

JACOBI STABILITY FOR GEOMETRIC DYNAMICS

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ABSTRACT. The paper investigates the KCC theory and the Jacobi stability for the Euler-Lagrange prolongation of a flow on a Riemannian manifold (geometric dynamics).

Section 1 explains the origin of Jacobi stability. Section 2 introduces the Kosambi-Cartan-Chern theory for second order differential systems of n equations with n unknown functions underlying the geometric roots. Section 3 introduces the fifth differential invariants associated to geometric dynamics and analyzes the Jacobi stability of geometric dynamics. Section 4 gives the basic ideas of the stability analysis of the linearized geometric dynamics of a given flow and shows that this Leapunov stability is different from the Jacobi stability.

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1. INTRODUCTION TO JACOBI STABILITY

The Kosambi-Cartan-Chern theory (KCC theory) of differential invariants and the Jacobi stability are useful tools for second order differential equations coming from physics, chemistry, biology, etc [3]. This theory started with variation equations in differential geometry and estimates the admissible perturbations from a steady state of a second order system of differential equations [13], i.e., the perturbations that does not change the steady state [3], [2].

Jacobi stability was studied initially for *geodesics* [3] in Riemann or Finsler geometry. By deviating the geodesics and using the KCC-covariant differential, it follows a differential system in variations. Automatically, it appears the second

KCC-invariant also called *the deviation curvature tensor*. The differential system in variations is assimilated with the system which describes a Jacobi field along the geodesics. In this way, starting from the definition of Jacobi stability for geodesics, it was proved that *the geodesics are Jacobi stable if and only if the real parts of the eigenvalues of the deviation curvature tensor are strictly negative everywhere and Jacobi unstable, otherwise*. Naturally, the Jacobi stability concept for geodesics in Riemann or Finsler geometry was extended to trajectories of a second order system of ordinary differential equations [13].

It is well known that linear stability analysis of a second order of ODEs involves changing this system into a first order system of ODEs and linearization via the Jacobian matrix. If a steady state is a stable one, then small perturbation away from this zero speed-point will not change the dynamics of the system. Around a stable state it appears a *basin of attraction* of the system, which is not specified by the linear analysis. On the other hand, the KCC analysis is based on the study of Leapunov stability of a *whole trajectories in a region*, therefore, in this case, the perturbations represent trajectories close-by to the reference trajectory. This geometrical method, even when derived at a stationary point, yields information about the behavior of trajectories in an open region surrounding this point.

The goal of this paper is two-fold: to study Jacobi stability of the Euler-lagrange extension of a flow, and to study the correlation between Jacobi stability and the stability of the linearized Euler-Lagrange extension (in the sense introduced by C. Udriște in [18]).

2. KCC THEORY AND JACOBI STABILITY

We recall first some basics of KCC theory (D. Kosambi [9], E. Cartan [7] and S. S. Chern [8]).

Let $x = (x^1, \dots, x^n)$, $\dot{x} = \frac{dx}{dt}$ and t be $2n + 1$ coordinates in an open connected subset $D \subset R^{2n+1}$. We consider a second order differential system

$$\frac{d^2 x^i}{dt^2} + g^i(x, \dot{x}, t) = 0, \quad i = \overline{1, n}, \quad (2.1)$$

where each function $g^i(x, \dot{x}, t)$ is smooth in a neighborhood of some initial condition $(x_0, \dot{x}_0, t_0) \in \mathbf{D}$ ([13], [2], [3]). Generally, the system (2.1) has no geometrical meaning since $\frac{d^2 x^i}{dt^2}$ ("acceleration") or $g^i(x, \dot{x}, t)$ ("force") is not a contravariant vector field with respect to diffeomorphisms of the form

$$\bar{x}^i = f^i(x^1, \dots, x^n), \quad i = \overline{1, n}, \quad \bar{t} = t.$$

