

NONLINEAR STABILITY IN GENERALIZED PHOTOGRAVITATIONAL RESTRICTED THREE BODY PROBLEM

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ABSTRACT. We have discussed the nonlinear stability of triangular equilibrium points in generalized photogravitational restricted three body problem. The problem is generalized in the sense that both primaries are taken as oblate spheroid. We have performed first and second normalization of the Hamiltonian of the problem. We have applied Arnold's theorem to examine the condition of non-linear stability. We have found three critical mass ratios where this theorem fails. The stability condition is different from the classical case due to radiation and oblateness of the primaries.

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1. INTRODUCTION

The nonlinear stability of an equilibrium point can be investigated by means of KAM theorem [the work of Kolmogorov (1975) extended by Arnold (1961), Moser (1962). Deprit and Deprit (1967) investigated the nonlinear stability of triangular points by applying Moser's modified version of Arnold's theorem (1961). Bhatnagar and Hallan (1983) studied the effect of perturbations on the nonlinear stability of triangular points. They normalized the Hamiltonian through the fourth order terms. But Robutel (1995) studied the stability of the planetary three body problem by expanding the Hamiltonian up to order six in Poincare variables. Niedzielska (1994) investigated the nonlinear stability of libration points in the photogravitational

restricted three body problem. Shevchenko and Sokolsky (1995) achieved the sixth order of normalization with the help of computer in the study of hyperboloidal precession of dynamically symmetric satellite. Ishwar B (1997) studied nonlinear stability in the generalized restricted three body problem. His problem is generalized in the sense that the infinitesimal body and one of the primaries have been taken as oblate spheroid. Subba Rao, P. V. and Sharma, R. K. (1997) examined effect of oblateness on the non-linear stability of L_4 in the restricted three body problem. Markellos, V.V. *et al.* (1996), Papadakis (1999) and Thakur, A.P. and Singh R.B. (1997) examined the stability of triangular libration points in circular RTBP. Hence we aim to study nonlinear stability of triangular points in the generalized photogravitational restricted three body problem. We have taken both primaries as source of radiation and oblate spheroid as well.

2. FIRST ORDER NORMALIZATION

Using dimensionless variables and synodic coordinate system (x, y) as Szebehely (1967) and Singh and Ishwar (1999) the equations of motion can be written as:

$$(1) \quad \ddot{x} - 2n\dot{y} = U_x \quad \ddot{y} + 2n\dot{x} = U_y$$

$$(2) \quad U = \frac{n^2(x^2 + y^2)}{2} + \frac{(1 - \mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1 - \mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3}$$

where $r_1^2 = (x - \mu)^2 + y^2$, $r_2^2 = (x + 1 - \mu)^2 + y^2$, $\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}$, q_1, q_2 are mass reduction factor due to radiation, A_1, A_2 are oblateness coefficients. The perturbed mean motion of the primaries, n is given by $n^2 = 1 + \frac{3}{2}(A_1 + A_2)$. The coordinates of triangular point L_4 obtained by Singh and Ishwar (1999) are

$$(3) \quad x_0 = -\frac{1}{2}\left[\gamma + \frac{2}{3}(q_1 - q_2) + A_1 q_1 - A_2 q_2\right]$$

$$(4) \quad y_0 = \frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}(A_1 + A_2) + \frac{1}{3}(A_1 q_1 + A_2 q_2) - \frac{2}{9}(1 - q_1) - \frac{2}{9}(1 - q_2)\right]$$

with $\gamma = 1 - 2\mu$. The subscript 0 denotes evaluation at L_4

$$(5) \quad L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) + \frac{(1 - \mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1 - \mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3}$$

with conditions $q_1 = 1 - \epsilon, q_2 = 1 - \epsilon', |\epsilon| \ll 1, |\epsilon'| \ll 1$. Shifting the origin to the triangular point L_4 and expanding in power series of x and y , we have

$$(6) \quad L_2 = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) - Ex^2 - Fy^2 - Gxy$$

$$(7) \quad L_3 = \frac{T_1 x^3}{3} + \frac{1}{2}T_2 x^2 y + T_3 x y^2 + \frac{T_4 y^3}{3}$$

