

ON THE ARITHMETIC DIFFERENCE OF AFFINE CANTOR SETS

B. HONARY, M. POURBARAT AND M.R. VELAYATI

DEPARTMENT OF MATHEMATICS
SHAHID BEHESHTI UNIVERSITY, TEHRAN, IRAN
E-MAIL: M-POURBARAT@CC.SBU.AC.IR

(Received: 05 December 2005)

ABSTRACT. In this paper we construct a recurrent compact set for the relative configurations set of special affine Cantor sets. This will produce a class of examples of Cantor sets such that their arithmetic difference contains an interval, while their thickness product is less than one.

AMS Classification: 58F14, 58F15

Keywords: Affine cantor set, Recurrent compact set, Thickness

INTRODUCTION

Regular Cantor sets play a fundamental role in dynamical systems and number theory. They usually appear when the stable and unstable manifolds of hyperbolic points of a surface diffeomorphism intersect basic sets [5]. The stable intersection of stable and unstable manifolds supports prevalence of non hyperbolic systems, and they have close relation with arithmetic difference of Cantor sets [1, 3]. A conjecture of J. Palis implies that the arithmetic difference of two affine Cantor sets either has measure zero or else contains intervals [4]. C. G. Moreira and J. Yoccoz have given an affirmative answer for generic pairs of regular Cantor sets [2]. To pave the way for a similar result for the difference of affine Cantor sets the main challenge is to prove the existence of recurrent compact set of relative configurations. The objective of this work is constructing a non empty recurrent compact set of relative configurations for affine Cantor sets. Then by a proposition of [2] we have; Any relative configuration contained in a recurrent compact set is stably intersecting.

In section 1 we present necessary definitions and symbols.

In section 2 we construct a homeomorphism between the space of relative configurations $\mathcal{C}$ and $\mathcal{S} \times \mathbb{R}$ (theorem A), and we transfer renormalization operators from space $\mathcal{C}$ to space $\mathcal{S} \times \mathbb{R}$.

C.G. Moreira has given example of Cantor sets K, K' with $T(K).T(K') < 1$ with their arithmetic difference contains an interval [1]. Here in section 3 using a theorem in [2] and renormalization operators on the space $\mathcal{S} \times \mathbb{R}$, we provide a

class of such examples of Cantor sets. In fact we construct a recurrent set of the relative configurations of affine Cantor sets with two expansive maps under special conditions (theorem B).

We are grateful to C.G. Moreira for his useful suggestions and his valuable advises.

1. BASIC DEFINITIONS

We mainly use notations similar to those of [2] which are restated here. We say that a Cantor set K is (regular) dynamically defined if:

- i) There are disjoint compact intervals $K_1, K_2, \dots, K_r$ such that $K \subseteq K_1 \cup \dots \cup K_r$ and the boundary of each K_i is contained in K ;
- ii) There is a $C^{1+\epsilon}$ expanding map ψ defined in a neighborhood of $K_1 \cup K_2 \cup \dots \cup K_r$ such that $\psi(K_i)$ is the convex hull of a finite union of some intervals K_j satisfying:
 - ii.1 For each i , $1 \leq i \leq r$ and n sufficiently big, $\psi^n(K \cap K_i) = K$;
 - ii.2 $K = \bigcap_{n=1}^{\infty} \psi^{-n}(K_1 \cup K_2 \cup \dots \cup K_r)$.

We say that $\{K_1, K_2, \dots, K_r\}$ is a Markov partition for K , and that $D = \bigcup_{i=1}^r K_i$ is the Markov domain of K .

We can reinterpret the definition above as follows.

Let A be a finite alphabet, $\mathcal{B}$ a subset of A^2 , and Σ the subshift of finite type $A^{\mathbb{Z}}$ with allowed transitions $\mathcal{B}$.

We will always assume that Σ is topologically mixing, and that every letter in A occurs in Σ .

