

STRONG EXPANSIVE HOMEOMORPHISMS HAVE NO ALMOST LYAPUNOV STABLE POINTS

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(Received: 12 April 2005)

ABSTRACT. Here, we obtain some result about strong expansive homeomorphisms on locally connected compact metric space X without any isolated point. In particular, we prove that strong expansive homeomorphisms have no almost Lyapunov stable point.

AMS Classification: 54H20; 54F50

Keywords: Strong expansive homeomorphisms, almost Lyapunov stable point

1. INTRODUCTION

One of the nice topological property of a hyperbolic set is expansivity. In fact, it is a property shared by a large class of dynamical systems exhibiting chaotic behaviour. Roughly, speaking an expansive dynamics is one in which two different trajectories can be distinguished by an observer with an instrument capable to distinguish points of a distance greater than a certain constant $\alpha > 0$, which is called the constant of expansiveness. The notion of expansivity was introduced by Utz in [4] with the term of unstable homeomorphisms. Lewowicz in [3] proved that the local stable and unstable sets for expansive homeomorphisms are arcwise connected and furthermore, they consists of a finite union of arcs. This leads to some characterization for expansive homeomorphisms. By using of the results in [1], Hertz found that the expansive homeomorphisms admit no stable points and that, there are nontrivial, uniformly sized, connected pieces within both the local stable and unstable sets of each point, see [3]. Also, he proved that any Lyapunov stable point of a expansive homeomorphism is periodic. Here, we introduce a stronger notion of expansivity, call it strong expansivity, which can be used for describing the

chaotic dynamics better than expansivity. One of the most important systems exhibit strong expansivity, is Lorenz attractor, called butterfly effect. Also, we define a weaker kind of stable points, which are almost Lyapunov stable points and prove that strong expansive homeomorphisms has no almost Lyapunov stable point

Before, we introduce some common notations.

Let (X, d) be a metric space and f be a homeomorphism on it. Put $\mathcal{O}_{n \leq m}(x) = \{x, f(x), \dots, f^m(x)\}$ and $\mathcal{O}_{|k| \leq n}(x) = \{f^{-n}(x), f^{-n+1}(x) \dots, f^n(x)\}$

A homeomorphism f on the compact metric space (X, d) is said to be *strong expansive*, if there exists an $\alpha > 0$ such that, for any $x \neq y$, there exists $N \in \mathbb{N}$ such that

$$d(\mathcal{O}_{|n| \leq N}(x), \mathcal{O}_{|n| \leq N}(y)) > \alpha$$

α is called the expansivity constant for f . we say that f is *expansive*, if there exists an $\alpha > 0$ such that, for any $x \neq y$, there is an integer n , such that, $d(f^n(x), f^n(y)) > \alpha$. strong expansive homemorphisms are special case of expansive homemorphisms.

We say that $x \in X$ is *almost Lyapunov stable* if for any $\epsilon > 0$ there exist a $\delta > 0$ such that $d(x, y) < \delta$ implies that for all $N \in \mathbb{N}$,

$$d(\mathcal{O}_{n \leq N}(x), \mathcal{O}_{n \leq N}(y)) < \epsilon$$

A point x is Lyapunov stable point for f , if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$, implies that, for any positive integer n , $d(f^n(x), f^n(y)) < \epsilon$. Almost Lyapunov stability is an extended notion of Lyapunov stability. In [1], Hertz proved that a expansive homeomorphism on compact locally connected metric space X without any isolated point has no Lyapunov stable point and so a strong expansive homeomorphism doesn't too. Here, we obtain stronger result for strong expansive homeomorphism. We prove the nonexistence of one weaker kind of Lyapunov stable points, call them almost Lyapunov stable points, for strong expansive homeomorphisms.

Before stating our results precisely, we introduce some definitions. Let X be a compact metric space, endowed with a metric d and $\mathcal{F}X$ composed of all the nonempty closed subsets of X . Let A and B be in $\mathcal{F}X$, put

$$d_H(A, B) = \inf\{\alpha > 0; A \subset U_\alpha(B) \text{ and } B \subset U_\alpha(A)\},$$

where $U_\alpha(\cdot)$ denotes the α -ball measured in d . d_H is a complete metric on $\mathcal{F}X$, called the Hausdorff metric (see [2]). Here the main goal of us is proving of the following fact.

From now on, for simplicity, we denote d_H by d too.