$$(8) \quad L_4 = N_1 x^4 + N_2 x^3 y + N_3 x^2 y^2 + N_4 x y^3 + N_5 y^4$$

In order to examine the stability of motion, as in Whittaker(1965), the following set of linear equations in the variables x, y can be written as:

$$(9) \quad \begin{aligned} -\lambda P_x &= \frac{\partial H_2}{\partial x} & \lambda x &= \frac{\partial H_2}{\partial P_x} \\ -\lambda P_y &= \frac{\partial H_2}{\partial y} & \lambda y &= \frac{\partial H_2}{\partial P_y} \\ \text{i.e. } AX &= 0 \end{aligned}$$

$$(10) \quad X = \begin{bmatrix} x \\ y \\ P_x \\ P_y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix}$$

Clearly $|A| = 0$ implies that

$$(11) \quad \lambda^4 + 2(E + F + n^2)\lambda^2 + 4EF - G^2 + n^4 - 2n^2(E + F) = 0$$

This is characteristic equation whose discriminant is

$$D = 4(E + F + n^2)^2 - 4\{4EF - G^2 + n^4 - 2n^2(E + F)\}$$

Stability is possible only when $D > 0$. This implies that [Singh and Ishwar(1999)]

$$(12) \quad \begin{aligned} \mu < \mu_{c_0} = 0.03852089 \cdots - 0.28500178 \cdots A_1 - 0.00891747 \cdots \epsilon \\ - 0.0627795 \cdots A_2 - 0.00891747 \cdots \epsilon' \end{aligned}$$

When $D > 0$, the roots $\pm i\omega_1$ and $\mp i\omega_2$ (ω_1, ω_2 being the long/short -periodic frequencies) are related to each other as

$$(13) \quad \begin{aligned} \omega_1^2 + \omega_2^2 &= \frac{1}{16} \left[16 + 8\epsilon + 8\epsilon' + \gamma(8\epsilon - 8\epsilon' + 24A_2 - 24A_1) \right] \\ 0 < \omega_2 &< \frac{1}{\sqrt{2}} < \omega_1 < 1 \end{aligned}$$

$$(14) \quad \begin{aligned} \omega_1^2 \omega_2^2 &= \frac{1}{16} \left[27 + 117A_1 + 117A_2 - 3\epsilon - 3\epsilon' + \gamma(24\epsilon - 24\epsilon') \right. \\ &\quad \left. - \gamma^2(27 + 117A_1 + 117A_2 - 27\epsilon - 27\epsilon') \right] \end{aligned}$$

which give

$$(15) \quad \begin{aligned} \lambda^2 &= \frac{16\omega_1^2 - 16\omega_2^2 + 27}{27} + \\ &\quad \left\{ \frac{208\omega_1^2(1 - \omega_1^2)}{81} + \frac{8\gamma\omega_1^2}{9} \right\} A_1 + \left\{ \frac{208\omega_1^2(1 - \omega_1^2)}{81} - \frac{8\gamma\omega_1^2}{9} \right\} A_2 \\ &\quad + \left\{ \frac{8(2\omega_1^4 - 3\omega_1^2 + 3)}{27} - \frac{8\gamma(\omega_1^2 - 3)}{27} \right\} \epsilon + \left\{ \frac{8(2\omega_1^4 - 3\omega_1^2 + 3)}{27} + \frac{8\gamma(\omega_1^2 - 3)}{27} \right\} \epsilon' \end{aligned}$$

Results given by (13) & (15) are different from others so obtained. But in the case $A_2 = \epsilon = \epsilon' = 0$, they agree with those found by Subba Rao and Sharma (1997).

For reducing H_2 to the normal form, we solve (9) and then, as in Whittaker (1965), use the transformation

$$(16) \quad X = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix}$$

where the elements of the dyadic J can be obtained by adopting the method of Breakwell and Pringle(1996) under the normality conditions. The transformations (16) transform H_2 to the form

$$(17) \quad H_2 = \frac{(P_1^2 - P_2^2 + \omega_1^2 Q_1^2 - \omega_2^2 Q_2^2)}{2}$$

Now, we apply a contact transformation defined by Whittaker(1965) as:

$$(18) \quad P_j = (2I_j\omega_j)^{1/2} \cos \phi_j, \quad Q_j = \left(\frac{2I_j}{\omega_j}\right)^{1/2} \sin \phi_j, \quad (j = 1, 2)$$

where ϕ_1, ϕ_2 and I_1, I_2 are the angle co-ordinates and the action momenta respectively. The second order part of Hamiltonian (17) becomes $H_2 = \omega_1 I_1 = -\omega_2 I_2$.