An *expansive map of type Σ* is a map g with the following properties:

- i) The domain of g is a disjoint union $\bigcup_{(a,b) \in \mathcal{B}} I(a,b)$, where, for each (a,b) , $I(a,b)$ is a compact subinterval of $I(a) := [0, 1] \times \{a\}$;
- ii) For each $(a,b) \in \mathcal{B}$ the restriction of g to $I(a,b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t .

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n} \left(\bigcup_{(a,b) \in \mathcal{B}} I(a,b) \right).$$

For $(a,b) \in \mathcal{B}$, let

$$f_{a,b} = [g|_{I(a,b)}]^{-1},$$

this is a contracting diffeomorphism from $I(b)$ onto $I(a,b)$. If $\underline{a} = (a_0, a_1, \dots, a_n)$ is a word of Σ , we put

$$f_{\underline{a}} = f_{a_0, a_1} \circ \dots \circ f_{a_{n-1}, a_n},$$

this is a contracting diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ denote by $I(\underline{a})$.

Let $\Sigma^- = \{(\theta_n)_{n \leq 0}; (\theta_i, \theta_{i+1}) \in \mathcal{B} \text{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \tilde{\underline{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \tilde{\underline{\theta}}) = \begin{cases} 1 & \theta_0 \neq \tilde{\theta}_0 \\ |I(\underline{\theta} \wedge \tilde{\underline{\theta}})| & \text{otherwise} \end{cases}$$

where $\underline{\theta} \wedge \tilde{\underline{\theta}} = (\theta_{-n}, \dots, \theta_0)$ if $\tilde{\theta}_{-j} = \theta_{-j}$ for $0 \leq j \leq n$ and $\tilde{\theta}_{-n-1} \neq \theta_{-n-1}$.

Regular Cantor set K is *affine* if Dg is constant on every $I(a,b)$.

Next, we define renormalization operators. Let $F^{\underline{\theta}}$ be the affine map from $I(\theta_0)$ onto $I(\theta_{-1}, \theta_0)$ with the same orientation of $f_{\theta_{-1}, \theta_0}$.

Let $\mathcal{A} = \{(\underline{\theta}, A)\}$, where $\underline{\theta} \in \Sigma^-$ and A is an *affine* embedding of $I(\theta_0)$ into $\mathbb{R}$. The *renormalization operators* $T_{\theta_1, \theta_0} : \mathcal{A} \rightarrow \mathcal{A}$ is defined as follows : if $(\theta_0, \theta_1) \in \mathcal{B}$,

$$T_{\theta_1, \theta_0}(\underline{\theta}, A) = (\underline{\theta}\theta_1, A \circ F^{\underline{\theta}\theta_1}).$$

Assume that we are given two sets of data $(A, \mathcal{B}, \Sigma, g)$, $(A', \mathcal{B}', \Sigma', g')$ defining regular Cantor sets K, K' .

We denote by $\mathcal{C}$ the quotient of $A \times A$ by the diagonal action on the left of affine group. An element of $\mathcal{C}$, represented by $(\underline{\theta}, A) \in \mathcal{A}$, $(\underline{\theta}', A') \in \mathcal{A}'$ is called a relative configuration of the limit geometries determined by $\underline{\theta}, \underline{\theta}'$.

A compact set $\mathcal{L}$ in $\mathcal{C}$ is *recurrent* if for every $u \in \mathcal{L}$, let $(\underline{\theta}, A)$, $(\underline{\theta}', A')$ represents u , then we can find $v \in \mathcal{L}^\circ$ and $\ell, \ell' \geq 0$ with $\ell + \ell' > 0$ such that there exists words $\underline{a} = (a_0, \dots, a_\ell)$, $\underline{a}' = (a'_0, \dots, a'_{\ell'})$ in Σ, Σ' respectively, with $a_0 = \theta_0$, $a'_0 = \theta'_0$, and $(T_{\underline{a}}(\underline{\theta}, A), T_{\underline{a}'}(\underline{\theta}', A')) = v$ belongs to $\mathcal{L}^\circ$. The following proposition is proved in [2].