The idea of KCC was to change the system (2.1) into an equivalent system (same solutions), but with geometrical meaning, and to show that it defines five tensor fields called *differential invariants*. For that we need the *KCC-covariant differential* of a contravariant vector field $\xi^i(x)$, $x \in D$ defined by

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + \frac{1}{2}g_{i,r}^i \xi^r, \quad (2.2)$$

where " ; " indicates partial differentiation with respect to $\dot{x}$. Using the KCC-covariant differential (2.2), the system (2.1) changes into

$$\frac{D\dot{x}^i}{dt} = \varepsilon^i, \quad \varepsilon^i = \frac{1}{2}g_{;r}^i \dot{x}^r - g^i, \quad (2.3)$$

where ε^i is a contravariant vector field on D called *the first KCC-invariant*. From a physical point of view, the contravariant vector field ε^i is interpreted as an *external force* [2].

Remark 2.1. The functions $g^i = g^i(x, \dot{x}, t)$ are 2 homogeneous in $\dot{x}$ if and only if $\varepsilon^i = 0$. In other words, $\varepsilon^i = 0$ is a necessary and sufficient condition for a semispray to be a spray [13]. It is obvious that for the geodesic spray of a Riemannian or Finsler manifold, the first invariant vanishes.

The KCC theory starts with the study of trajectories which are slightly deviated upon a certain trajectory of (2.1). Precisely, one studies the dynamics of the system in variations. For this purpose, the trajectories $x^i(t)$ of (2.1) are varied into nearby ones with respect to x according to $\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta$, with $0 < \eta \ll 1$. In this way, one gets the variational equations

$$\frac{d^2\xi^i}{dt^2} + g_{;r}^i \frac{d\xi^r}{dt} + g_{;r}^i \xi^r = 0, \quad (2.4)$$

where " ; " indicates partial differentiation with respect to x . Using now the KCC-covariant differential (2.2), one obtains

$$\frac{D^2\xi^i}{dt^2} = \frac{d^2\xi^i}{dt^2} + \frac{1}{2}g_{;r}^i \frac{d\xi^r}{dt} + \frac{1}{2} \left[(\dot{x}^j g_{;j;r}^i + \ddot{x}^j g_{;j;r}^i + \frac{\partial g_{;r}^i}{\partial t}) \xi^r + \frac{d\xi^r}{dt} + \frac{1}{2}g_{;j}^r \xi^j \right].$$

In this way, the differential system (2.4) rewrites in the covariant form

$$\frac{D^2\xi^i}{dt^2} = -\frac{1}{2}g_{;r}^i \frac{d\xi^r}{dt} - g_r^i \xi^r + \frac{1}{2}(-g^j g_{;j;r}^i \xi^r + \dot{x}^j g_{;j;r}^i \xi^r + \frac{\partial g_{;r}^i}{\partial t} \xi^r + g_{;r}^i \frac{d\xi^r}{dt} + \frac{1}{2}g_{;r}^i g_{;j}^r \xi^j)$$

or equivalently,

$$\frac{D^2\xi^i}{dt^2} = P_r^i \xi^r, \quad (2.5)$$

where the tensor field

$$P_j^i = -g_{;j}^i - \frac{1}{2}g^r g_{;r;j}^i + \frac{1}{2}\dot{x}^r g_{;r;j}^i + \frac{1}{4}g_{;r}^i g_{;j}^r + \frac{1}{2}\frac{\partial g_{;j}^i}{\partial t}$$

is called the *deviation curvature tensor* or *the second KCC-invariant* of the differential system (2.1).

The deviation curvature tensor plays an important role in Jacobi stability of the second order system (2.1). In order to understand how was born the concept of *Jacobi stability for second order differential systems*, let us look at differential geometry counterpart.