Main theorem: *Let (X, d) be a locally connected compact metric space without any isolated point(In particular X can be a topological manifold.) and f be a strong expansive homeomorphism on it, then f has no almost lyapunov stable point.*

2. SOME PRELIMINARY FACTS

In this section, we assume that f is strong expansive homeomorphism on X .

Given point $x \in X$, we put $\omega(x, f) = \{\lim f^{n_k}(x); n_k \rightarrow \infty\}$ and $\alpha(x, f) = \omega(x, f^{-1})$. We say that $x \in X$ is ω -recurrent (or α -recurrent), if $x \in \omega(x, f)$ (resp. $x \in \alpha(x, f)$). We define almost stable set of a almost Lyapunov stable point as follows.

Definition 2.1. For any almost Lyapunov stable point $x \in X$ and each $\epsilon > 0$, put

$$A_\epsilon = \{y \in M; d(\mathcal{O}_{n \leq N}(x), \mathcal{O}_{n \leq N}(y)) \leq \epsilon \text{ for all } N \in \mathbb{N}\}$$

Remark 2.2. $x \in X$ is almost Lyapunov stable if and only if for any $\epsilon > 0$, x is an interior point of A_ϵ .

In the following we show that any strong expansive homomorphism has a uniform expansiveness property.

Theorem 2.3. *If f is strong α -expansive then, for any $\epsilon > 0$, there exist N_ϵ such that, for any $x, y \in M$*

$$d(x, y) \geq \epsilon \implies \sup_{n \leq N_\epsilon} d(\mathcal{O}_{|m| \leq n}(x), \mathcal{O}_{|m| \leq n}(y)) > \alpha$$

Proof. By contradiction, suppose that there are $\epsilon_0 > 0$ and a sequence $N_m \in \mathbb{N}$, x_m and y_m in M , such that $d(x_m, y_m) \geq \epsilon_0$ and

$$\sup_{n \leq N_m} d(\mathcal{O}_{|m| \leq n}(x_m), \mathcal{O}_{|m| \leq n}(y_m)) \leq \alpha.$$

Let $x_m \rightarrow x$ and $y_m \rightarrow y$, so $d(x, y) \geq \epsilon$. On the other hand, for any $N \in \mathbb{N}$, $d(\mathcal{O}_{|n| \leq N}(x), \mathcal{O}_{|n| \leq N}(y)) \leq \alpha$ and thus by strong expansivity of f , $x = y$. Which is contradiction. $\square$

Now, let $y \in A_\alpha(x)$. Assume that $\epsilon > 0$ be given. We have $d(\mathcal{O}_{n \leq N}(x), \mathcal{O}_{n \leq N}(y)) \leq \alpha$. Now let $n \geq N_\epsilon$ then

$$\sup_{N \leq N_\epsilon} d(\mathcal{O}_{m \leq N}(f^n(x)), \mathcal{O}_{m \leq N}(f^n(y))) \leq \alpha$$

and so by the previous theorem, $d(f^n(x), f^n(y)) < \epsilon$. Thus for any $n \geq N_\epsilon$

$$(1) \quad f^n(A_\alpha(x)) \subseteq B_\epsilon(f^n(x)).$$

In particular for any $y \in A_\alpha(x)$,

$$(2) \quad \omega(x) = \omega(y).$$

Theorem 2.4. *An almost Lyapunov stable point which is recurrent is periodic.*

Proof. Since $A_\alpha(x)$ contain an open neighborhood of x , so there is a $0 < \delta < \alpha$ such that, $B_\delta(x) \subseteq A_\alpha(x)$. If $s \geq N_{\delta/4}$ be large enough such that, $f^s(x) \in B_{\delta/4}(x)$, then

$$f^s(B_\delta(x)) \subseteq f^s(A_\alpha(x)) \subseteq B_{\delta/4}(f^s(x)) \subseteq B_{\delta/2}(x).$$

Thus $f^s(B_\delta(x)) \subseteq B_{\delta/2}(x)$ and therefore we have a periodic point z in $B_{\delta/2}(x) \subseteq A_\delta(x) \subseteq A_\alpha(x)$. By (2), $x \in \omega(x) = \omega(z) = \mathcal{O}(z)$ and hence, x is periodic. $\square$

Remark 2.5. The point x is almost Lyapunov stable if and only if A_α contain an open neighborhood of x .