3. SECOND ORDER NORMALIZATION

For transforming the Hamiltonian H to Birkhoff's normal form, we have utilized Henrard's method [Deprit and Deprit Bartholomé (1967)]. We have expanded the coordinates (x, y) of the infinitesimal body in double d'Alembert series:

$$(19) \quad x = \sum_{j \geq 1} B_j^{1,0}(\phi_1, \phi_2, I_1, I_2), \quad y = \sum_{j \geq 1} B_j^{0,1}(\phi_1, \phi_2, I_1, I_2)$$

where the homogeneous components $B_j^{1,0}$ and $B_j^{0,1}$ are of degree j , I_1, I_2 are to be taken as constants of integration, while ϕ_1 and ϕ_2 are linear functions of time.

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y obtained by (16).That is

$$(20) \quad B_1^{1,0} = J_{13}\sqrt{2\omega_1 I_1} \cos \phi_1 + J_{14}\sqrt{2\omega_2 I_2} \cos \phi_2$$

$$(21) \quad B_1^{0,1} = J_{21}\sqrt{\frac{2I_1}{\omega_1}} \sin \phi_1 + J_{22}\sqrt{\frac{2I_2}{\omega_2}} \sin \phi_2 + J_{23}\sqrt{2I_1\omega_1} \cos \phi_1 + J_{24}\sqrt{2I_2\omega_2} \cos \phi_2$$

To find the other components $B_2^{1,0}, B_2^{0,1}, B_3^{1,0}, B_3^{0,1}$, etc. we have considered the Lagrange's equation of motion. Consequently we have

$$(22) \quad \left. \begin{aligned} \ddot{x} - 2n\dot{y} + (2E - n^2)x + Gy &= \frac{\partial L_3}{\partial x} + \frac{\partial L_4}{\partial x} \\ \ddot{y} + 2n\dot{x} + (2F - n^2)y + Gx &= \frac{\partial L_3}{\partial y} + \frac{\partial L_4}{\partial y} \end{aligned} \right\}$$

Proceeding as in Deprit and Deprit Bartholomé (1967), we have obtained the second order components $B_2^{1,0}, B_2^{0,1}$ as solutions of the partial differential equations

$$(23) \quad \begin{aligned} \Delta_1 \Delta_2 B_2^{1,0} &= \Phi_2, \quad \Delta_1 \Delta_2 B_2^{0,1} = -\Psi_2 \quad \text{with} \\ \Delta_i &= D^2 + \omega_i^2, (i = 1, 2), D = \omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2} \end{aligned}$$

$$(24) \quad \Phi_2 = (D^2 + S_1)X_2 + (S_2D + S_3)Y_2$$

$$(25) \quad \Psi_2 = (S_2D - S_3)X_2 + (D^2 - S_4)Y_2$$

$$S_1 = 2F - n^2, S_2 = 2[1 + \frac{3}{4}(A_1 + A_2)], S_3 = -G, S_4 = n^2 - 2E$$

$$(26) \quad X_2 = \frac{\partial L_3}{\partial x}, \quad \text{evaluated for } x = B_1^{1,0}, y = B_1^{0,1}$$

$$(27) \quad Y_2 = \frac{\partial L_3}{\partial y} \quad \text{evaluated for } x = B_1^{1,0}, y = B_1^{0,1}$$

From (6),(7), (8),(26) and (27) we have

$$(28) \quad X_2 = (B_1^{1,0})^2 T_1 + B_1^{1,0} B_1^{0,1} T_2 + (B_1^{0,1})^2 T_3$$

$$(29) \quad Y_2 = (B_1^{0,1})^2 T_4 + 2B_1^{1,0} B_1^{0,1} T_3 + \frac{1}{2}(B_1^{1,0})^2 T_2$$