Geometric explanation. Let (R^n, F) be a Finsler manifold ([3], [5]), where the fundamental function F induces a norm on R^n , denoted by $\|\cdot\|$. Let $\gamma : x^i = x^i(s)$

be a fixed geodesic of the manifold (M, F) , i.e., a curve described by the differential equations

$$\frac{d^2 x^i}{ds^2} + G^i(x, y) = 0, \quad (2.6)$$

where $G^i(x, y) = \Gamma_{jk}^i(x, y)y^j y^k$ is the geodesic spray coefficient and Γ_{jk}^i are the Christoffel symbols of F ([13]). Using now the KCC-covariant differential (2.2) and deviating the geodesic trajectories given by (2.6) in the same manner as above, we get the system in variations in the covariant form

$$\frac{D^2 \xi^i}{dt^2} = P_j^i \xi^j, \quad (2.7)$$

with

$$P_j^i = -G_{,j}^i - \frac{1}{2} G^r G_{,r;j}^i + \frac{1}{2} y^r G_{,r;j}^i + \frac{1}{4} G_{,r}^i G_{,j}^r + \frac{1}{2} \frac{\partial G_{,j}^i}{\partial s},$$

where “,” indicates partial differentiation with respect to x , and “;” indicates partial differentiation with respect to y .

The definition of *Jacobi stability for geodesics* is

Definition 2.1. Let $\gamma(s) : x^i = x^i(s)$ be a geodesic, i.e., a trajectory of the differential system (2.6). The trajectory $\gamma(s)$ is said to be *Jacobi stable* if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\bar{x}^i(s) - x^i(s)\| < \varepsilon$ holds for all $s \geq s_0$ and all trajectories $\bar{x}^i(s)$ for which

$$\|\bar{x}^i(s_0) - x^i(s_0)\| < \delta(\varepsilon)$$

and

$$\left\| \frac{d\bar{x}^i}{ds}(s_0) - \frac{dx^i}{ds}(s_0) \right\| < \delta(\varepsilon).$$

The geometrical meaning of Jacobi stability is: if any other geodesic γ , with sufficiently close initial condition at s_0 , remains close to $\gamma(s)$ for all $s \geq s_0$, then $\gamma(s)$ is *Jacobi stable*. If every geodesic is stable in this sense, then the Finsler manifold (R^n, F) is called *Jacobi stable*.

One can observe that the equation (2.7) is the Jacobi fields equation along the geodesics (2.6). On the other hand, the Jacobi equation (2.7) on the Finsler manifold (R^n, F) can be written in the scalar form

$$\frac{d^2 v}{ds^2} + K \cdot v = 0,$$

where $\xi^i = v(s)\eta^i$ is a Jacobi field along $\gamma : x^i = x^i(s)$, η^i is the unit normal vector field along γ and K is the *flag curvature* of (R^n, F) . The sign of K influences the geodesic rays [5]. Indeed,

- if $K > 0$, then the geodesics bunch together, i.e., they are *Jacobi stable*;
- if $K < 0$, then the geodesics disperse, i.e., they are *Jacobi unstable*.

Theorem 2.2. ([3], [2]) *The geodesics (solutions of (2.6)) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P_i^j are strict negative everywhere, and Jacobi unstable, otherwise.*

Remark 2.2. The negativity of flag curvature K is equivalent with the positivity of the eigenvalues of the deviation curvature tensor P_j^i and the positivity of flag curvature is equivalent with the negativity of the eigenvalues of P_j^i .

The notion of Jacobi stability for geodesics presented above was extended at second order systems of ODEs (2.1).

Definition 2.2. ([3], [2]) *The trajectories of the system (2.1) are called Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P_i^j are strictly negative everywhere; otherwise, they are called Jacobi unstable.*

The equations (2.5) are *generalized Jacobi field equations* for the equations (2.1).

Remark 2.3. The second order system of ODEs (2.1) is Jacobi stable if and only if the system in variations (in the covariant form) (2.5) is Leapunov stable.

Remark 2.4. Jacobi stability analysis is based on the study of Leapunov stability of whole trajectories in a region without considering the velocity. This theory, even when derived at the stationary point, yield information about the behavior of trajectories in an open region surrounding that point.

