduced by Conley (see [3]) has remarkable connections to the structure of attractors. In fact, he considered a weaker version of indecomposability for each attractor, that was chain-transitivity.In 1977, Easton introduced another notion of recurrence which was stronger than chain recurrency, see [4]. He defined strong chains and strong chain-transitive sets. He obtained a relation between strong chain-transitivity and Lipschitz ergodicity. Also, he gave a specific example of a Anosov transformation f of the torus T which was strong chain-transitive on all of T. However, if the dynamical systems preserves a finite Borel measure, by a consequence of Poincare recurrence theorem, the chain recurrent set is all of X. In this case, it may be justified to consider those isolated invariant sets on which the dynamical system is strong chain transitive to be important. So, this is a motivation for studying the strong chain transitive sets. In this article, we consider the components of strong chain transitive sets which we call them strong chain class and show that an isolated strong chain class S of generic homeomorphisms f has a generic continuation, in C°-topology. Furthermore, this implies that the persistence of generic Lipschitz ergidicity near S.

2. Preliminaries and statement of results

Let X be a compact metric space with metric d and $\mathcal{H}(X)$ be the space of all homeomorphisms on X with the usual C°-topology. Recall that a subset is residual, if it contains a countable intersection of open and dense sets. We say that a property is generic, if it holds on a residual subset. Suppose that $\mathcal{F}X$ composed of all the nonempty closed subsets of X. Let A and B be in $\mathcal{F}X$. Put

$$d_H(A,B) = \inf\{\alpha > 0 : A \subset U_\alpha(B) \text{ and } B \subset U_\alpha(A)\},$$

where $U_\alpha(\cdot)$ denotes the α-ball measured in d. By [6], d_H is a complete metric on $\mathcal{F}X$, called Hausdorff metric. Let (X_i, d_i) be metric spaces, $i = 1, 2$, and let X_2 be compact. Suppose that $(\mathcal{F}X, d_H)$ is the set of closes subsets of X_2 equipped with Hausdorff metric. A mapping $h : X_1 \longrightarrow \mathcal{F}X_2$ is said to be lower semi-continuous at x, if for each $\alpha > 0$, there is a $\beta > 0$ such that $h(x) \subseteq U_\alpha(f(y))$, whenever $d_1(x,y) < \beta$. The map h is called lower semi-continuous on X_1, if it is lower semi-continuous at x, for all $x \in X_1$. For given $\epsilon > 0$, an ϵ-chain is a sequence of points $x_0, \ldots, x_n \in X$ such that $d(f(x_{k-1}), x_k) < \epsilon$, for $k = 1, \ldots, n$. An invariant set Λ is said to be chain-transitive, if for each $p, q \in \Lambda$ and $\epsilon > 0$, there exists an ϵ-chain $x_0, \ldots, x_n$ such that $x_0 = p$ and $x_n = q$. Define Λ to be strong chain transitive if given $p, q \in \Lambda$ and $\epsilon > 0$, there exists an ϵ-chain $x_0, \ldots, x_n$ from p to q such that $\sum_{k=1}^{n} d(f(x_{k-1}), x_k) < \epsilon$. Such a chain is called a ϵ-strong chain. If for each $\epsilon > 0$, there exists an ϵ-chain from x to itself, then x is said to be chain-recurrent. However, if there is an ϵ-strong chain from x to itself, then x is called a strong chain-recurrent point, we denote the sets of all chain-recurrent points and strong chain-recurrent points of f by $CR(f)$ and $SCR(f)$, respectively. Now, we define two binary relations which are equivalence relations and their equivalence classes are the main object of this article. For each $x, y \in CR(f)$, we say that $x \sim_C y$ if and only if for any $\epsilon > 0$, there is an ϵ-chain of f from x to y and conversely from y to x. Similarly, if $x, y \in SCR(f)$ then $x \sim_S y$ if and only if there exists an

ϵ-strong chain from x to y and from y to x, for each $\epsilon > 0$. Clearly $\sim_C$ and $\sim_S$ define equivalence relations on $CR(f)$ and $SCR(f)$, respectively. The equivalence classes of $\sim_C$ and $\sim_S$ are called chain recurrence classes and strong chain recurrence classes or chain classes and strong chain classes, briefly. In [5], the second author deduced that an isolated chain class S of f has a continuation S_g of chain classes, where g varies in a residual set close to f in C°-topology. Here, we show the same results for an isolated strong chain class. Moreover, we deduce the persistency of Lipschitz ergodic behaviour at S. In other words, we have the following results.