Substituting the values of X_2, Y_2 from (28),(29) in(24) and (25) we have

$$(30) \quad \begin{aligned} \Phi_2 &= (\bar{A}_1 D^2 + A_2 D + A_3)(B_1^{1,0})^2 + \\ & (B_1 D^2 + B_2 D + B_3)(B_1^{0,1})^2 + (C_1 D^2 + C_2 D + C_3)B_1^{1,0} B_1^{0,1} \end{aligned}$$

$$(31) \quad \begin{aligned} \Psi_2 &= (\bar{A}'_1 D^2 + A'_2 D + A'_3)(B_1^{1,0})^2 + (B'_1 D^2 + B'_2 D + B'_3)(B_1^{0,1})^2 + \\ & (C'_1 D^2 + C'_2 D + C'_3)B_1^{1,0} B_1^{0,1} \end{aligned}$$

$$(32) \quad \begin{aligned} \text{Now,} \quad D \cos(m\phi_1 + n\phi_2) &= (n\omega_2 - m\omega_1) \sin(m\phi_1 + n\phi_2) \\ D \sin(m\phi_1 + n\phi_2) &= (m\omega_1 - n\omega_2) \cos(m\phi_1 + n\phi_2) \\ D^2 \cos(m\phi_1 + n\phi_2) &= -(m\omega_1 - n\omega_2)^2 \cos(m\phi_1 + n\phi_2) \\ D^2 \sin(m\phi_1 + n\phi_2) &= -(m\omega_1 - n\omega_2)^2 \sin(m\phi_1 + n\phi_2) \end{aligned}$$

By the condition(i) of KAM theorem each of $\omega_1, \omega_2, \omega_1 \pm 2\omega_2, \omega_2 \pm 2\omega_1$ is different from zero and consequently none of the divisors $\Delta_{0,0}, \Delta_{0,2}, \Delta_{2,0}, \Delta_{1,1}, \Delta_{1,-1}$, is zero, where

$$(33) \quad \Delta_{m,n} = [\omega_1^2 - (\omega_1 m - \omega_2 n)^2] [\omega_2^2 - (\omega_1 m - \omega_2 n)^2]$$

From (23)

$$(34) \quad B_2^{1,0} = \frac{\Phi_2}{\Delta_1 \Delta_2}, \quad B_2^{0,1} = -\frac{\Psi_2}{\Delta_1 \Delta_2}$$

Making use of (20), (21) in (31), (32) and then using (33), we get Φ_2, Ψ_2 . Further utilizing (33) in (34), we get

$$(35) B_2^{1,0} = r_1 I_1 + r_2 I_2 + r_3 I_1 \cos 2\phi_1 + r_4 I_2 \cos 2\phi_2 + r_5 I_1 \sin 2\phi_1 \\ + r_6 I_2 \sin 2\phi_2 + r_7 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + r_8 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ + r_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) + r_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2)$$

and

$$(36) B_2^{0,1} = s_1 I_1 + s_2 I_2 + s_3 I_1 \cos 2\phi_1 + s_4 I_2 \cos 2\phi_2 + s_5 I_1 \sin 2\phi_1 \\ + s_6 I_2 \sin 2\phi_2 + s_7 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + s_8 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ + s_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) + s_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2)$$

Further, as Bhatnagar and Hallan (1983), we have proved that by the transformation $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ the third order part H_3 of the Hamiltonian in $I_1^{1/2} I_2^{1/2}$ is zero.

4. SECOND ORDER COEFFICIENTS IN THE FREQUENCIES

In order to find the second order coefficients $f_{2,0}, f_{0,2}, g_{2,0}, g_{0,2}$ in the polynomials f_2 and g_2 we have done as in Deprit and Deprit (1967). Proceeding as (23), we find

$$(37) \quad \triangle_1 \triangle_2 B_3^{1,0} = \Phi_3 - 2f_2 P - 2g_2 Q, \quad \triangle_1 \triangle_2 B_3^{0,1} = \Psi_3 - 2f_2 U - 2g_2 V$$

$$(38) \quad \text{where, } \Phi_3 = \left[D^2 - \left\{ 1 + \frac{3}{2}(A_1 + A_2) - F \right\} \right] X_3 + \\ \left[2 \left\{ 1 + \frac{3}{4}(A_1 + A_2) \right\} D - G \right] Y_3$$