The *third, fourth and fifth differential invariants* of the system (2.1) are respectively

$$R_{jk}^i = \frac{1}{3}(P_{j;k}^i - P_{k;j}^i), \quad B_{jkl}^i = R_{jk;l}^i, \quad D_{jkl}^i = g_{;j;k;l}^i.$$

A basic result of the KCC-theory which points out the role of the five invariants is the following :

Theorem 2.3. ([2]) *Two second order differential systems*

$$\frac{d^2 x^i}{dt^2} + g^i(x, \dot{x}, t) = 0, \quad \text{and} \quad \frac{d^2 y^i}{dt^2} + \bar{g}^i(y, \dot{y}, t) = 0, \quad i = \overline{1, n},$$

on D , first characterized by the differential invariants $\varepsilon^i, P_j^i, R_{jk}^i, B_{jkl}^i$ and D_{jkl}^i , and the second characterized by $\bar{\varepsilon}^i, \bar{P}_j^i, \bar{R}_{jk}^i, \bar{B}_{jkl}^i$ and $\bar{D}_{jkl}^i$, can be locally transformed, via $\bar{x}^i = f^i(x^1, \dots, x^n)$, $i = \overline{1, n}$, $\bar{t} = t$, one into another, if and only if the five invariants $\varepsilon^i, P_j^i, R_{jk}^i, B_{jkl}^i, D_{jkl}^i$ are tensors equivalent with $\bar{\varepsilon}^i, \bar{P}_j^i, \bar{R}_{jk}^i, \bar{B}_{jkl}^i$ respectively $\bar{D}_{jkl}^i$. In particular, there are local coordinates $(\bar{x})$ for which $\bar{g}^i(x, \dot{x}, t) = 0$ if and only if $\varepsilon^i, P_j^i, R_{jk}^i, B_{jkl}^i$ and D_{jkl}^i vanish.

Remark 2.5. The eigenstructure of the deviation curvature tensor is an alternative to the classical Floquet Theory [12], [4]. One can see that Jacobi stability is a weaker condition than the stability of periodic orbits given by the sign of the eigenvalues of characteristic multipliers.

3. KCC THEORY AND JACOBI STABILITY FOR GEOMETRIC DYNAMICS

Let us analyze the Jacobi stability of geometric dynamics (the Euler-Lagrange extension of a flow on a Riemannian manifold). We will achieve this analysis for the stationary points generated by the stationary points of the flow. In order to do this, we remind some basic elements of *geometric dynamics which describes a geodesic motion in a gyroscopic field of forces* [14], [15], [20].

This theory can be equally applied to any kind of flow on a manifold endowed with a geometric structure capable to produce *square of length* (density of energy) and

derivatives. As an example, we can use a Euclidean-Lagrangian structure associated to the metric δ_{ij} or a Riemannian-Lagrangian structure associated to the metric g_{ij} .

Let us consider the vector field $X \in \chi(D)$, $D \subset R^n$ and the Riemannian (Euclidean) manifold (R^n, δ_{ij}) . To change the flow

$$\frac{dx^i}{dt} = X^i(x), \quad i = \overline{1, n} \quad (3.1)$$

into a geodesic motion under a gyroscopic field of forces, we have to prolongate the system by derivation, and after modifying this prolongation in a suitable way, to obtain an Euler-Lagrange system

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \left(\frac{dx^i}{dt} \right)} = 0$$

associated to the *least squares Lagrangian*

$$L = \frac{1}{2} \left\| \frac{dx^i}{dt} - X^i(x) \right\|_\delta^2, \quad i = \overline{1, n}. \quad (3.2)$$