Proposition. *Any relative configuration (of limit geometries) contained in a recurrent compact set is stably intersecting.*

In the end of this section we suppose that $\mathcal{S} = \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. One can see that, the fibers of the quotient map $\mathcal{C} \rightarrow \mathcal{S}$ are one-dimensional and have a canonical affine structure. Moreover, this bundle map is trivializable. We choose an explicit trivialization $\mathcal{C} \cong \mathcal{S} \times \mathbb{R}$ in order to have a coordinate in each fiber.

2. TRANSFER OF RENORMALIZATION OPERATORS ON THE SPACE $\mathcal{S} \times \mathbb{R}$

We assume that K is an affine Cantor set together with the Markov partition $\{I(n, m)\}_{m, n \in A}$ and an expansive map:

$$\Phi|_{I(n, m)} = p_{(n, m)} \cdot x + q_{(n, m)},$$

where $A = \{1, 2, 3, \dots, N\}$. Moreover, we assume that $I(n) = [a_n^1, a_n^2]$, $I(n, m) = [a_{n, m}^1, a_{n, m}^2]$, therefore, for every $(n, m) \in \mathcal{B}$, we have

$$\Phi(I(n, m)) = I(m).$$

In case $P_{(\theta_0, \theta_1)} > 0$, $F^{\underline{\theta}\theta_1}$ is as below

$$F^{\underline{\theta}\theta_1} : I(\theta_1) \rightarrow I(\theta_0, \theta_1),$$

$$F^{\underline{\theta}\theta_1}(x) = \frac{a_{\theta_0, \theta_1}^2 - a_{\theta_0, \theta_1}^1}{a_{\theta_1}^2 - a_{\theta_1}^1}(x - a_{\theta_1}^1) + a_{\theta_0, \theta_1}^1.$$

Also, in the opposite orientation, we have

$$F^{\underline{\theta}\theta_1}(x) = \frac{a_{\theta_0, \theta_1}^2 - a_{\theta_0, \theta_1}^1}{a_{\theta_1}^2 - a_{\theta_1}^1}(x - a_{\theta_1}^1) + a_{\theta_0, \theta_1}^2.$$

Therefore, in both cases we have

$$F^{\underline{\theta}\theta_1}(x) = \frac{1}{p_{(\theta_0, \theta_1)}}x - \frac{q_{(\theta_0, \theta_1)}}{p_{(\theta_0, \theta_1)}}.$$

In this section we first construct the homeomorphism between $\mathcal{S} \times \mathbb{R}$ and $\mathcal{C}$, then we transfer all renormalization operators to $\mathcal{S} \times \mathbb{R}$ and find the recurrent set in $\mathcal{S} \times \mathbb{R}$.

Theorem A. The map

$$L : \mathcal{C} \rightarrow \mathcal{S} \times \mathbb{R}$$

$$[(\underline{\theta}, ax + b), (\underline{\theta}', a'x + b')] \mapsto (\underline{\theta}, \underline{\theta}', \frac{a'}{a}, \frac{b' - b}{a})$$

is a homeomorphism between the space of relative configurations $\mathcal{C}$ and $\mathcal{S} \times \mathbb{R}$.

Proof. L is well defined:

Let

$$[(\underline{\theta}, ax + b), (\underline{\theta}', a'x + b')] = [(\underline{\theta}, a_1x + b_1), (\underline{\theta}', a'_1x + b'_1)].$$

Then, there exists $c, d \in \mathbb{R}$ such that

$$a_1x + b_1 = c(ax + b) + d, \quad a'_1x + b'_1 = c(a'x + b') + d$$

and therefore

$$\frac{a'}{a} = \frac{a'_1}{a_1}, \quad \frac{b' - b}{a} = \frac{b'_1 - b_1}{a_1}.$$

L is onto:

$$\forall (\underline{\theta}, \underline{\theta}', s, t) \in \mathcal{S} \times \mathbb{R} \quad L([(\underline{\theta}, x), (\underline{\theta}', sx + t)]) = (\underline{\theta}, \underline{\theta}', s, t).$$

L is one to one:

Suppose that

$$L([(\underline{\theta}, x), (\underline{\theta}', sx + t)]) = L([(\underline{\theta}, x), (\underline{\theta}', s'x + t')]),$$

then $s = s'$, $t = t'$.