$$(39) \quad \Psi_3 = \left[D^2 - \left\{ 1 + \frac{3}{2}(A_1 + A_2) - 2E \right\} \right] Y_3 - \\ \left[2 \left\{ 1 + \frac{3}{4}(A_1 + A_2) D \right\} + G \right] X_3$$

$$(40) \quad P = \frac{\partial}{\partial \phi_1} \left[\left\{ \omega_1^2 \frac{\partial^2}{\partial \phi_1^2} - 1 - \right. \right. \\ \left. \left. \frac{3}{2}(A_1 + A_2) + 2F \right\} \left\{ \omega_1 \frac{\partial B_1^{1,0}}{\partial \phi_1} - \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \right. \\ \left. + \left\{ 2 \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) \omega_1 \frac{\partial}{\partial \phi_1} - G \right\} \left\{ \omega_1 \frac{\partial B_1^{0,1}}{\partial \phi_1} + \left(1 + \frac{3}{2}A_1 + \frac{3}{2}A_2 \right) B_1^{0,1} \right\} \right]$$

$$\begin{aligned}
 (41) \quad Q = & \frac{\partial}{\partial \phi_2} \left[\left\{ \omega_2^2 \frac{\partial^2}{\partial \phi_2^2} - 1 - \right. \right. \\
 & \left. \frac{3}{2}(A_1 + A_2) + 2F \right\} \left\{ -\omega_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} - \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \\
 & \left. + \left\{ -2 \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) \omega_2 \frac{\partial}{\partial \phi_2} - G \right\} \left\{ -\omega_2 \frac{\partial B_1^{0,1}}{\partial \phi_2} + \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 (42) \quad U = & \frac{\partial}{\partial \phi_1} \left[\left\{ \omega_1^2 \frac{\partial^2}{\partial \phi_1^2} - 1 - \right. \right. \\
 & \left. \frac{3}{2}(A_1 + A_2) + 2E \right\} \left\{ \omega_1 \frac{\partial B_1^{0,1}}{\partial \phi_1} + \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{1,0} \right\} \\
 & \left. - \left\{ 2 \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) \omega_1 \frac{\partial}{\partial \phi_1} + G \right\} \left\{ \omega_1 \frac{\partial B_1^{0,1}}{\partial \phi_1} - \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 (43) \quad V = & \frac{\partial}{\partial \phi_2} \left[\left\{ \omega_2^2 \frac{\partial^2}{\partial \phi_2^2} - 1 - \right. \right. \\
 & \left. \frac{3}{2}(A_1 + A_2) + 2E \right\} \left\{ -\omega_2 \frac{\partial B_1^{0,1}}{\partial \phi_2} + \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \\
 & \left. + \left\{ -2 \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) \omega_2 \frac{\partial}{\partial \phi_2} + G \right\} \left\{ -\omega_2 \frac{\partial B_1^{0,1}}{\partial \phi_2} + \left(1 + \frac{3}{4}A_1 + \frac{3}{4}A_2 \right) B_1^{0,1} \right\} \right]
 \end{aligned}$$

$$(44) \quad X_3 = \frac{\partial}{\partial x} (L_3 + L_4) = l'_1 x^2 + l'_2 xy + l'_3 y^2 + l'_4 x^3 + l'_5 x^2 y + l'_6 xy^2 + l'_7 y^3$$

$$\begin{aligned}
 (45) \quad Y_3 = & \frac{\partial}{\partial x} (L_3 + L_4) = m'_1 x^2 + m'_2 xy + m'_3 y^2 + m'_4 x^3 + \\
 & m'_5 x^2 y + m'_6 xy^2 + m'_7 y^3
 \end{aligned}$$

with

$$l'_1 = T_1, l'_2 = T_2, l'_3 = T_3, l'_4 = 4N_1, l'_5 = 3N_2, l'_6 = 2N_3, l'_7 = N_4$$

$$m'_1 = \frac{T_1}{2}, m'_2 = 2T_3, m'_3 = T_4, m'_4 = N_2, m'_5 = 2N_3, m'_6 = 3N_4, m'_7 = 4N_5$$

