

LARGE DEVIATION RESULTS FOR PERIODIC POINTS OF A RATIONAL MAP

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ABSTRACT. In this note we show a large deviation result for the periodic points of a hyperbolic rational map on the Riemann sphere. This extends the well known equidistribution result of Lyubich in this setting. We also consider convergence results for more general weighted averages of orbital measures with respect to Hölder continuous functions.

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1. INTRODUCTION

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere and let $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree $d > 1$. Let $\mathbb{J} \subset \widehat{\mathbb{C}}$ denote the Julia set of T and we denote by $\text{Fix}_n(T)$ the set of periodic points of period n in $\mathbb{J}$, i.e.,

$$\text{Fix}_n(T) := \{z \in \mathbb{J} : T^n(z) = z\}.$$

Note that the cardinality of the set $\text{Fix}_n(T)$ is at most $d^n + 1$. There is a particularly elegant result due to Lyubich which shows that periodic points are distributed according to a particular invariant measure μ_0 [9].

More generally, let us define the pressure of a real-valued continuous function $f : \mathbb{J} \rightarrow \mathbb{R}$ by

$$P(f) := \sup \left\{ h(\nu) + \int f d\nu : \nu \in \mathcal{M}_T \right\}.$$

Here $h(\nu)$ denotes the entropy of T with respect to the measure ν and $\mathcal{M}_T$ denotes the space of all T -invariant probability measures on $\mathbb{J}$. When f is Hölder continuous

there is a unique equilibrium state μ_f , which is defined to be the unique T -invariant probability measure realising the above supremum. Given $z \in \widehat{\mathbb{C}}$ it is convenient to denote

$$(1) \quad f^n(z) := f(z) + f(Tz) + \cdots + f(T^{n-1}z).$$

We say that $T : \mathbb{J} \rightarrow \mathbb{J}$ is *hyperbolic* if there exists $C > 0$ and $\lambda > 1$ such that for all $z \in \mathbb{J}$ and $n \geq 1$ we have that $|(T^n)'(z)| \geq C\lambda^n$. Our main result is the following.

Theorem 1.1. *Let T be a hyperbolic rational map of degree at least 2 and let $f : \mathbb{J} \rightarrow \mathbb{R}$ be a Hölder continuous function. Then for any weak* neighbourhood $\mathcal{U} \subset \mathcal{M}_T$ around μ_f the weighted proportion of the measures,*

$$\mu_{z,n} := \frac{1}{n} [\delta_z + \delta_{Tz} + \cdots + \delta_{T^{n-1}z}] \notin \mathcal{U}, \text{ where } z \in \text{Fix}_n(T),$$

tends to zero exponentially fast, in the sense that there exists $c > 0$ and $0 < \eta < 1$ such that

$$\frac{\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \notin \mathcal{U}}} e^{f^n(z)}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \leq c\eta^n, \text{ for } n \geq 0.$$

A similar conclusion holds for quite general locally expanding maps and Hölder continuous maps.

Although, at first sight, Theorem 1 may seem fairly technical in nature it incorporates a number of simple results that we shall now state as corollaries. For example, in the special case that the function f is identically zero we have that $P(0) = \log d$ and the equilibrium state μ_0 is the unique measure of maximal entropy for T . Theorem 1 then reduces to the following corollary.

Corollary 1.2. *For any weak* open neighbourhood $\mathcal{U} \subset \mathcal{M}_T$ around μ_0 , the proportion of the points $z \in \text{Fix}_n(T)$ such that $\mu_{z,n} \notin \mathcal{U}$ tends to zero exponentially fast, i.e., there exists $c > 0$ and $0 < \eta < 1$ such that for $n \geq 0$,*

$$\frac{1}{d^n + 1} \# \{z \in \text{Fix}_n(T) : \mu_{z,n} \notin \mathcal{U}\} \leq c\eta^n \text{ for } n \geq 0.$$

The next corollary gives a traditional equidistribution result for periodic orbits with respect to weightings by the Hölder continuous function.

Corollary 1.3. *Let $f : \mathbb{J} \rightarrow \mathbb{R}$ be a Hölder continuous function and let μ_f be the equilibrium state for f . Then the probability measures*

$$\mu_{f,n} := \frac{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \mu_{z,n}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}}$$

converge to μ_f in the weak topology as $n \rightarrow \infty$.*

Both of the above corollaries can be viewed as partial generalisations of the Lyubich's theorem on the distribution of periodic points of a general rational map.