From the structure of L , we have L and L^{-1} are continuous. $\square$

Now we transfer the renormalization operators of relative configurations $\mathcal{C}$ to the space $\mathcal{S} \times \mathbb{R}$. To do this, let $((\theta_1, \theta_0), (\theta'_1, \theta'_0)) \in \mathcal{B} \times \mathcal{B}'$, then we have

$$\begin{aligned} & (\underline{\theta}, \underline{\theta}', s, t) \xrightarrow{L^{-1}} \left[(\underline{\theta}, x), (\underline{\theta}', sx + t) \right] \xrightarrow{(T_{\theta_1, \theta_0}, T'_{\theta'_1, \theta'_0})} \\ & \left[\left(\underline{\theta}, \theta_1, \frac{1}{p(\theta_0, \theta_1)}x - \frac{q(\theta_0, \theta_1)}{p(\theta_0, \theta_1)} \right), \left(\underline{\theta}', \theta'_1, \frac{s}{p(\theta'_0, \theta'_1)}x - \frac{sq(\theta'_0, \theta'_1)}{p(\theta'_0, \theta'_1)} + t \right) \right] \\ & \xrightarrow{L} \left(\underline{\theta}, \theta_1, \underline{\theta}', \theta'_1, \frac{p(\theta_0, \theta_1)}{p'(\theta'_0, \theta'_1)}s, p(\theta_0, \theta_1)t - \frac{q'(\theta'_0, \theta'_1)}{p'(\theta'_0, \theta'_1)} \times p(\theta_0, \theta_1)s + q(\theta_0, \theta_1) \right), \end{aligned}$$

we denote $L \circ (T_{\theta_1, \theta_0}, T'_{\theta'_1, \theta'_0}) \circ L^{-1}$ by $T_{((\theta_1, \theta_0), (\theta'_1, \theta'_0))}$. Moreover,

$$T_{((\theta_1, \theta_0), id)} : (\underline{\theta}, \underline{\theta}', s, t) \longrightarrow (\underline{\theta}, \underline{\theta}', p_{\theta_1, \theta_0}s, p_{\theta_1, \theta_0}t + q_{\theta_1, \theta_0}),$$

$$T_{(id, (\theta'_1, \theta'_0))} : (\underline{\theta}, \underline{\theta}', s, t) \longrightarrow (\underline{\theta}, \underline{\theta}', \theta'_1, \frac{s}{p'_{\theta'_1, \theta'_0}}, t - \frac{q'_{\theta'_1, \theta'_0}}{p'_{\theta'_1, \theta'_0}} \cdot s).$$

3. CONSTRUCTION OF RECURRENT SETS

Here we introduce family of Cantor sets that have recurrent compact set in relative configurations.

Theorem B. Suppose that K, K' are two Cantor sets with convex hull $[0, 1]$, with two expanding maps Φ, Φ' such that;

I) $p_0, p_1, p'_0, p'_1 > 0$ and $p'_1 \leq p_1$,

II) $1 - \frac{1}{p_0} - \frac{1}{p_1}$ is less than the two following numbers

$$\frac{p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil}{p_1(p'_0 p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil - p'_0 - p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil)}$$

and

$$\inf \left\{ \frac{p_1^{\alpha(n)} (p_1^{\alpha(n-1)} - 1)}{p_1 p_0^n (p_1^{\alpha(n)} - p_1^{\alpha(n-1)})} \mid \alpha(n) = \left\lceil \log_{p'_1} p_0^n \right\rceil, n \in \mathbb{N} \right\}.$$

Then K, K' contain a non empty recurrent compact set in relative configurations.