These are the partial differential equations which are satisfied by the third order components $B_3^{1,0}, B_3^{0,1}$ and the second order polynomials f_2, g_2 in the frequencies. We do not require to find out the components $B_3^{1,0}$ and $B_3^{0,1}$. We find the coefficients of $\cos \phi_1, \sin \phi_1, \cos \phi_2$ and $\sin \phi_2$ in the right hand sides of (37). They are the critical terms, since $\Delta_{1,0}, \Delta_{0,1} = 0$. We eliminate these terms by choosing properly the coefficients in the polynomials

$$(46) \quad f_2 = f_{2,0}I_1 + f_{0,2}I_2, \quad g_2 = g_{2,0}I_1 + g_{0,2}I_2$$

Further, we find that

$$(47) \quad f_{2,0} = \frac{(\text{coefficient of } \cos \phi_1 \text{ in } \Phi_3)}{2(\text{coefficient of } \cos \phi_1 \text{ in } P)} = A$$

$$(48) \quad f_{0,2} = g_{2,0} = \frac{(\text{coefficient of } \cos \phi_1 \text{ in } \Phi_3)}{2(\text{coefficient of } \cos \phi_2 \text{ in } Q)} = B$$

$$(49) \quad g_{0,2} = \frac{(\text{coefficient of } \cos \phi_2 \text{ in } \Phi_3)}{2(\text{coefficient of } \cos \phi_2 \text{ in } Q)} = C$$

where

$$(50) \quad A = A_{1,1} + (A_{1,2} + A_{1,3}\gamma)A_1 + (A_{1,4} + A_{1,5}\gamma)A_2 + (A_{1,6} + A_{1,7}\gamma)\epsilon + (A_{1,8} + A_{1,9}\gamma)\epsilon'$$

$$(51) \quad B = B_{1,1} + (B_{1,2} + B_{1,3}\gamma)B_1 + (B_{1,4} + B_{1,5}\gamma)B_2 + (B_{1,6} + B_{1,7}\gamma)\epsilon + (B_{1,8} + B_{1,9}\gamma)\epsilon'$$

$$(52) \quad C = C_{1,1} + (C_{1,2} + C_{1,3}\gamma)C_1 + (C_{1,4} + C_{1,5}\gamma)C_2 + (C_{1,6} + C_{1,7}\gamma)\epsilon + (C_{1,8} + C_{1,9}\gamma)\epsilon'.$$

5. STABILITY

5.1. The condition (i) of KAM theorem fails when $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$.
Case(i) when

$$(53) \quad \omega_1 = 2\omega_2$$

then from (53) and (13) we have

$$(54) \quad \omega_1 = \frac{2}{\sqrt{5}} \left\{ 1 - \frac{3A_1\gamma}{4} + \frac{3A_2\gamma}{4} + \frac{(1+\gamma)\epsilon}{4} + \frac{(1-\gamma)\epsilon'}{4} \right\}$$

$$(55) \quad \omega_2 = \frac{1}{\sqrt{5}} \left\{ 1 - \frac{3A_1\gamma}{4} + \frac{3A_2\gamma}{4} + \frac{(1+\gamma)\epsilon}{4} + \frac{(1-\gamma)\epsilon'}{4} \right\}.$$

Putting these values in (15), we get

$$(56) \quad \begin{aligned} & (675 + 2925A_1 + 2925A_2 - 675\epsilon - 675\epsilon')\mu^2 - \\ & (675 + 2829A_1 + 3021A_2 - 928\epsilon - 407\epsilon')\mu \\ & + (16 - 48A_1 + 48A_2 - 268\epsilon) = 0. \end{aligned}$$

Solving for μ , we have

$$(57) \quad \mu = \mu_{c1} = 0.0242938 \dots - 0.1790727 \dots A_1 - 0.0368505 \dots A_2 - 0.3828284 \dots \epsilon + 0.014776 \dots \epsilon'.$$

Case(ii) when

$$(58) \quad \omega_1 = 3\omega_2$$

proceeding as case(i), we have

$$(59) \quad \begin{aligned} & (225 + 975A_1 + 975A_2 - 225\epsilon - 225\epsilon')\mu^2 - \\ & (225 + 957A_1 + 993A_2 - 319\epsilon - 131\epsilon')\mu \\ & + (3 - 9_1 + 9A_2 - 84\epsilon) = 0. \end{aligned}$$

