

HOMOTHETIC MOTION OF SUBMANIFOLDS ON THE PLANE IN E^3

YILMAZ TUNÇER[†], M. KEMAL SAĞEL AND YUSUF YAYLI

USAK UNIVERSITY SCIENCE & ART FAC. MATH. DEPT. USAK-TURKEY
E-MAIL:YTUNCERAKU@HOTMAIL.COM

(Received: 29 November 2006)

ABSTRACT. In this study, we obtained an equation of homothetic motion of any regular surface M on its tangent plane at the contact points, along pole curves which are trajectories of instantaneous rotation centers and we gave some remarks for the homothetic motions will be both sliding and rolling at every moments. In addition, we establish a suprising relationship between curvatures of the moving and fixed pole curves.

AMS Classification: 70B14, 53A17, Secondary:53A05, 53A04

Keywords: Homothetic motion, Darboux frame, Darboux vector, pole curve, sliding and rolling.

1. INTRODUCTION

We know that the angular velocity vector has an important role in kinematic of two rigid bodies. Mathematicians and phycisists are interpreted the motion of two rigid body in various ways. K. Nomizu studied the 1-parameter motions of any orientable surface M on the tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations about the angular velocity vector of rolling without sliding in [1]. H. H. Hacisalihoğlu showed some properties of 1-parameter homothetic motions in [3]. In this study, we shall define the homothetic motion of any smooth surface M on its tangent plane and we shall give some results and conditions using any vector fields and Frenet frames along smooth pole curves on M and on tangent plane, for both rolling and sliding motion.

2. PRELIMINARIES

The homothetic motion in Euclidean space of 3–dimensions is generated by the transformation

$$(1) \quad \begin{aligned} F : E^3 &\longrightarrow E^3 \\ X &\longrightarrow Y = hAX + C \end{aligned}$$

where $A \in SO(3)$, $C \in IR_1^3$ and h is a homothetic scale. The elements of A , C and h are continuously differentiable functions of time-dependent parameter t and the elements of X are coordinates of a point in the body. It is well known that all of the homothetic motions includes both rolling and sliding at every moments and they are regular motions[4]. By differentiating both side of (1), we obtain

$$(2) \quad \dot{Y} = hA\dot{X} + (\dot{h}A + h\dot{A})X + \dot{C}$$

where $(\dot{h}A + h\dot{A})X + \dot{C}$ is sliding velocity of X . X is called a pole point if its sliding velocity is vanish and locus of points of (X) called moving pole curve[4]. We take B as hA , then the equation of moving pole curve is $X = -\dot{B}^{-1}\dot{C}$. Substitution X with $X = -\dot{B}^{-1}\dot{C}$ in (1), we obtain fixed pole curve $Y = -B\dot{B}^{-1}\dot{C} + C$. Now we examine the matrix $B\dot{B}^{-1}$.

$$B\dot{B}^{-1} = hA \left(\dot{h}^{-1}A^{-1} + h^{-1}\dot{A}^{-1} \right) = \underbrace{h\dot{h}^{-1}I_3}_{\varphi} + \underbrace{\dot{A}A^{-1}}_S$$

where φ and S is sliding part and rolling part of (1)[5]. For $S \neq 0$, there is a uniquely determined vector $w(t)$ such that $S(U)$ equal to the cross product $w(t) \times U$ for every vector U . The vector $w(t)$ is called the angular velocity vector at instant t and the homothetic motion (1) called is rolling if $w(t)$ lies in the tangent space of the moving curve X and F is spinning if $w(t)$ normal space of the tangent space of X at the contact points of moving and fixed pole curves at instant t [1].

3. SLIDING AND ROLLING OF M ON Σ

Let us consider the orientable surface M and the tangent plane Σ of M at contact points $p \in M$ along moving and fixed smooth pole curves $X(t)$ on M and $Y(t)$ on Σ which are starting at p . We shall take a rectangular coordinate system in E^3 and the unit vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of IR^3 . We wish to roll and slide M on Σ along smooth pole curves $X(t)$ and $Y(t)$. We can define homothetic motion M on Σ as following.

$$(3) \quad \begin{aligned} F : M &\longrightarrow \Sigma \\ X &\longrightarrow Y = BX + C, \quad B = hA \end{aligned}$$