Proof. By theorem A, it is sufficient to show the existence of a recurrent compact set for the following maps;

$$\begin{aligned} \mathbb{R}^* \times \mathbb{R} &\longrightarrow \mathbb{R}^* \times \mathbb{R} \\ (s, t) &\xrightarrow{T_{(0, id)}} (p_0 s, p_0 t) & (s, t) &\xrightarrow{T_{(1, id)}} (p_1 s, p_1 t - p_1 + 1) \\ (s, t) &\xrightarrow{T_{(id, 0)}} \left(\frac{1}{p_0} s, t\right) & (s, t) &\xrightarrow{T_{(id, 1)}} \left(\frac{1}{p_1} s, t + \frac{p'_1 - 1}{p_1} s\right). \end{aligned} \quad (*)$$

For all $n \in \mathbb{N} \cup \{0\}$, we choose the natural number $\alpha(n)$ such that $1 \leq \frac{p_1 p_0^n}{p'_1 \alpha(n)} < p'_1$.

Let

$$\alpha = \inf \left\{ \frac{p_1^{\alpha(n)} (p_1^{\alpha(n-1)} - 1)}{p_1 p_0^n (p_1^{\alpha(n)} - p_1^{\alpha(n-1)})} \mid n \in \mathbb{N} \right\},$$

where $\frac{1}{0}$ means ∞ , and

$$\beta = \sup \left\{ \frac{p_1^{\alpha(n-1)} (p_1^{\alpha(n)} - 1)}{p_1 p_0^{n-1} (p_1^{\alpha(n)} - p_1^{\alpha(n-1)})} \mid n \in \mathbb{N}, \alpha(n) \neq \alpha(n-1) \right\}.$$

It is clear that $\beta < \infty$. Let $\varepsilon > 0$ be such that

$$\text{i) } \varepsilon < \min \left\{ \frac{1}{p_0 + 1}, \frac{1}{p_1 + 1} \right\},$$

$$\text{ii) } 1 - \frac{1}{p_0} - \frac{1}{p_1} < \frac{p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil}{p_0(p'_0 p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil - p'_0 - p'_1 \left\lceil \log_{p'_1} p_1 \right\rceil)} - \frac{\varepsilon}{p_0},$$

$$\text{iii) } 1 - \frac{1}{p_0} - \frac{1}{p_1} < \alpha - \varepsilon(\beta + \frac{1}{p_0}).$$

We claim that

$$U = V \times [\varepsilon, 1 - \varepsilon] := \left[1 - \frac{1}{p_0} - \frac{1}{p_1} + \frac{2\varepsilon}{p_0}, 2p'_0 \left(1 - \frac{1}{p_0} - \frac{1}{p_1} + \frac{2\varepsilon}{p_0} \right) \right] \times [\varepsilon, 1 - \varepsilon]$$

is a recurrent set for the maps $(*)$. For every $n \in \mathbb{N} \cup \{0\}$ and $x \geq \frac{1-\varepsilon}{p_0}$, let

$$S_1^n(x) = \left(1 - x - \frac{1}{p_1} \right) \frac{p_1'^{\alpha(n)}}{p_1'^{\alpha(n)} - 1} + \varepsilon \frac{p_1'^{\alpha(n)}}{p_1 p_0^n (p_1'^{\alpha(n)} - 1)}$$

$$S_2^n(x) = S_1^n(x) + \frac{p_1'^{\alpha(n)}}{p_1 p_0^n (p_1'^{\alpha(n)} - 1)} - \varepsilon \frac{p_1'^{\alpha(n)}}{p_1 p_0^n (p_1'^{\alpha(n)} - 1)}.$$

Lemma 1. For all $n \in \mathbb{N}$ we have $S_1^{n-1}(x) < S_2^n(x)$.