Solving for μ , we have

$$(60) \quad \begin{aligned} \mu = \mu_{c2} = & 0.013516 \dots - 0.099383 \dots A_1 - \\ & 0.019383 \dots A_2 - 0.3641983 \dots \epsilon + 0.0079002 \dots \epsilon'. \end{aligned}$$

5.2. Normalized Hamiltonian up to fourth order is

$$(61) \quad H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1 I_2 + CI_2^2) + \dots$$

Calculating the determinant D occurring in condition (ii) of KAM theorem, we have

$$D = -(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2)$$

Putting the values of A, B and C and if $u = \omega_1\omega_2$, we have

$$(62) \quad \begin{aligned} D = & \frac{644u^4 - 541u^2 + 36}{8(4u^2 - 1)(25u^2 - 4)} + (D_2 + D_3\gamma)A_1 + \\ & (D_4 + D_5\gamma)A_2 + (D_6 + D_7\gamma)\epsilon + (D_8 + D_9\gamma)\epsilon' \end{aligned}$$

$$(63) \quad D_2 = \frac{39176u^4 - 14359u^2 + 492}{48(4u^2 - 1)(25u^2 - 4)}$$

$$(64) \quad \begin{aligned} D_3 = & 3(1593600u^{10} + 2122096u^8 - 13052000u^6 + \\ & 5408175u^4 - 840076u^2 + 23616)/(16(4u^2 - 1)^2(25u^2 - 4)^2(16u^2 + 117)) \end{aligned}$$

$$(65) \quad D_4 = \frac{59944u^6 + 14269689u^4 - 12215187u^2 + 166284}{1962(4u^2 - 1)^2(25u^2 - 4)^2}$$

$$(66) \quad \begin{aligned} D_5 = & (55977008u^{10} + 16915980u^8 - 694540170u^6 + \\ & 171574880u^4 - 31023250u^2 + 732927)/(208(4u^2 - 1)^2(25u^2 - 4)^2(16u^2 + 117)) \end{aligned}$$

$$(67) \quad D_6 = \frac{195560u^6 - 549558u^4 + 24575u^2 + 48}{432(4u^2 - 1)^2(25u^2 - 1)}$$

$$(68) \quad \begin{aligned} D_7 = & (48947712u^{12} + 617901664u^{10} - 454674128u^8 + 2454064u^6 \\ & - 73692732u^4 + 8118909u^2 - 71472)/(96(4u^2 - 1)^3(25u^2 - 4)^2(16u^2 + 117)) \end{aligned}$$

$$(69) \quad D_8 = \frac{19440u^8 - 578504u^6 + 272091u^4 - 33970u^2 + 216}{654(4u^2 - 1)^2(25u^2 - 4)}$$

$$(70) \quad \begin{aligned} D_9 = & (252817792u^{12} + 2444733120u^{10} - 1258751060u^8 - 45728424u^6 \\ & + 32733556u^4 - 5336499u^2 + 631382)/(48(4u^2 - 1)^3(25u^2 - 4)^2(16u^2 + 117)) \end{aligned}$$

The second condition of KAM theorem is satisfied if, in the interval $0 < \mu < \mu_\infty$, the mass parameter does not take the value μ_{c3} , which makes $D = 0$. To find μ_{c3} , we note that when $A_1 = A_2 = \epsilon = \epsilon' = 0$ then from (62), D becomes zero if and only if

$$644u^4 - 541u^2 + 36 = 0$$

This implies that

$$(71) \quad u^2 = \frac{541 - \sqrt{199945}}{1288} = .0728632 = u_0(\text{say})$$

Writing $u^2 = \frac{27(1-\gamma^2)}{16}$ and $\gamma = 1 - 2\mu$ and then solving above, we find

$$(72) \quad \gamma = \gamma_0 = 0.9781726 \dots \quad \text{and} \quad \mu = \mu_0 = 0.0109136 \dots$$