Since $F(M)$ is tangent to Σ at the contact points we can write $Bx = he_3$. Suppose that (X) and (Y) (moving and fixed) pole curves on M and Σ , respectively, and we denote the normal vector fields of M along the curve (X) as ξ . Let b_1 , b_2 and a_1 , a_2 be orthonormal vector fields along (X) and (Y) so that, at the contact points, the relation between these orthonormal systems as follows

$$b_1 = hB^{-1}a_1, b_2 = hB^{-1}a_2$$

Hence $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ will be moving and fixed system for (X) and (Y) , respectively. Let T , N and B are tangent, principal normal and binormal vector fields of (X) , respectively. Since $\xi(t) \in Sp\{N, B\}$ then we can write

$$(4) \quad \xi(t) = \cos \psi(t)N + \sin \psi(t)B$$

We must construct the frames for determine the orthogonal matrix $A\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ along the pole curves (X) and (Y) , respectively. During this operations we make use of Darboux frame along (X) and tangent and principal normal vector field of (Y) at contact points on M and Σ , respectively. we can find an orthogonal matrix Q using the vector fields T , N and B such as (5)

$$(5) \quad \begin{bmatrix} T \\ X\Lambda T \\ X \end{bmatrix} = [Q] \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

It is well known that system of $\{T, N, B\}$ rotates according to e_1, e_2, e_3 , then we can write (6) for $P \in SO(3)$.

$$(6) \quad \begin{bmatrix} T \\ N \\ B \end{bmatrix} = [P] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Let angle between b_1 and T (b_2 and $X\Lambda T$, respectively) be θ . Since $Sp\{b_1, b_2\}$ and $Sp\{T, X\Lambda T\}$ are the same tangent spaces then we can write (7).

$$(7) \quad \begin{bmatrix} b_1 \\ b_2 \\ X \end{bmatrix} = [R] \begin{bmatrix} T \\ X\Lambda T \\ X \end{bmatrix}$$

Using the equalities (5), (6) and (7), we can obtain the matrix $A_1 = [P]^T [Q]^T [R]^T$. The matrix A_1 transforms b_1 to e_1 , b_2 to e_2 and (X) to e_3 , respectively. We denote the skew symmetric matrix $\frac{dA_1^T}{dt} A_1$ as W_1 then W_1 will be follows.

$$(8) \quad W_1 = \begin{bmatrix} 0 & \theta' + k_1 \sin \psi & \begin{Bmatrix} k_1 \cos \theta \cos \psi - \\ (k_2 + \psi') \sin \theta \end{Bmatrix} \\ -\theta' - k_1 \sin \psi & 0 & -\begin{Bmatrix} k_1 \sin \theta \cos \psi - \\ (k_2 + \psi') \cos \theta \end{Bmatrix} \\ -\begin{Bmatrix} k_1 \cos \theta \cos \psi + \\ (k_2 + \psi') \sin \theta \end{Bmatrix} & \begin{Bmatrix} k_1 \sin \theta \cos \psi + \\ (k_2 + \psi') \cos \theta \end{Bmatrix} & 0 \end{bmatrix}$$

Remark 3.1. b_1 and b_2 vector fields are parallel according to the connection of M along curve (X) if and only if $\theta' + k_1 \sin \psi = 0$ is satisfies.

Proof. Let $\overline{D}$ be Levi Civita connection and S_M be shape operator of M . From (4), (5) and (7), we can write $b_1(t)$ as follows.

$$b_1 = \cos \theta T + \sin \theta \sin \psi N - \sin \theta \cos \psi B$$

Using Gauss equation

$$\overline{D}_T b_1 = D_T b_1 + \langle S_M(T), b_1 \rangle \xi$$

we obtain following equation.

$$\overline{D}_T b_1 = -\{\theta' + k_1 \sin \psi\} \sin \theta T + \cos \theta \{\theta' + k_1 \sin \psi\} \{\sin \psi N - \cos \psi B\}$$

It is easily see that $\overline{D}_T b_1 = 0$. Similarly, $\overline{D}_T b_2 = 0$ can proof. $\square$

On the other hand, let us $\{\overline{T}, \overline{N}, e_3\}$ and $\overline{k}$ be orthonormal frame and curvature along the planar pole curve (Y), respectively. We can construct the orthonormal system $\{a_1, a_2, e_3\}$ along (Y) as above. Since e_3 is normal to Σ then we can write (9).