Proof . If n is such that $\alpha(n) = \alpha(n-1)$, then because of $\varepsilon < \frac{1}{p_0+1}$, the above relation is hold. Otherwise obviously $S_1^{n-1}(x) < S_2^n(x)$ is equivalent to

$$x > 1 - \frac{1}{p_1} - \frac{p_1'^{\alpha(n)} (p_1'^{\alpha(n-1)} - 1)}{p_1 p_0^n (p_1'^{\alpha(n)} - p_1'^{\alpha(n-1)})} + \frac{p_1'^{\alpha(n-1)} (p_1'^{\alpha(n)} - 1)}{p_1 p_0^{n-1} (p_1'^{\alpha(n)} - p_1'^{\alpha(n-1)})} \varepsilon.$$

Now, as $x \geq \frac{1-\varepsilon}{p_0}$, by (iii) we see that this last expression always holds. This completes the proof of the lemma.

We define

$$E(x) = \bigcup_{n=0}^{\infty} E_n(x), \quad E_n(x) = [S_1^n(x), S_2^n(x)], \quad n = 0, 1, 2, \dots$$

$$\tilde{E}(x) = \left(1 - x - \frac{1}{p_1}, \infty \right).$$

Lemma 2. For all $s \in \tilde{E}(x)$, there exists a composition of functions $(*)$, say T , such that

- a) $\pi_1 T(s, x) > 1 - x - \frac{1}{p_1}$,
- b) $\pi_2 T(s, x) \in [\varepsilon, 1 - \varepsilon]$.

Proof . Because $\varepsilon < \frac{1}{p_0+1}$, by using (ii) above we obtain

$$p'_0 \left(1 - x - \frac{1}{p_1} \right) < \frac{p_1'^{\left\lceil \log_{p'_1} p_1 \right\rceil}}{p_1'^{\left\lceil \log_{p'_1} p_1 \right\rceil} - 1} (1 - x) = S_2^0(x).$$

From lemma 1 we have $\tilde{E}(x) = \bigcup_{n=0}^{\infty} p_0'^n E(x)$. As $s \in \tilde{E}(x)$, then there exists

$$n_1, n_2 \in \mathbb{N} \cup \{0\} \text{ such that } \frac{1}{p_0'^{n_1}} s \in E_{n_2}(x). \text{ Let } T = T_{(id, 1)}^{\alpha(n_2)} \circ T_{(0, id)}^{n_2} \circ T_{(id, 0)}^{n_1} \circ T_{(1, id)}.$$

To prove (a)

$$\pi_1 \circ T(s, x) = \frac{p_1 p_0'^{n_2}}{p_1'^{\alpha(n_2)} p_0'^{n_1}} s \geq \frac{1}{p_0'^{n_1}} s > 1 - x - \frac{1}{p}.$$

To prove (b), because $\alpha(n_2) \neq 0$

$$\pi_2 T(s, x) = (p_0^{n_2} p_1 x + p_0^{n_2} - p_0^{n_2} p_1 + \frac{p_1'^{\alpha(n_2)} - 1}{p_1'^{\alpha(n_2)}} \times \frac{p_1 p_0^{n_2}}{p_0'^{n_1}} s).$$

Moreover, $S_1^{n_2}(x) \leq \frac{1}{p_0'^{n_1}} s \leq S_2^{n_2}(x)$, since $\frac{1}{p_0'^{n_1}} s \in E_{n_2}(x)$. By multiplying the inequality to $\frac{p_1'^{\alpha(n_2)} - 1}{p_1'^{\alpha(n_2)}} \times p_1 p_0^{n_2}$ and adding $p_0^{n_2} p_1 x + p_0^{n_2} - p_0^{n_2} p_1$ to both sides we obtain $\varepsilon \leq \pi_2 T(s, x) \leq 1 - \varepsilon$.