When $A_1, A_2, \epsilon, \epsilon'$ are not zero, we assume that D is zero if

$$(73) \quad \mu = \mu_0 + \alpha A_1 + \alpha' A_2 + \alpha'' \epsilon + \alpha''' \epsilon'$$

$$(74) \quad \gamma = \gamma_0 - 2(\alpha A_1 + \alpha' A_2 + \alpha'' \epsilon + \alpha''' \epsilon') \quad \text{with} \quad \gamma_0 = 1 - 2\mu_0$$

$$(75) \quad u^2 = u_0 + (u_1 + \alpha u_2)A_1 + (u_3 + \alpha' u_4)A_2 + (u_5 + \alpha'' u_6)\epsilon + (u_7 + \alpha''' u_8)\epsilon'$$

with

$$(76) \quad u_1 = u_3 = \frac{117(1 - \gamma_0^2)}{16}, u_5 = \frac{a + 24\gamma_0}{16}, u_7 = \frac{a - 24\gamma_0}{16},$$

$$a = 24 - 27(1 - \gamma_0^2), u_2 = u_4 = u_6 = u_8 = \frac{27\gamma_0}{4}$$

and $\alpha, \alpha', \alpha'', \alpha'''$ are to be determined. From (62), D is zero when

$$(77) \quad D = \frac{644u^4 - 541u^2 + 36}{8(4u^2 - 1)(25u^2 - 4)} + (D_2 + D_3\gamma)A_1 +$$

$$(D_4 + D_5\gamma)A_2 + (D_6 + D_7\gamma)\epsilon + (D_8 + D_9\gamma)\epsilon' = 0.$$

Making use of (76) in (77) and equating to zero the coefficients of A_1, A_2, ϵ and ϵ' , we get

$$(78) \quad \alpha = -\frac{(1288u_0 - 541)u_1 + 8(D_2^0 + D_3^0\gamma_0)(4u_0 - 1)(25u_0 - 4)}{u_2(1288u_0 - 541)}$$

$$(79) \quad \alpha' = -\frac{(1288u_0 - 541)u_3 + 8(D_4^0 + D_5^0\gamma_0)(4u_0 - 1)(25u_0 - 4)}{u_4(1288u_0 - 541)}$$

$$(80) \quad \alpha'' = -\frac{(1288u_0 - 541)u_5 + 8(D_6^0 + D_7^0\gamma_0)(4u_0 - 1)(25u_0 - 4)}{u_6(1288u_0 - 541)}$$

$$(81) \quad \alpha''' = -\frac{(1288u_0 - 541)u_7 + 8(D_8^0 + D_9^0\gamma_0)(4u_0 - 1)(25u_0 - 4)}{u_8(1288u_0 - 541)}$$

where D_n^0 ($n = 2, 3, 4, \dots, 9$) are D_n given by(63) to (71)as evaluated for the unperturbed problem. Numerical computation yields, $\alpha = -0.1036878 \dots, \alpha' = 0.5456468 \dots, \alpha'' = -0.4748319 \dots, \alpha''' = -0.1432604 \dots$. We then have

$$\mu_{c3} = \mu_0 + \alpha A_1 + \alpha' A_2 + \alpha'' \epsilon + \alpha''' \epsilon'$$

$$= 0.0109136 \dots - 0.1036878 \dots A_1 + 0.5456468 \dots A_2 -$$

$$0.4748319 \dots \epsilon - 0.1432604 \dots \epsilon'$$

Hence in the interval $0 < \mu < \mu_\infty$, both the conditions of KAM theorem are satisfied and therefore the triangular point is stable except for three mass ratio μ_{ci} ($i = 1, 2, 3$).

6. CONCLUSION

We conclude that triangular point is stable in the nonlinear sense for all ratios in the range of linear stability except for three mass ratios;

$$\begin{aligned}\mu_{c1} &= 0.0242938 \dots - 0.1790727 \dots A_1 - 0.0368505 \dots A_2 - \\ &\quad 0.3828284 \dots \epsilon + 0.014776 \dots \epsilon' \\ \mu_{c2} &= 0.013516 \dots - 0.099383 \dots A_1 - 0.019383 \dots A_2 - 0.3641983 \dots \epsilon + \\ &\quad 0.0079002 \dots \epsilon' \\ \mu_{c3} &= 0.0109136 \dots - 0.1036878 \dots A_1 + 0.54565468 \dots A_2 - \\ &\quad 0.4748319 \dots \epsilon - 0.1432604 \dots \epsilon'\end{aligned}$$

at which KAM theorem fails. When $A_1 = A_2 = \epsilon = \epsilon' = 0$, then the values of $\mu_{ci}(i = 1, 2, 3)$ agree with those found by Deprit and Deprit (1967) and other researchers.

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