$$(9) \quad \begin{bmatrix} \overline{T} \\ e_3 \Lambda \overline{T} \\ e_3 \end{bmatrix} = [\overline{Q}] \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix}$$

where $[\overline{Q}] = I_3$. Since $\{\overline{T}, \overline{N}, e_3\}$ rotates according to the system $\{e_1, e_2, e_3\}$, for $\overline{P} \in SO(3)$ we obtain (10).

$$(10) \quad \begin{bmatrix} \overline{T} \\ \overline{N} \\ e_3 \end{bmatrix} = [\overline{P}] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Let angle between a_1 and $\overline{T}$ (a_2 and $e_3 \Lambda \overline{T}$, respectively) be $\overline{\theta}$. Since $Sp\{a_1, a_2\}$ and $Sp\{\overline{T}, e_3 \Lambda \overline{T}\}$ are the same space then we can write (11)

$$(11) \quad \begin{bmatrix} a_1 \\ a_2 \\ e_3 \end{bmatrix} = [\overline{R}] \begin{bmatrix} \overline{T} \\ e_3 \Lambda \overline{T} \\ e_3 \end{bmatrix}$$

Using (9), (10) and (11), we obtain the matrix $A_2 = [\overline{P}]^T [\overline{Q}]^T [\overline{R}]^T$. The matrix A_2 transforms a_1 to e_1 , a_2 to e_2 and e_3 to e_3 . We denote the skew symmetric matrix $\frac{dA_2^T}{dt} A_2$ as W_2 so W_2 will be follows.

$$(12) \quad W_2 = \begin{bmatrix} 0 & \overline{\theta}' + \overline{k}_1 & 0 \\ -(\overline{\theta}' + \overline{k}_1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark 3.2. $a_1(t)$ and $a_2(t)$ are parallel vector fields according to the connection of Σ along curve (Y) if and only if $\overline{\theta}' + \overline{k}_1 = 0$ is satisfies.

Proof. Since the shape operator of Σ is $S_\Sigma = 0$, Levi Civita connection of Σ and Riemann connection of E^3 are same connections. Using (9) and (11) we can write

$$a_1 = \cos \overline{\theta} \overline{T} - \sin \overline{\theta} \overline{N}$$

and since $D_{\overline{T}} a_1 = 0$,

$$D_{\overline{T}} a_1 = \frac{da_1}{dt} = -(\overline{\theta}' + \overline{k}_1) \sin \overline{\theta} \overline{T} - (\overline{\theta}' + \overline{k}_1) \cos \overline{\theta} \overline{N}$$

we obtain $\overline{\theta}' + \overline{k}_1 = 0$. Hence, $a_1(t)$ is parallel vector field along (Y). Similarly, we can show that $a_2(t)$ is parallel vector field along (Y) too. $\square$

Therefore, we can find the matrix A using A_1 and A_2 as $A = A_2 A_1^T$ so that A transforms b_1 to a_1 , b_2 to a_2 and (X) to e_3 , respectively. The skew symmetric matrix $S = \frac{dA}{dt} A^T$ is instantaneous rotation matrix and S represents a linear transformation as $S : T_{Y(t)}\Sigma \longrightarrow Sp\{e_3\}$. It is easily see that the matrix $S = A_2(-W_2 + W_1)A_2^T$ corresponds the transformation S . Since the matrix S is skew symmetric, S determines a vector $w \in Sp\{a_1, a_2, \eta\}$. At any contact point $p \in M$, we find

$$w = \left\{ \begin{array}{l} k_1 \cos \psi \sin \theta \\ -(\psi' + k_2) \cos \theta \end{array} \right\} a_1 + \left\{ \begin{array}{l} k_1 \cos \psi \cos \theta \\ +(\psi' + k_2) \sin \theta \end{array} \right\} a_2 - (\bar{\theta}' + \bar{k}_1 - \theta' - k_1 \sin \psi) e_3$$

Remark 3.3. We obtain all of the results section 2 in [8], if we take $h = 1$ and b_1 , b_2 , a_1 and a_2 are parallel vector fields.