This completes the proof of lemma 2.

Now, because V does not contain zero, U is compact. To prove that U is a recurrent set, we check 3 cases;

Case 1) $(s, t) \in V \times [\varepsilon, \frac{1-\varepsilon}{p_0}]$. Choose $n_1 \in \mathbb{N} \cup \{0\}$ such that $\frac{p_0 s}{p_0'^{n_1}} \in V^0$. Clearly $T_{(id,0)}^{n_1} \circ T_{(0,id)}(s, t) \in U^0$,

Case 2) $(s, t) \in V \times \{1 - \varepsilon\}$. Let $n_2 \in \mathbb{N} \cup \{0\}$ such that $\frac{p_1 s}{p_0'^{n_2}} \in V^0$. Because $\varepsilon < \frac{1}{1+p_1}$, then $T_{(id,0)}^{n_2} \circ T_{(1,id)}(s, t) \in U^0$,

Case 3) I $(s, t) \in V \times [\frac{1-\varepsilon}{p_0}, 1 - \varepsilon)$, then as $s \in V$ and $t \geq \frac{1-\varepsilon}{p_0}$, thus

$$s \geq 1 - \frac{1}{p_0} - \frac{1}{p_1} + \frac{2\varepsilon}{p_0} > 1 - t - \frac{1}{p_1} \Rightarrow s \in \tilde{E}(t).$$

From Lemma 2, there is T such that $\pi_1 T(s, t) > 1 - t - \frac{1}{p_1}$ and $\pi_2 T(s, t) \in [\varepsilon, 1 - \varepsilon]$. Let n_3 be an integer in $\mathbb{N} \cup \{0\}$ such that $\frac{\pi_1 T(s, t)}{p_0'^{n_3}} \in V^0$. If $\pi_2 T(s, t) \in (\varepsilon, 1 - \varepsilon)$, then $T_{(id,0)}^{n_3} \circ T(s, t) \in U^0$.

If $\pi_2 T(s, t)$ is equal to ε or $1 - \varepsilon$ then it comes back into U^0 with a transformation of case 1 or case 2 (respectively), thus U is a recurrent set for maps $(*)$. This completes the proof.

To see that conditions of theorem B are realistic, we introduce affine Cantor sets K and K' , for them the numbers p_0, p_1, p_0', p_1' satisfy conditions of theorem B.

Example. Consider the affine Cantor sets

$$K : \quad \underline{\quad 2^{-\frac{3}{2}} \quad} \quad \underline{\quad 2^{-\frac{4}{5}} \quad} \quad K' : \quad \underline{\quad 2^{-4} \quad} \quad \underline{\quad 2^{-\frac{3}{4}} \quad}$$

Here, $p_0 = 2^{\frac{3}{2}}, p_1 = 2^{\frac{4}{5}}, p_0' = 2^4, p_1' = 2^{\frac{3}{4}}$. It is easy to see that conditions (I) and (II) of theorem B are satisfied, since $\alpha(n) = 2n + 1$. For this two Cantor sets we have $T(K) \cdot T(K') < 1$. Therefore it is not possible to judge about their difference by gap lemma [3].

REFERENCES

- [1] Moreira C.G., *Stable intersections of Cantor sets and homoclinic bifurcations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), no 6, 741-781.
- [2] Moreira C.G. and Yoccoz J.C., *Stable intersections of regular Cantor sets with large Hausdorff dimension*, Ann. of Math.(2) 154 (2001), no 1, 45-98.
- [3] Newhouse Sheldon E., *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms*, Inst. Hautes Etudes Sci. Publ. Math. 50 (1979), 101-151.
- [4] Palis J., *A global view of dynamics and conjecture on denseness of finitude of attractors*, Astérisque 261 (2000), 339-351.
- [5] Palis J. and Takens F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations: fractal dimensions and infinitely many attractors*, Cambridge Univ. Press, 1992.