STRUCTURAL CONSIDERATION OF ISOTHEORY

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Abstract. Santilli's extension of Lie bracket was the reason of isotheory. In this paper by the use of isomanifold, and top isospaces the Lie-Santilli algebra from isotheory is deduced. The notion of top isospaces as a special class of isomanifolds is studied. It is proved that: the set of left invariant isovector fields on a top isospaces with finite numbers of identities, is a Lie-Santilli algebra.

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1. Introduction

One of the critical problem in differential geometry and mathematical physics is: constructing one metric from another one on a differentiable manifold. This will be an open problem in the case of metrics with different signatures [2]. In this case another approach to the problem of unity presented by Santilli [9, 10, 13, 11, 12]. His suggestion was replacing "manifolds" with "isomanifolds". Santilli isotopies which are axiom preserving maps [6, 9, 11, 12, 14, 15] have essential roles in his mechanism.

A nonempty set S with a binary operation $\hat{\times}$ is called a set of Santilli isounites if satisfies the following conditions:

i) There is $1 \in S$ such that $s\hat{\times}1 = 1\hat{\times}s = s$ for all $s \in S$;

ii) For each $s \in S$ there is $s^{-1} \in S$ such that $s\hat{\times}s^{-1} = s^{-1}\hat{\times}s = 1$.

Each element of S is called a Santilli isounit.

If $(F, +, \times)$ is a field and $\hat{I} \in S$, then $\hat{F} = \{r \times \hat{I} : r \in F\}$ with the product $\hat{\times} = \times\hat{I}^{-1}\times$ and addition $\hat{+} = +$ is a field which is called an isofield [9]. We also use of the notation $F(\hat{I})$ for $\hat{F}$.

Santilli's isounits are means for constructing isomorphic fields. This method can

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apply to vector spaces. In fact a vector space $(\hat{V}, \hat{\odot}, \hat{+})$ is an isovector space on the isofield $\hat{F}$ [8], corresponding to the vector space V on the field F, if there is a one to one and onto map

$$\begin{array}{rcl} h_V : V & \rightarrow & \hat{V} \\ v & \longmapsto & h_V(v) \end{array}$$

such that for all v, $w \in V$ and $a \in F$,

$$(a \times \hat{I}) \hat{\odot} h_V(v) = h_V(a \odot v) \text{ and } h_V(v) \hat{+} h_V(w) = h_V(v + w)$$

where $\hat{I}$ is the Santilli isounit of $\hat{F}$.

If (V, F) is a Banach space then $(\hat{V}, \hat{F})$ is called a Banach-Santilli isospace [6] corresponding to (V, F).

If $\hat{M}$ is a Hausdorff space, $\hat{\mathcal{A}} = \{(\hat{U}_\gamma, \hat{\varphi}_\gamma) : \gamma \in \Gamma\}$ and $\{\hat{V}_\gamma : \gamma \in \Gamma\}$ is a set of Banach-Santilli isospaces on $\hat{F}$ corresponding to the Banach space V on F, then $(\hat{M}, \hat{\mathcal{A}}, V)$ is called a ($\hat{C}^r$ isodifferentiable) isomanifold [5, 8, 15] if

i) $\{\hat{U}_\gamma : \gamma \in \Gamma\}$ is an isoopen covering of $\hat{M}$;

ii) For each $\gamma \in \Gamma$, $\hat{\varphi}_\gamma : \hat{U}_\gamma \rightarrow \hat{\varphi}_\gamma(\hat{U}_\gamma)$ is a homeomorphism where $\hat{\varphi}_\gamma(\hat{U}_\gamma)$ is an open subset of $\hat{V}_\gamma$;

iii) $h_\beta^{-1} o \hat{\varphi}_\beta^{-1} o \hat{\varphi}_\alpha o h_\alpha : h_\alpha^{-1}(\hat{U}_\alpha \cap \hat{U}_\beta) \rightarrow h_\beta^{-1}(\hat{U}_\alpha \cap \hat{U}_\beta)$ is a C^r mapping, where $h_\beta = h_{V_\beta}$ and $h_\alpha = h_{V_\alpha}$.

In isotheory the identities of isofields for local charts have essential roles for constructing isotopies. To extend this theory even for points of $\mathbf{R}$, then one method is: to associate an identity to each element of $\mathbf{R}$. This method was a reason for us to introduce generalized groups [3]. In fact a non-empty set G admitting an operation called multiplication, is called a generalized group or completely simple semigroup [1] if it satisfies the following axioms:

(i) $(xy)z = x(yz)$ for all x, y, z in G (associative law);

(ii) For each x in G there exists a unique z in G such that $xz = zx = x$, we denote z by $e(x)$ (existence and uniqueness of identity);

(iii) For each x in G there exists y in G such that $xy = yx = e(x)$ (existence of inverse).

We are going to introduce a new class of isomanifold by the use of a class of completely simple semigroups. In the third section a method for constructing new Lie-Santilli algebras is presented.