In this case, w vector field will be as follows.

$$w = \left\{ \begin{array}{l} k_1 \cos \psi \sin \left(\int_I k_1 \sin \psi dt \right) - (\psi' + k_2) \cos \left(\int_I k_1 \sin \psi dt \right) \\ + \left\{ \begin{array}{l} k_1 \cos \psi \cos \left(\int_I k_1 \sin \psi dt \right) + (\psi' + k_2) \sin \left(\int_I k_1 \sin \psi dt \right) \end{array} \right\} a_2 \end{array} \right\} a_1$$

where I is an integer for the regular curve (X) . It is obvious that $\vec{w}|_p$ lies in the tangent space of the curves (X) and (Y) at contact points p .

Remark 3.4. M slides and rolls on Σ along geodesics.

Remark 3.5. M slides and rolls on Σ along arbitrary (X) and (Y) curves on which are satisfies the conditions $\theta' + k_1 \sin \psi = 0$ and $\bar{\theta}' + \bar{k}_1 = 0$, respectively.

Theorem 3.6. *The transformation $F : M \longrightarrow \Sigma$ is only sliding motion without rolling at the flat points of (X) of the surface M .*

Proof. Let's S_M be shape operator of M along the curve (X) , then we can write

$$S_M \left(\frac{dX}{dt} \right) = \frac{d\xi}{dt}$$

and let p be a flat point on (X) of M . At the p we have

$$S_M \left(\frac{dX}{dt} \Big|_p \right) = 0$$

and differentiating of $B\xi = he_3$ with respect to t we have

$$S(e_3) = -A \frac{d\xi}{dt} \Big|_p$$

where $\frac{d\xi}{dt}$ will be as follow at the points p .

$$\frac{d\xi}{dt} \Big|_p = \underbrace{\left\{ \begin{array}{l} k_1 \cos \theta \cos \psi \\ + (k_2 + \psi') \sin \theta \end{array} \right\}}_{\lambda_1|_p} a_1 \Big|_p - \underbrace{\left\{ \begin{array}{l} k_1 \sin \theta \cos \psi \\ - (k_2 + \psi') \cos \theta \end{array} \right\}}_{\lambda_2|_p} a_2 \Big|_p$$

Finally, we obtain $\lambda_1(p) = 0$ and $\lambda_2(p) = 0$ so that $S = 0$. Consequently, rolling part of F will be vanish. Hence the homothetic motion F is sliding without rolling at the flat points on (X) of M . $\square$

We can give an important following theorem.

Theorem 3.7. *Let (X) be a smooth pole curve on a surface M which does not go through a flat point of M . There exists a unique homothetic motion of M on its tangent plane at $p = X(t_0)$ such that $Y(t) = F(X(t))$ is the locus of points of contact.*

We can give following theorem as the main result of this study.

Theorem 3.8. *Let $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are any vector field systems along the curves (X) and (Y) , respectively. Hence $F : M \longrightarrow \Sigma$ is a homotetic motion if and only if*

$$\bar{\theta}' + \bar{k}_1 - \theta' - k_1 \sin \psi = 0$$

satisfies.

Thus, we obtained the condition of rolling of any regular surface in E^3 upon its tangent plane, along the centers of instantaneous points which are called trajectory of contact points of moving curve (X) and the fixed curve (Y) on moving regular surface and on its tangent plane, respectively.

Remark 3.9. If M is a unit sphere then in this case $\psi' + k_2 = 0$, the angular velocity vector will be as follows.

$$w_p = k_1 \cos \psi \sin \theta a_1|_p + k_1 \cos \psi \cos \theta a_2|_p - (\bar{\theta}' + \bar{k}_1 - \theta' - k_1 \sin \psi) e_3|_p$$

Unit sphere sliding and rolling on plane if and only if

$$\bar{\theta}' + \bar{k}_1 - \theta' + k_1 \sin \left(\int_I k_2 dt \right) = 0$$

where I is an integer for the regular spherical curve (X) .

Example: Let $X(t) = \left(\sin \left(\frac{t}{\sqrt{2}} \right), \frac{t}{\sqrt{2}}, 1 - \cos \frac{t}{\sqrt{2}} \right)$, $t \in [0, \pi]$ is unit speed curve on $x^2 + (1 - z)^2 = 1$ and $Y(t) = \left(\frac{t^2}{2}, 0, 0 \right)$ is any curve on Σ . We obtain