Clearly if x is almost Lyapunov stable point, then $A_\alpha(x)$ contain an open neighborhood of x . Conversely, let $A_\alpha(x)$ contain an open neighborhood of x and $\epsilon > 0$ be given. Let $\delta > 0$ be such that $B_\delta(x) \subseteq A_\alpha(x)$. Now by (1), for any $s \geq N_\epsilon$, $f^s(B_\delta(x)) \subseteq f^s(A_\alpha(x)) \subseteq B_\epsilon(f^s(x))$. So for sufficiently small δ and any $y \in B_\delta(x)$, $y \in A_\epsilon$.

Theorem 2.6. *The set of almost Lyapunov stable points is open.*

Proof. Let x be an almost Lyapunov stable point and let $\delta \leq \alpha/2$. If $y, z \in \text{int}(A_\delta(x))$, then $z \in A_\alpha(y)$ and so $\text{int}(A_\delta(x)) \subseteq A_\alpha(y)$. Thus $y \in \text{int}(A_\delta(x)) \subseteq A_\alpha(y)$. By remark 2.5, y is almost Lyapunov stable point and thus $\text{int}(A_\delta(x))$ is neighborhood of x which consist of almost Lyapunov stable points. $\square$

Finally we prove the main theorem of this article.

3. PROOF OF THE MAIN THEOREM

By the previous results, it is enough to show that x is recurrent. Let $z \in \alpha(x)$. First, we obtain $\delta > 0$ such that for any $k \geq 0$,

$$B_\delta(z) \subseteq A_\alpha(f^{-n_k}(x))$$

In which $z = \lim_{k \rightarrow \infty} f^{-n_k}(x)$. By contradiction, assume that there is no such an δ . So $f^{-n_k}(x) \in B_{1/k}(z)$ and $B_{1/k}(z) \not\subseteq A_\alpha(f^{-n_k}(x))$. But $f^{-n_k}(x) \in B_{1/k}(z)$ and thus $f^{-n_k}(x) \in A_\alpha(f^{-n_k}(x)) \cap B_{1/k}(z)$. Since f is continuous and $B_{1/k}$ is connected, so there is $y_k \in M$ such that $f^{-n_k}(y_k) \in B_{1/k}(z) \cap \partial(A_\alpha(f^{-n_k}(x)))$. By the continuity of f , there is a sequence of even positive integers N_k with the following property

$$(3) \quad d(\mathcal{O}_{|n| \leq N_k} f^{-n_k}(y), \mathcal{O}_{|n| \leq N_k} f^{-n_k}(x)) = \alpha$$

and for any $m < N_k$

$$d(\mathcal{O}_{|n| \leq N_k} f^{-n_k}(y), \mathcal{O}_{|n| \leq N_k} f^{-n_k}(x)) < \alpha.$$

Put $s_k = \frac{N_k}{2}$. Let $f^{s_k - N_k}(y_k) \rightarrow y^*$ and $f^{s_k - N_k}(x) \rightarrow x^*$. Therefore, for any $N \in \mathbb{N}$, we have

$$d(\mathcal{O}_{|n| \leq N}(x^*), \mathcal{O}_{|n| \leq N}(y^*)) \leq \alpha$$

By the expansiveness of f , $x^* = y^*$. On the other hand

$$\begin{aligned} \mathcal{O}_{|n| \leq N_k}(f^{-n_k}(y_k)) &= \mathcal{O}_{|n| \leq s_k}(f^{s_k - N_k}(y_k)) \rightarrow \overline{\mathcal{O}(y^*)} \\ \mathcal{O}_{|n| \leq N_k}(f^{-n_k}(x)) &= \mathcal{O}_{|n| \leq s_k}(f^{s_k - N_k}(x)) \rightarrow \overline{\mathcal{O}(x^*)} \end{aligned}$$

which is contradiction to (3).

Now for any $\epsilon > 0$, we can choose $n_k > m_k > N_\epsilon$ such that,

$$f^{-n_k}(x) \in B_\delta(z) \subseteq A_\alpha(f^{-m_k}(x))$$

Using (1), $f^{n_k - m_k}(x) \in B_\epsilon(x)$ and so $x \in \omega(x)$.

Therefore, if x is an almost Lyapunov stable point then it must be periodic. Now let δ be small enough so that $d(x, y) < \delta$ implies that $y \in A_\alpha(x)$. For sufficiently near y , we can assume y be almost Lyapunov stable and so periodic. By (2),

$$\mathcal{O}(y) = \omega(y) = \omega(x) = \mathcal{O}(x)$$

and so x must be an isolated point, contradiction by assumption on X .

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