Theorem 2.1. *There exists a residual subset $\mathcal{R}$ of $\mathcal{H}(X)$ such that if $f \in \mathcal{R}$ and S is an isolated strong chain class of f, then there exists a neighborhood V of S such that and for sufficiently close $g \in \mathcal{R}$ to f,*

$$SCR(g) \cap V = S_g$$

is a strong chain class of g and furthermore $S_g \to S$ in Hausdorff topology as $g \to f$ in C°-topology.

Corollary 2.2. *If $f \in \mathcal{R}$ has an isolated strong chain class S then f exhibits a generic persistence Lipschitz ergodicity at S.*

3. Proof of results

We denote the set of points that are ends of ϵ-strong chain beginning at x by $SC_\epsilon(x, f)$ and define

$$SC(x, f) = \bigcap_{\epsilon > 0} SC_\epsilon(x, f).$$

If $0 < \delta < \epsilon$, then it is easy to verify that $\overline{SC_\delta(x, f)} \subset SC_\epsilon(x, f)$, from which it follows that $SC(x, f)$ is closed. Clearly all periodic points are strong chain recurrent. The strong chain recurrent set also contains all points that are Ω-recurrent (i.e., contained in their own omega limit sets) or non-wandering (a point p is non-wandering if each neighborhood of p contains a point x such that the forward orbit of $f(x)$ intersects the neighborhood). In fact, if we denote the non-wandering set by $\Omega(f)$ then we have

$$\Omega(f) \subseteq SCR(f) \subset CR(f).$$

Note that there is an obvious transitivity. We see that if $y \in SC(x, f)$ and $z \in SC(y, f)$, then $z \in SC(x, f)$. Also, by continuity of f, if p is a strong chain recurrent point, then $p \in SC(f(p), f)$. Since $f(p) \in SC(p, f)$, we see that

$$(*) \qquad p \in SCR(f) \Leftrightarrow p \in SC(f(p), f) \Leftrightarrow f(SC(p, f)) = SC(p, f).$$

We will also speak of $SC(f)$-invariant sets, such a set is defined by the property that it contains $SC(x, f)$ for every x in the set. We define the strong chain omega limit set of a point x to be the set of all points $y \in X$ with the property that for any $\epsilon > 0$ and any integer n, there is an ϵ-strong chain of length at least n that begins in x and ends at y. This set is denoted by $\omega SC(x, f)$.

Lemma 3.1. *$\omega SC(x, f)$ is the largest invariant subset of $SC(x, f)$.*

Proof. Since we can follow an ϵ-strong chain from x to p by one of arbitrary length from p to p, so we have

$$p \in SCR(f) \cap SC(x, f) \Rightarrow p \in \omega SC(x, f).$$

By continuity of f, $SC(x, f) \setminus \omega SC(x, f) \subset O(x, f)$, i.e., $SC(x, f) = O(x, f) \cup \omega SC(x, f)$. Clearly $\omega SC(x, f)$ is forward invariant, and the previous equation shows that there is no larger invariant subset of $SC(x, f)$, so we only need to verify that it is invariant. To do this, suppose $y \in \omega SC(x, f)$. For each $n \geq 1$, let ξ_n be a finite $\frac{1}{n}$-strong chain of length at least n that begins at x and ends at y. Let $z(n)$ be the next to last point in ξ_n. If z is an accumulation point of the sequence $z(n)$ then it is easy to see that $f(z) = y$ and that $z \in \omega SC(x, f)$. □

Proposition 3.2. *Let $f \in \mathcal{H}(X)$. If S is a strong chain component for f, then S is a closed and f-invariant subset of X. In particular, if $x \in S$ then $\omega(x, f) \subset S$.*

Proof. It is easy to see that S is closed, so it is sufficient to show that $f(S) = S$. Fix a point $z \in S$. By its definition, S is the set of all points x such that $x \in SC(z, f)$ and $z \in SC(x, f)$. Since every point of S is strong chain recurrent, it follows from this characterization and $(*)$ that $f(S) \subseteq S$. The fact that $f(S) = S$ follows from the same argument that established in Lemma 3.1. □