The *Euler-Lagrange extension* (geometric dynamics) of the flow (3.1), on the Riemannian manifold $(R^n, g_{ij} = \delta_{ij})$, is

$$\frac{d^2 x^i}{dt^2} = \delta^{ir} \frac{\partial f}{\partial x^r} + \delta^{ir} \Omega_{rs} \frac{dx^s}{dt}, \quad (3.3)$$

where

$$f = \frac{1}{2} \|X\|_\delta^2, \quad \Omega_{ij} = \frac{\partial (\delta_{js} X^s)}{\partial x^i} - \frac{\partial (\delta_{is} X^s)}{\partial x^j}, \quad i, j, k = \overline{1, n}.$$

We mention that the function f is called *the energy of the vector field X* . The gyroscopic character of the Euler-Lagrange extension is entailed by the gyroscopic force (2-form) Ω with the components Ω_{ij} . The previous Euler-Lagrange extension of a flow can be written as

$$\frac{d^2 x^i}{dt^2} - g^i(x, \dot{x}, t) = 0, \quad \forall i = \overline{1, n},$$

where

$$g^i(x, \dot{x}, t) = -\delta^{ir} \frac{\partial f}{\partial x^r} - \delta^{ir} \Omega_{rs} \frac{dx^s}{dt}.$$

Remark 3.1. The trajectories of the Euler-Lagrange extension (3.3), include the initial field lines and curves which are transversal to the field lines. From this point of view, the field lines of the starting system (3.1) can be looked at as geodesics in a proper geometric structure.

In the context of the computer modelling techniques via Newton method, it was proved in [19] that the numeric methods used for the trajectories of the Euler-Lagrange extension are contractive, even when the numerical methods used for the field lines of the initial system are convergent or divergent. So, even when the starting flow is dissipative or chaotic, the iterative method used for the Euler-Lagrange prolongation is convergent.

Remark 3.2. The Euler-Lagrange extension (3.3) is equivalent to a system in variations,

$$\frac{d^2\xi^i}{dt^2} = \delta^{ir} \frac{\partial f}{\partial \xi^r} + \delta^{ir} \Omega_{rs} \frac{d\xi^s}{dt}.$$

Proposition 3.1. *The Euler-Lagrange system (3.3) defines the following five KCC differential invariants*

$$\begin{aligned}\varepsilon^i &= \frac{1}{2} \Omega_r^i \dot{x}^r + \frac{\partial f}{\partial x^i}, \\ P_j^i &= \frac{\partial^2 f}{\partial x^i \partial x^j} + \left(\frac{\partial \Omega_r^i}{\partial x^j} - \frac{1}{2} \frac{\partial \Omega_j^i}{\partial x^r} \right) \dot{x}^r + \frac{1}{4} \Omega_r^i \Omega_j^r, \\ R_{jk}^i &= \frac{1}{2} \left(\frac{\partial \Omega_k^i}{\partial x^j} - \frac{\partial \Omega_j^i}{\partial x^k} \right), \\ B_{jkl}^i &= 0, D_{jkl}^i = 0.\end{aligned}$$

Proof. We have $g_{;r}^i = -\delta^{ij} \Omega_{jr}$, so the first KCC invariant of the Euler-Lagrange extension is

$$\varepsilon^i = -\frac{1}{2} \delta^{ij} \Omega_{jr} \dot{x}^r + \delta^{ir} \frac{\partial f}{\partial x^r} + \delta^{ir} \Omega_{rs} \frac{dx^s}{dt} = \delta^{ir} \left(\frac{1}{2} \Omega_{rs} \dot{x}^s + \frac{\partial f}{\partial x^r} \right).$$

Also, in this case, we have

$$g_{,r}^i = -\delta^{ik} \frac{\partial^2 f}{\partial x^k \partial x^r} - \delta^{ik} \frac{\partial \Omega_{kj}}{\partial x^r} \dot{x}^j, g_{,r;j}^i = -\delta^{ik} \frac{\partial \Omega_{kj}}{\partial x^r}, g_{;r;j}^i = 0, \frac{\partial g_{;j}^i}{\partial t} = 0.$$