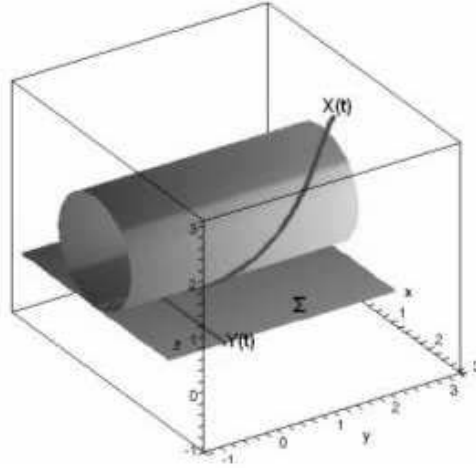
$$T = \frac{1}{\sqrt{2}} \left(\cos \left(\frac{t}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}} \right), \sin \left(\frac{t}{\sqrt{2}} \right) \right), N = \left(-\sin \left(\frac{t}{\sqrt{2}} \right), 0, \cos \left(\frac{t}{\sqrt{2}} \right) \right), \\ B = \frac{1}{\sqrt{2}} \left(\cos \left(\frac{t}{\sqrt{2}} \right), -1, \sin \left(\frac{t}{\sqrt{2}} \right) \right), k_1 = \frac{1}{2}, k_2 = -\frac{1}{2}, \psi = \pi$$

and

$$\bar{T} = (1, 0, 0), \bar{N} = (0, 1, 0), \bar{B} = (0, 0, 1), \bar{k}_1 = 0, \bar{k}_2 = 0$$

for (X) and (Y) curves, respectively. Since $\|dY/dt\| = h$ we find $h = t$ and using $\frac{dY}{dt} = B \frac{dX}{dt}$ we obtain $\bar{\theta}(t) = \theta = \frac{\pi}{4}$ so the motion will be,

$$(13) \quad Y(t) = \begin{bmatrix} \frac{t}{\sqrt{2}} \cos\left(\frac{t}{\sqrt{2}}\right) & \frac{t}{\sqrt{2}} & \frac{t}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \\ \frac{t}{\sqrt{2}} \cos\left(\frac{t}{\sqrt{2}}\right) & -\frac{t}{\sqrt{2}} & \frac{t}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \\ t \sin\left(\frac{t}{\sqrt{2}}\right) & 0 & -t \cos\left(\frac{t}{\sqrt{2}}\right) \end{bmatrix} X(t) + \begin{bmatrix} -\frac{t}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \\ \frac{t^2}{2} - \frac{t}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \\ -t + t \cos\left(\frac{t}{\sqrt{2}}\right) \end{bmatrix}$$



the matrix S and the vector W will

$$S = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

and

$$(14) \quad W|_P = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$$

respectively. Both (X) and (Y) are the moving and fixed pole curves according to the motion (12) and the vector field given (13) lies in both $T_{X(t)}(M)$ and Σ at the contact points of the curves and it satisfies the condition given theorem (2.8). Thus the motion (12) is a homothetic motion.

REFERENCES

- [1] Appell, P., *Traite de Mecanique Rationnelle*, Tome I, Gauthiers-Villars, Paris, 1919.
- [2] Cheng, M. G., *Hypersurfaces in Euclidean spaces*, Proc. Fifth Pacific Rim Geom. Conference, Tohoku University, Sensai, Japan. Tohoku Math. Publ. 20(2001), 33-42.
- [3] Clifford, W. And McMahon, J.J., *The Rolling Of One Curve or manifold Upon Another*, Am. Math. Mon. 68, 338-341, 23A 2134, 1961.
- [4] Hacsalihoğlu, H.H., *On The Rolling Of One Curve or manifold Upon Another*, Proceedings Of The Royal Irish Academy, Vol. 71, Sec. A, Number 2, Dublin, 1971.

- [5] Karakas, B., On Differential Geometry and Kinematics of Submanifolds, Phd Thesis, Atatrk University, Ankara, Turkey, 1982.
- [6] Karger, A., Novak, J., Space Kinematics And Lie Grups, Gordon and Breach Science Publishers, Prague, Czechoslovakia, 1978.
- [7] Muller, H. R., Zur Bewegungssgeometrie In Raumen Hherer Dimension, Mh. Math. 70 Band, 1 Heft. pp. 47-57, 1966.
- [8] Nomizu, K., "Kinematics And Differential Geometry Of Submanifolds", Tohoku Math. Journ. 30.(1978), 623-637.
- [9] Yayli, Y. , Hamilton Motions and Lie Grups, Phd Thesis, Gazi University Science Institute, Ankara, Turkey, 1988.