Lemma 3.3. *The mapping $f \mapsto CR(f)$is lower semi-continuous.*

Proof. It is sufficient to show that, if $g_m \to f$ and $x_m \in CR(g_m)$, $m \in \mathbb{N}$, convergent to x, then $x \in CR(f)$. For, let $\epsilon > 0$ be given. Since $x_m \in CR(g_m)$, so there is an $\frac{1}{m}$-chain of g_m from x_m to x_m, say ξ_m. By compactness of ξ_m's, there is a subsequence of $(\xi_m)_{m\in\mathbb{N}}$ which is convergent to a subset ξ of X, in Hausdorff topology. By uniform continuity of f, we can choose $0 < \delta < \frac{\epsilon}{4}$ such that

$$d(x, y) < \delta \text{ implies that } d(f(x), f(y)) < \frac{\epsilon}{8}. \tag{1}$$

Also, let $m \in \mathbb{N}$ be large enough such that

$$\frac{1}{m} < \frac{\epsilon}{8} \quad , \quad d(g_m, f) < \frac{\epsilon}{8} \quad \text{and} \quad d_H(\xi_m, \xi) < \delta. \tag{2}$$

Suppose that $\xi_m = \{p_1^m, \ldots, p_{j_m}^m\}$. Choose $x_1, \ldots, x_{j_m} \in \xi$ such that $d(x_i, p_i^m) < \delta$. Since $p_1^m = p_{j_m}^m = x_m$ and $x_m \to x$, then we can choose $x_1 = x_{j_m} = x$. Now, by (1) and (2) we have

$$\begin{aligned} d(f(x_i), x_{i+1}) &< d(f(x_i), f(p_i^m)) + d(f(p_i^m), g_m(p_i^m)) + \\ &\quad d(g_m(p_i^m), p_{i+1}^m) + d(p_{i+1}^m, x_{i+1}) \\ &< \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{1}{m} + \delta < \epsilon, \end{aligned}$$

i.e. $\{x_1, \ldots, x_{j_m}\}$ is a ϵ-chain from x to x and this implies that $x \in CR(f)$. □

In the sequel we need the following topological lemma. For the proof, see [6].

Topological Lemma I. *If X and Y are metric spaces with Y compact, and if $f : X \longrightarrow \mathcal{F}Y$ is either upper or lower semi-continuous then the set of continuity* *points of f is residual subset of X.*

By Topological Lemma I, we have the following result.

Corollary 3.4. *There exists a residual subset R, of $\mathcal{H}(X)$ such that the mapping $f \mapsto CR(f)$ varies continuously on R.*

Also we need the following result. For the proof see [1].

Proposition 3.5. *There exists a residual subset R' of $\mathcal{H}(X)$ such that for each $f \in R'$, we have $\Omega(f) = CR(f)$.*

Already, we see that $\Omega(f) \subseteq SCR(f) \subseteq CR(f)$. We use this fact, Corollary 3.4 and Proposition 3.5 to obtain the following results.

Proposition 3.6. *There exists a residual set R_2 of $\mathcal{H}(X)$ such that the mapping $f \mapsto SCR(f)$ varies continuously on R_2.*

Lemma 3.7. *Let U be any open set in X. Then there exists a residual subset R_U of $\mathcal{H}(X)$ such that if $f \in R_U$ for which finitely many of its strong chain classes are contained in U, then there exists an open subset $\mathcal{U}$ of R_U contains f, such that for each $g \in \mathcal{U}$, the number of the strong chain classes of g contained in U is also finite.*

Proof. First, we note that if $f \in R_2$ has only finitely strong chain classes contained in U, then the number of chain classes of f in U is also finite. Let $C_1, \ldots, C_k$ be distinct strong chain classes of f in U. Suppose that $\delta > 0$ be small enough such that

$$U_\delta(C_i) \cap U_\delta(C_j) = \varnothing \ , \ i \neq j \ , \ 1 \leq i, j \leq k.$$

Put $V = \cup_{i=1}^{k} U_\delta(C_i)$. Since the mapping $f \mapsto SCR(f)$ is continuous on R_2, so for g sufficiently close to f, the number of strong chain classes of g contained in $V \subset U$ is at least k. Therefore, there exists a neighborhood $\mathcal{N}$ of f in R_2 such that each $g \in \mathcal{N}$ has at least k distinct strong chain classes in U. Now, put $R_2 = S_{fin} \cup S_\infty$ in which S_{fin} in the set of all $f \in R_2$ with finite distinct strong chain classes contained in U and $S_\infty = R_2 \setminus S_{fin}$. Put $R_U = R_2 \setminus (cl(S_\infty) \cap S_{fin})$. We show that R_U is residual in R_2. Let A_n be the set of all $f \in R_2$ which has at most n distinct strong chain classes in U. By the above statement A_n is closed in R_2. Also $S_{fin} = \cup_{n \in \mathbb{N}} A_n$. Therefore,