In conclusion, the deviation curvature tensor of the second order system of ODEs (3.3) is

$$P_j^i = -\delta^{ir} \left[-\frac{\partial^2 f}{\partial x^r \partial x^j} + \left(-\frac{\partial \Omega_{rs}}{\partial x^j} + \frac{1}{2} \frac{\partial \Omega_{rj}}{\partial x^s} \right) \dot{x}^s \right] + \frac{1}{4} \delta^{ik} \Omega_{kr} \delta^{rk} \Omega_{kj}.$$

and the third, the fourth and the fifth KCC invariants of the Euler-Lagrange extension (3.3) become

$$\begin{aligned}R_{jk}^i &= \frac{1}{3} (P_{j;k}^i - P_{k;j}^i) = -\frac{\delta^{ir}}{6} \left(\frac{\partial \Omega_{rj}}{\partial x^k} - \frac{\partial \Omega_{rk}}{\partial x^j} \right), \\ B_{jkl}^i &= R_{jk;l}^i = 0, \\ D_{jkl}^i &= g_{;j;k;l}^i = 0.\end{aligned}$$

Remark 3.4. The equations

$$\frac{D^2 \xi^i}{dt^2} = P_j^i \xi^j,$$

are the Jacobi fields equations associated to the Euler-Lagrange equations (3.3).

In the following, we will state the Jacobi stability theorem for geometric dynamics (around a stationary point generated by a stationary point of the initial flow):

Theorem 3.1. Let $\frac{dx^i}{dt} = X^i(x)$, $i = \overline{1, n}$ be a flow and $x^* = (x^{*1}, \dots, x^{*n})$ be a stationary point of the given flow. Let

$$\frac{d^2x^i}{dt^2} = \delta^{ir} \frac{\partial f}{\partial x^r} + \delta^{ir} \Omega_{rs} \frac{dx^s}{dt}, i = \overline{1, n},$$

the Euler-Lagrange extension with the stationary point $(x^*, \dot{x}^* = 0)$. Then, the matrix whose eigenvalues describe the Jacobi stability of geometric dynamics (3.3) is

$$P = H + \frac{1}{4}\Omega^2$$

or

$$P = \frac{1}{4} \left[J^2 - (J^T)^2 - JJ^T + 3J^T J \right],$$

where $P = (P_j^i(x^*, 0))$ is the matrix associated to the deviation curvature tensor, $H = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(x^*) \right)$ is the Hessian of the energy f , $\Omega = (\Omega_j^i(x^*))$ is the matrix of gyroscopic force and $J = \left(\frac{\partial X^i}{\partial x^j}(x^*) \right)$ is the Jacobi matrix of the initial flow.

Proof. The deviation curvature tensor in a stationary point $(x^*, \dot{x}^* = 0)$ is given by

$$P_j^i(x^*, 0) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x^*) + \frac{1}{4} \Omega_r^i(x^*) \Omega_j^r(x^*), \forall i, j = \overline{1, n}$$

In short,

$$P = H + \frac{1}{4}\Omega^2.$$

Because $f = \frac{1}{2} \sum_{k=1}^n (X^k)^2$, and x^* is a zero of X , we get

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(x^*) = \sum_{k=1}^n \frac{\partial X^k}{\partial x^i}(x^*) \frac{\partial X^k}{\partial x^j}(x^*),$$

or, equivalently $H = J^T J$. Also, we have

$$\begin{aligned} \Omega_r^i(x^*) \Omega_j^r(x^*) &= \frac{\partial X^i}{\partial x^r}(x^*) \frac{\partial X^r}{\partial x^j}(x^*) - \frac{\partial X^i}{\partial x^r}(x^*) \frac{\partial X^j}{\partial x^r}(x^*) \\ &\quad - \frac{\partial X^r}{\partial x^i}(x^*) \frac{\partial X^r}{\partial x^j}(x^*) + \frac{\partial X^r}{\partial x^i}(x^*) \frac{\partial X^j}{\partial x^r}(x^*). \end{aligned}$$

or, in the matriceal form,

$$\Omega^2 = \frac{1}{4} \left[J^2 - JJ^T - J^T J - (J^T)^2 \right].$$

From these we get the deviation curvature tensor P .