Theorem 1.4 (Lyubich's Theorem). *Let T be a rational map of degree at least 2. Define*

$$\mu_n := \frac{1}{d^n + 1} \sum_{z \in \text{Fix}_n(T)} \delta_z,$$

where δ_z is the Dirac delta measure. Then, the sequence $\{\mu_n\}$ converges in the weak* topology to a T -invariant measure μ_0 .

Following the publication of Lyubich's original result, it was subsequently demonstrated that this measure maximised the entropy [3] (and Mañé showed the existence of a unique measure of maximal entropy in [5]). While Lyubich's theorem can be established using normality of sequences of analytic functions, the Montel-Carathéodory theorem and the Denjoy-Wolff theorem (see [9]), the stronger results we presented above seem to require a different argument.

Lyubich also proved a complementary result to Theorem 4 on the distribution of preimages of points. The corresponding large deviation in that case was studied in [7]. In that context, it was possible to avoid the hyperbolicity assumption using results of Przytycki [8].

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2. PROPERTIES OF RATIONAL TRANSFORMATIONS

Let us briefly review some of the basic results on rational maps which we will need later. Let $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ denote a rational map, i.e., a quotient of two relatively prime polynomials, $P(z)$ and $Q(z)$ of the form

$$(2) \quad T(z) = \frac{P(z)}{Q(z)} = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n},$$

where $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in \mathbb{C}$, with $a_0 \neq 0$ and $b_0 \neq 0$. The degree of the rational map is $d = \max(m, n)$. Henceforth, we shall always assume the degree of the rational map is at least two (i.e., $d \geq 2$). We now give one of several possible definitions of the Julia set.

Definition: The Julia set $\mathbb{J} \subset \widehat{\mathbb{C}}$ of a rational map T is the closure of the set of all periodic points $z \in \text{Fix}_n(T)$ such that $|(T^n)'(z)| > 1$.

It is obvious from the definition that $\mathbb{J}$ is a closed T -invariant set. We shall be interested in the restriction of the rational map to its Julia set, i.e., $T : \mathbb{J} \rightarrow \mathbb{J}$.

The following basic result will be particularly useful to us.

Lemma 2.1 (cf. [3],[4]).

1. $h(T) = \log d$.
2. The map $\nu \mapsto h(\nu)$ is upper semi-continuous in weak* topology.
3. There is a unique measure of maximal entropy μ_0 for $T : \mathbb{J} \rightarrow \mathbb{J}$.

We shall write $\mathcal{C}(\mathbb{J})$ for the space of all real-valued continuous functions defined on $\mathbb{J}$ and $\mathcal{C}^\alpha(\mathbb{J})$ for the space of real-valued α -Hölder continuous function defined on $\mathbb{J}$. The next two results are standard.

Lemma 2.2 ([6]). *Let T be a hyperbolic rational map. Let $f \in \mathcal{C}^\alpha(\mathbb{J})$ be a Hölder continuous function, then f has a unique equilibrium state μ_f .*

For general T there is a generalization of this result for certain functions f due to Denker and Urbanski [2].

Lemma 2.3 ([6]). *Let $z \in \mathbb{J}$, $f \in \mathcal{C}^\alpha(\mathbb{J})$ and $g \in \mathcal{C}(\mathbb{J})$. Then,*

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} = P(f)$$

$$(4) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in \text{Fix}_n(T)} e^{(f^n(z) + g^n(z))} \leq P(f + g).$$

Finally, we recall a quite general result (cf. [11], pp. 221-222).

Lemma 2.4 ([7]). *If $\nu \in \mathcal{M}_T$, then*

$$h(\nu) = \inf \left\{ P(f) - \int f d\nu : f \in \mathcal{C}(\mathbb{J}) \right\}.$$

3. PROOF OF THEOREM 1

In this section, we will give the proof Theorem 1 using the results from the previous section. The basic methodology is fairly classical, but the application to Julia sets is of topical interest. Let $f \in \mathcal{C}^\alpha(\mathbb{J})$ be a Hölder continuous function with unique equilibrium state μ_f . We can define a function $Q : \mathcal{C}(\mathbb{J}) \rightarrow \mathbb{R}$ by

$$(5) \quad Q(g) = P(f + g) - P(f)$$

where $g \in \mathcal{C}(\mathbb{J})$. Let us denote by $I : \mathcal{M}_T \rightarrow \mathbb{R}$, the Legendre transform of Q given by

$$(6) \quad I(\nu) = \sup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \int g d\nu - Q(g) \right\},$$

for any measure $\nu \in \mathcal{M}_T$. For any weak* closed subset $\mathcal{K} \subset \mathcal{M}_T$ we define:

$$(7) \quad \rho = \inf_{\nu \in \mathcal{K}} I(\nu).$$

The next lemma is useful in relating ρ to the periodic points of the rational map T .