$$\begin{aligned} cl(S_\infty) \cap S_{fin} &= cl(S_\infty) \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right) \\ &= \bigcup_{n \in \mathbb{N}} (cl(S_\infty) \cap A_n), \end{aligned}$$

where $cl(S_\infty) \cap A_n$ is closed in R_2. Now, we show that the interior of $cl(S_\infty) \cap A_n$ is empty, for each $n \in \mathbb{N}$. By contradiction, let there exists $n \in \mathbb{N}$, such that $cl(S_\infty) \cap A_n$, has nonempty interior. So there exists an open set $\mathcal{V}$ of R_2 such that $\mathcal{V} \subseteq cl(S_\infty) \cap A_n$. Since $\mathcal{V} \subseteq cl(S_\infty)$, so S_∞ is dense in $\mathcal{V}$ which is contradiction with $\mathcal{V} \subseteq A_n \subset S_{fin}$. Therefore, R_U is residual in $\mathcal{H}(X)$. Now, if $f \in R_U$ with finite distinct strong chain classes in U, then f does not belongs to $cl(S_\infty)$ and so

f belongs to interior S_{fin}. This implies that, there exists an open set $\mathcal{W}_f$ in R_2 such that $\mathcal{W}_f \subset S_{fin}$. Clearly $g \in \mathcal{W}_f$ has only finitely many strong chain classes in U. □

In the sequel, we use the following Topological Lemma, for the proof see [6].

TOPOLOGICAL LEMMA II. *Let X be a Bair topological space and $\Gamma : X \longrightarrow \mathbb{N}$ be a lower semi-continuous map, then there exists a residual set $\mathcal{N}$ of X such that $\Gamma|_{\mathcal{N}}$ is locally constant.*

Let $\Gamma_U(f)$ be the number of distinct strong chain classes of f which are contained in U, if this number is finite. Then we have the following result.

Lemma 3.8. *Let U be an open set in X. Then, there exists a residual subset T_U of $\mathcal{H}(X)$ such that if $f \in T_U$ and $\Gamma_U(f)$ is finite then Γ_U is constant in a neighborhood of f in T_U.*

Proof. Let R_U be the residual subset of $\mathcal{H}(X)$ given by previous lemma. As before, we put

$$R_U = R_{fin} \cup R_\infty$$

where R_{fin} is the set of all $f \in R_U$ with only finitely many strong chain classes of f are contained in U and $R_\infty = R_U \setminus R_{fin}$. By the previous lemma R_{fin} is open in R_U and the mapping $f \mapsto \Gamma_U(f)$ is lower semi-continuous on it and therefore by Topological Lemmas, there exists a residual subset $R'_{fin} \subset R_{fin}$ such that Γ_U is locally constant on it and also the mapping $f \mapsto SCR(f)$ is continuous on it. Now put $T_u = R'_{fin} \cup R_\infty$. Then T_U satisfies the statement of the lemma. □

Now, we prove the main results.

PROOF OF THEOREM 2.1. Let $\{U_n\}$ be the set of all finite union of elements of some open base of X. Put

$$\mathcal{R} = \cap_{n \in \mathbb{N}} \mathcal{R}_{U_n}$$

Let $f \in \mathcal{R}$ and S is an isolated strong chain class of f. Choose $V \in \{U_n\}$ such that $V \cap SCR(f) = S$. Now $\Gamma_V(f) = 1$ and therefore for sufficiently near $g \in \mathcal{R}$, $\Gamma_V(g) = 1$. Thus $SCR(g) \cap V = S_g$ is only strong chain class of g which is contained in V. Clearly $S_g \to S$, by the choosing of $\mathcal{R}$. □

Let $f \in \mathcal{H}(X)$. We say that f is Lipschitz ergodic if any Lipschitz function $G : X \longrightarrow R$ such that $G(f(x)) = G(x)$, for each $x \in X$, is constant. Now, we use a result of Easton (see [4]).

PROPOSITION A. *Suppose that Λ is a closed invariant set on which f is strong chain transitive. Then f restricted to Λ is Lipschitz ergodic.*

Since each strong chain class is a closed invariant strong chain transitive set, then f restricted to each strong chain class is Lipschitz ergodic. If we use it and Theorem 2.1, we obtain the following result.

Corollary 2.2. *If $f \in \mathcal{R}$ has an isolated strong chain class S then f exhibits a generic persistence of Lipschitz ergoidicity at S.*

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