4. LINEARIZED GEOMETRIC DYNAMICS

Here, we will give the basic ideas of the stability analysis of the linearized geometric dynamics of a given flow (3.1), following the ideas from [18]. It is well known that the linearization of the system (3.1) around the stationary point 0 has the form

$$\frac{dx}{dt} = Ax, \quad (4.1)$$

where A is the associated Jacobi matrix calculated at 0. The linearized dynamical system (4.1) and the Riemannian metric δ produce the Lagrangian

$$L = \frac{1}{2} \left\| \frac{dx}{dt} - Ax \right\|^2. \quad (4.2)$$

The linearization of the Euler-Lagrange equations (3.3) at the stationary point generated by the initial stationary point, is represented in fact, by the Euler-Lagrange equations for the Lagrangean (4.2), i.e.,

$$\frac{d^2x}{dt^2} = (A - A^T) \frac{dx}{dt} + A^T Ax. \quad (4.3)$$

The stationary point of this system is $x(t) = 0$, $\frac{dx}{dt}(t) = 0$.

Remark 4.1. Obviously, the second order system (3.3) can be linearized also around a nonzero critical point of the density of energy f (which is not necessarily a zero of the vector field).

The solutions of the second order system (4.3) have the form $x = ue^{\lambda t}$, $t \in \mathbb{R}$, where u is a nonzero solution of the linear system

$$(-A^T A - \lambda(A - A^T) + \lambda^2 I) u = 0, \quad (4.4)$$

and λ is a solution (called *latent value*) of the equation

$$\det(\lambda I - A) \det(\lambda I + A^T) = 0.$$

Consequently, if λ is an eigenvalue of the real matrix A , then λ and $-\lambda$ are latent values of λ -matrix in parenthesis, so the sum of all latent values of the λ -matrix is zero.

Let λ be a complex latent value and u be the associated complex latent vector. Multiplying in the left of (4.4) with $\bar{u}^T$ (conjugate transpose of u) we find the second order algebraic equation

$$m\lambda^2 + ig\lambda + n = 0, \quad (4.5)$$

where

$$m = \bar{u}^T u > 0, ig = -\bar{u}^T (A - A^T) u, n = -\bar{u}^T A^T A u < 0.$$

The discriminant of the equation (4.5) is

$$\Delta = -g^2 - 4mn. \quad (4.6)$$

From [18] it is known

Theorem 4.1. a) The ODEs system (4.4) has saddle point properties around the equilibrium point if and only if $\Delta > 0$.

b) If $\Delta < 0$, no saddle point properties exist.

Remark 4.2. Let μ be a parameter of the system. If there is a number μ_0 such that $\Delta(\mu_0) = 0$ and $\Delta(\mu) > 0$ (respectively < 0) for $\mu < \mu_0$ (respectively $\mu > \mu_0$), and $\frac{d\Delta}{d\mu}(\mu_0) \neq 0$ then, when μ passes μ_0 , the system will undergo a bifurcation and loose its saddle point properties.

Let J the Jacobi matrix associated to the initial flow (3.1), at the stationary point x^* . One can notice that the stability of the linearized geometric dynamics (4.3) at $(x^*, \dot{x}^* = 0)$ is given by the structure of the eigenvalues of the associated Jacobi matrix at the equilibrium point x^* , while Jacobi stability of geometric dynamics at the same point is determined by the structure of the eigenvalues of the matrix $\frac{1}{4} [J^2 - (J^T)^2 - JJ^T + 3J^T J]$. So, it is obvious that Jacobi stability of geometric dynamics and the stability of the linearized geometric dynamics leads us to different results.

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