2. ISOTOPIC FORM OF TOP SPACES

A normal topological generalized group $(T, .)$ [4, 6] is called a top space [7] if:

i) The topological space T is a smooth manifold of dimension t;

ii) The mapping $m_1 : T \rightarrow T$ defined by $m_1(u) = u^{-1}$ and the mapping $m_2 : T \times T \rightarrow T$ defined by $m_2(u_1, u_2) = u_1 u_2$ are smooth maps.

Clearly each Lie group is a top space.

Definition 2.1. *An isosmooth isomanifold $\hat{T}$ is called a top isospace if there is a binary operation $\hat{.}$ on $\hat{T}$ such that $(\hat{T}, \hat{.})$ is a normal completely simple semigroup and the mapping $\hat{m}_1 : \hat{T} \rightarrow \hat{T}$ defined by $\hat{m}_1(\hat{u}) = \hat{u}^{-1}$ and the mapping $\hat{m}_2 : \hat{T} \times \hat{T} \rightarrow \hat{T}$ defined by $\hat{m}_2(\hat{u}_1, \hat{u}_2) = \hat{u}_1 \hat{.} \hat{u}_2$ are isosmooth maps.*

Theorem 2.2. *If $(\hat{T}, .)$ is a top isospace, then there is a top space $(T, .)$ and an isotopy $f : T \to \hat{T}$ such that $\hat{T}$ is an isomanifold corresponding to T with the isotopy f and $f(a.b) = f(a)\hat{.}f(b)$ for all $a, b \in T$.*

Proof. Since $\hat{T}$ is an isomanifold then there is a manifold T with a one to one and onto isotopy $f : T \to \hat{T}$ such that $\hat{T}$ is an isomanifold corresponding to T. If we define the binary operation . on T by $a.b = f^{-1}(a)\hat{.}f^{-1}(b)$ then $(T, .)$ satisfies the conditions of theorem.□

Example 2.3. *The isomanifold $\hat{T} = \mathbf{R}(\hat{I}_1) \times \mathbf{R}(\hat{I}_2) \times ... \times \mathbf{R}(\hat{I}_n)$ with the isounits $\hat{I}_1, \hat{I}_2, ..., \hat{I}_n$ and the product*

$$((x^1 \times \hat{I}_1, \ldots, x^n \times \hat{I}_n), (y^1 \times \hat{I}_1, \ldots, y^n \times \hat{I}_n)) \longmapsto$$

$$(\frac{nx^1 + \sum_{i=1}^{n} y^i}{n} \times \hat{I}_1, \ldots, \frac{nx^n + \sum_{i=1}^{n} y^i}{n} \times \hat{I}_n)$$

is a top isospace corresponding to the top space $\mathbf{R}^n$ with the product

$$((x^1, \ldots, x^n), (y^1, \ldots, y^n)) \longmapsto (\frac{nx^1 + \sum_{i=1}^{n} y^i}{n}, \ldots, \frac{nx^n + \sum_{i=1}^{n} y^i}{n}).$$

Theorem 2.4. *The mapping $\hat{e} : \hat{T} \to \hat{T}$ which assigns to each $\hat{u}$ of $\hat{T}$ the identity element $\hat{e}(\hat{u})$ is an isosmooth mapping.*

Proof. There exists a top space $(T, .)$ such that $(\hat{T}, \hat{.})$ is a top isospace corresponding to it with the one to one and onto isotopy $f : T \to \hat{T}$. Since the diagram

$$\begin{array}{ccc} T & \xrightarrow{e} & T \\ f \downarrow & & \downarrow f \\ \hat{T} & \xrightarrow{\hat{e}} & \hat{T} \end{array}$$

commutes and e is a smooth map [6] then $\hat{e}$ is an isosmooth map.□

3. ISOVECTOR FIELDS

Let $(\hat{M}, \hat{\mathcal{A}})$ be an isosmooth isomanifold corresponding to a smooth manifold M. Then an isocurve [8] $\hat{\alpha} : (-\epsilon, \epsilon) \to \hat{M}$ is isodifferentiable if $\hat{\varphi} o \hat{\alpha} : (-\epsilon, \epsilon) \to \hat{U}$ is isodifferentiable, for all $(\hat{U}, \hat{\varphi}) \in \hat{\mathcal{A}}$ with $\hat{\alpha}(-\epsilon, \epsilon) \cap \hat{U} \neq \emptyset$.
The isotangent vector corresponding to $\hat{\alpha}$ at $\hat{p} = \hat{\alpha}(0)$ is the set of isodifferentiable isocurves $\hat{\beta} : (-\epsilon, \epsilon) \to \hat{M}$ such that $\hat{\beta}(0) = \hat{p}$ and $\hat{d}(\hat{\varphi} o \hat{\beta})_0 = \hat{d}(\hat{\varphi} o \hat{\alpha})_0$. This isotangent vector denoted by $[\hat{\alpha}]$. The set $T\hat{M}_{\hat{p}} = \{[\hat{\alpha}] \mid \hat{\alpha}$ is isodifferentiable and $\hat{p} = \hat{\alpha}(0)\}$ with the addition $[\hat{\alpha}] + [\hat{\beta}] = [\hat{\gamma}]$, where $\hat{\gamma} = \hat{\varphi}^{-1} o \eta$ and $\eta : (-\epsilon, \epsilon) \to \hat{V}$ defined by $\eta(t) = \hat{t} \hat{\times} (\hat{d}(\hat{\varphi} o \hat{\alpha})_0 + \hat{d}(\hat{\varphi} o \hat{\beta})_0) + \hat{\varphi}(\hat{p})$, and isoscalar isomultiplication