Lemma 3.1. *With the above assumptions,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[\frac{\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \in \mathcal{K}}} e^{f^n(z)}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \right] \leq -\rho$$

Proof: From the definition of ρ it is clear that for any $\epsilon > 0$ and for every $\nu \in \mathcal{K}$, there exists a function $g \in \mathcal{C}(\mathbb{J})$ such that $\int g d\nu - Q(g) > \rho - \epsilon$. Hence,

$$\mathcal{K} \subset \bigcup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \nu \in \mathcal{M}_T : \int g d\nu - Q(g) > \rho - \epsilon \right\}.$$

Since $\mathcal{K}$ is weak* compact we can choose a finite family $g_1, \dots, g_k \in \mathcal{C}(\mathbb{J})$ such that

$$\mathcal{K} \subset \bigcup_{i=1}^k \left\{ \nu \in \mathcal{M}_T : \int g_i d\nu - Q(g_i) > \rho - \epsilon \right\}.$$

We can then bound

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[\frac{\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \in \mathcal{K}}} e^{f^n(z)}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left[\log \sum_{i=1}^k \sum_{z \in S(g_i, n, \rho - \epsilon) \cap \text{Fix}_n(T)} e^{f^n(z)} - \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \right] \end{aligned}$$

where we denote

$$S(g_i, n, \rho - \epsilon) := \left\{ z : \int g_i d \left(\frac{1}{n} (\delta_z + \delta_{Tz} + \dots + \delta_{T^{n-1}z}) \right) - Q(g_i) > \rho - \epsilon \right\}.$$

Therefore,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[\frac{\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \in \mathcal{K}}} e^{f^n(z)}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \right] \\
& \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left[\log \sum_{i=1}^k \sum_{\substack{z \in \text{Fix}_n(T) \\ g_i^n(z) > n[Q(g_i) + (\rho - \epsilon)]}} e^{-g_i^n(z)} e^{f^n(z) + g_i^n(z)} \right. \\
& \quad \left. - \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \right] \\
& \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left[\log \sum_{i=1}^k e^{-n[Q(g_i) + (\rho - \epsilon)]} \sum_{z \in \text{Fix}_n(T)} e^{f^n(z) + g_i^n(z)} \right. \\
& \quad \left. - \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \right] \\
& = \sup_{1 \leq i \leq k} \left[-Q(g_i) - \rho + \epsilon + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z) + g_i^n(z)} \right. \\
& \quad \left. - \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \right] \\
& \leq \sup_{1 \leq i \leq k} [-Q(g_i) - \rho + \epsilon + P(f + g_i) - P(f)] \\
& = -\rho + \epsilon,
\end{aligned}$$

where we have used lemma 7 in the penultimate step. Since ϵ is arbitrary, the proof is complete. $\bullet$

We next want to ensure that if $\mathcal{K}$ does not contain μ_f then $\rho > 0$. This will follow from the next lemma.

Lemma 3.2 ([7]).

1. If $\nu \neq \mu_f$, then $I(\nu) > 0$.
2. The map $\nu \longrightarrow I(\nu)$ is lower semi-continuous on $\mathcal{M}_T$.

Proof: We include the short proof of this lemma for the reader's convenience. Part 1 follows from:

$$\begin{aligned}
 I(\nu) &= \sup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \int g d\nu - Q(g) \right\} \\
 &= \sup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \int g d\nu - P(f+g) + P(f) \right\} \\
 &= \sup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \int (g-f) d\nu - P(g) + P(f) \right\} \\
 &= \sup_{g \in \mathcal{C}(\mathbb{J})} \left\{ \int g d\nu - P(g) \right\} + P(f) - \int f d\nu \\
 &= - \inf_{g \in \mathcal{C}(\mathbb{J})} \left\{ P(g) - \int g d\nu \right\} + P(f) - \int f d\nu. \\
 &= -h(\nu) + P(f) - \int f d\nu > 0.
 \end{aligned}$$

For part 2, the lower semi continuity of I follows from its definition. •

We can now complete the proof of Theorem 1. Let $\mathcal{U} \subset \mathcal{M}_T$ be an open weak* neighbourhood of μ_f . In particular, $\mathcal{K} = \mathcal{M}_T \setminus \mathcal{U}$ is weak* closed and so compact. Thus, it can be seen from the definition of ρ (in equation(7)) and lemma 10) that $\rho > 0$. Theorem 1 now follows from lemma 9.

4. PROOF OF THE COROLLARIES

Proof: (of corollary 2) Let $f \equiv 0$. Then the equilibrium state for this function is precisely the measure of maximal entropy, μ_0 , for the rational map $T : \mathbb{J} \longrightarrow \mathbb{J}$. The corollary then follows immediately from Theorem 1. •

Proof: (of corollary 3) For any arbitrary $\epsilon > 0$, let us define an open neighbourhood $\mathcal{U}$ of μ_f (in the weak* topology) by

$$\mathcal{U} = \left\{ \nu \in \mathcal{M}_T : \left| \int g d\nu - \int g d\mu_f \right| < \epsilon \right\}$$

where g is a continuous function defined on $\mathbb{J}$. Then,

$$\begin{aligned}
& \frac{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \frac{g^n(z)}{n}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \\
&= \frac{1}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \left[\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \in \mathcal{U}}} e^{f^n(z)} \frac{g^n(z)}{n} + \sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \notin \mathcal{U}}} e^{f^n(z)} \frac{g^n(z)}{n} \right] \\
&= \frac{1}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \left[\sum_{\substack{z \in \text{Fix}_n(T) \\ \mu_{z,n} \in \mathcal{U}}} e^{f^n(z)} \left\{ \int g d\mu_f + E_n(z) \right\} \right] + O(\eta^n)
\end{aligned}$$

where $|E_n(z)| < \epsilon$ and $0 < \eta < 1$ as given by Theorem 1.

Hence, by adding appropriate constants to g (if necessary),

$$\begin{aligned}
\int g d\mu_f - \epsilon &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \frac{g^n(z)}{n}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \\
&\leq \limsup_{n \rightarrow \infty} \frac{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \frac{g^n(z)}{n}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \\
&\leq \int g d\mu_f + \epsilon
\end{aligned}$$

i.e.,

$$\frac{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)} \frac{g^n(z)}{n}}{\sum_{z \in \text{Fix}_n(T)} e^{f^n(z)}} \longrightarrow \int g d\mu_f \text{ as } n \rightarrow \infty,$$

as required. •

5. FINAL REMARKS

In this note we have concentrated on large deviation properties for periodic points of rational maps. However, to give a broader perspective we conclude by describing some related measure theoretic results.

We say that $T : \mathbb{J} \rightarrow \mathbb{J}$ satisfies a central limit theorem with respect to the measure μ_f and a Hölder continuous function $h : \mathbb{J} \rightarrow \mathbb{R}$ with $\int h d\mu_f = 0$ if there exists $\sigma > 0$ such that

$$\lim_{n \rightarrow +\infty} \mu_f \{x \in \mathbb{J} : h^n(x) < \sqrt{nt}\} = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^t \exp(-x^2/2\sigma^2) dx,$$

for each $t \in \mathbb{R}$. These results were established by Denker, Przytycki and Urbanski by showing stretched exponential convergence for the transfer operator, in the Hölder norm [1]. However, we can consider a stronger statistical properties, as follows. We can define a process $\{\psi_n(t) : t \in [0, 1]\}$ by

$$\frac{1}{\sigma\sqrt{n}} h^{[nt]}(x), \text{ for } t \in [0, 1] \text{ and } n \geq 1.$$

If the $\psi_n(t)$ converge weakly to the standard Brownian motion then say that we have a Functional Central Limit Theorem. Comparing the bounds of [1] with the hypothesis of Theorem 1 in [10] we conclude the following result.

Theorem 5.1. *Let T be a rational map of degree at least 2 and let $f : \mathbb{J} \rightarrow \mathbb{R}$ be a Hölder continuous function such that $P(f) > \sup_{x \in \mathbb{J}} f(x)$. Let μ_f be the unique equilibrium state for f and a Hölder continuous function $h : \mathbb{J} \rightarrow \mathbb{R}$ with $\int h d\mu_f = 0$ and $\sigma > 0$. Then $T : \mathbb{J} \rightarrow \mathbb{J}$ satisfies a Functional Central Limit Theorem.*

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