$$\begin{array}{rcl} \hat{\odot} : \hat{F} \times T\hat{M}_{\hat{p}} & \longrightarrow & T\hat{M}_{\hat{p}} \\ (\hat{c}, [\hat{\alpha}]) & \longmapsto & [\hat{c\alpha}] \end{array}$$

where $\hat{c\alpha} : (-\epsilon, \epsilon) \to \hat{M}$ is defined by $\hat{c\alpha} = \hat{\varphi} o \theta$, and $\theta(t) = \hat{t} \hat{\times} \hat{c} \hat{\times} \hat{d}(\hat{\varphi} o \hat{\alpha})_0 + \hat{\varphi}(\hat{p})$, is an isovector space on $\mathbf{R}(\hat{I})$. The mapping $g : TM_p \to T\hat{M}_{\hat{p}}$ defined by $g([\alpha]) =$

$[h_V^{-1} o\varphi o\hat{\alpha}]$ is a Santilli isotopy, and the following diagram

$$\begin{array}{ccc} \mathbf{R}(\hat{I}) \times T\hat{M}_{\hat{p}} & \xrightarrow{\hat{\odot}} & T\hat{M}_{\hat{p}} \\ (i \times \hat{I}) \times g \uparrow & & \uparrow g \\ \mathbf{R} \times TM_p & \xrightarrow{\odot} & TM_p \end{array}$$

commutes, where $i \times \hat{I} : \mathbf{R} \to \mathbf{R}(\hat{I})$ is the map $r \mapsto r \times \hat{I}$. Thus $(T\hat{M}_{\hat{p}}, +, \hat{\odot})$ is an isovector space corresponding to the vector space $(TM_p, +, \hat{\odot})$. So we can consider $\bigcup_{\hat{p}} T\hat{M}_{\hat{p}}$ as an isomanifold corresponding to the tangent bundle of M.

Definition 3.1. *An isosmooth isovector field on $\hat{M}$ is an isosmooth map $\hat{X} : \hat{M} \to \bigcup_{\hat{p}} T\hat{M}_{\hat{p}}$ such that $\hat{X}(\hat{p}) \in T\hat{M}_{\hat{p}}$ for all $\hat{p} \in \hat{M}$.*

We now consider isovector fields on top isospaces to create new Lie-Santilli algebra.
An isovector field $\hat{X}$ on a top isospace $\hat{T}$ is called a left invariant isovector field if $(l_{\hat{g}})_{*\hat{p}}(\hat{X}(\hat{p})) = (\hat{X} \circ l_{\hat{g}})(\hat{p})$ where $\hat{g}, \hat{p} \in \hat{T}$ and $l_{\hat{g}} : \hat{T} \to \hat{T}$ is the map $l_{\hat{g}}(\hat{x}) = \hat{g}\hat{.}\hat{x}$, and $(l_{\hat{g}})_{*\hat{p}}$ is the isoderivative of $l_{\hat{g}}$ at $\hat{p}$.

Theorem 3.2. *If $\hat{T}$ is a top isospace and the cardinality of $\hat{e}(\hat{T})$ is finite, then the set of left invariant isovector fields on $\hat{T}$ is a Lie-Santilli algebra.*

Proof. There exist a top space T and isotopies $f : T \to \hat{T}$, $g : T(T) \to \bigcup_{f(p)} T\hat{T}_{f(p)}$ such that for given isovector field $\hat{X}$ on $\hat{T}$ there is a unique vector field X on T so that $\hat{X}(f(p)) = g(X(p))$. $\hat{X}$ is a left invariant isovector field if and only if X is a left invariant vector field. Since the set of left invariant vector fields on the top space T is a Lie-algebra with the Lie bracket operation, then this operation creates a Lie-Santilli operation on the set of left invariant isovector fields.$\square$

4. CONCLUSION

The extension of Lie brackets by Santilli was the motivation of isotheory. In the other word

$$A\ collection\ of\ Lie - Santilli\ brackets \Rightarrow Isotheory.$$

In this paper we showed that:

$$Isotheory \Rightarrow A\ class\ of\ Lie - Santilli\ brackets.$$

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