MULTIFRACTAL ANALYSIS FOR GIBBS GROUND STATES

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Abstract. We consider a family of Gibbs states $\{\mu_q\}_{q\geq 1}$ describing the multifractal spectrum of local entropies. An accumulation point μ_∞ of this sequence may be called a *Gibbs ground state* or a *zero temperature limit*, because the interpretation of q as the inverse of the temperature. We prove that, in more general systems than symbolics, that μ_∞ is a maximizing measure. We also do geometric and combinatorial approaches as well as some interpretations in a particular case.

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Introduction

Let $f : X \to X$ be a homeomorphism with (X, d) a compact metric space. The following dynamical metric can be introduced in X :
Let $d_n(x, y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0, 1, ..., n-1\right\}$ and we denote by $B_{n,\varepsilon}(x)$ the ball of center x and radius ε in the metric d_n. If μ is an $f-$invariant measure, then the *upper and lower local entropies* are

$$\overline{h_\mu}(x, f) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup -\frac{1}{n}\log\mu\left(B_{n,\varepsilon}(x)\right)$$

$$\underline{h_\mu}(x, f) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\inf -\frac{1}{n}\log\mu\left(B_{n,\varepsilon}(x)\right).$$

By the Brin-Katok theorem [4], the local entropy does exist $\mu - a.e$, i.e. $\overline{h_\mu}(x, f) = \underline{h_\mu}(x, f) := h_\mu(x, f)$, for $\mu-a.e.$ $x \in X$. The multifractal *local entropies spectrum of f*, or more precisely of the dynamical system (X, f), is specified by a decomposition in level sets

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$$K_{\alpha} = \{x : h_{\mu}(x, f) = \alpha\},$$

and a map G defined on the sets K_{α}. Then a further decomposition $X = \left(\bigcup_{\alpha} K_{\alpha}\right) \cup Y$, where Y is the set in which $h_{\mu}(x, f)$ is not defined, is introduced. The set Y is called the irregular part of the spectrum and by the Brin-Katok theorem has zero measure for any $f-$invariant measure. The general concept of multifractal analysis was introduced in [1]. The description of multifractal spectra is given by families of measures $\{\mu_{\alpha}\}_{\alpha \in \mathbf{R}}$ such that $\mu_{\alpha}(K_{\alpha}) = 1$, thus is needed a parametrization $\alpha(q)$ with $\mu_q\left(K_{\alpha(q)}\right) = 1$ and $\mu_q(K_{\alpha}) = 0$ if $\alpha \neq \alpha(q)$. In this way it has a correspondence between the level sets of the decomposition and the family of full measures $\{\mu_q\}$. In the special case of the local entropies an one-parameter family of measures $\{\mu_q\}_{q \in \mathbf{R}}$ is introduced as the Gibbs state of each member of certain family of potentials $\{\varphi_q\}$. By a potential we shall mean a continuous map $\varphi : X \to \mathbf{R}$. The parameterization is obtained with $\alpha(q) := -T^{'}(q)$ where T is the "free energy " in the terminology of the Ruelle thermodynamic formalism, being q interpreted as the inverse of the temperature.

Since X is compact the sequence $\{\mu_q\}_{q \geq 1}$ has a weak* accumulation point μ_{∞}, and by the above mentioned interpretation of the parameter q may be pertinent to name the limit $q \to \infty$ as the *zero limit temperature* and μ_{∞} as a *ground state.*

There is an interesting set of problems about zero temperature states, among them we can mention:

1. It should be noticed that in general $\{\mu_q\}$ does not converge, but if there is an unique maximizing measure for the potential $\varphi = \varphi_{q=1}$, i.e. a measure μ such that $\int \varphi d\mu \geq \int \varphi d\nu$ for any measure ν, then $\{\mu_q\}$,weakly, converges to μ. The problem is when there are several maximizing measures and we only know that there is an accumulation point.

2. If $h_{\mu_{\infty}}(f)$ is the measure theoretical entropy of the limit μ_{∞}, it is interesting to relate the entropy of the limit measure with the limit of the entropies h_{μ_q}.

3. Another question is if the $h_{\mu_{\infty}}$ is the maximal entropy among those of maximizing measures for some potential.

In [11] has been proved, in a symbolic context but for subshifts with infinite alphabet, that an accumulation point of the sequence $\{\mu_q\}$ considered therein is maximizing. By the other way in [3] was proved that if the potential depends on a finite number of coordinates then $\{\mu_q\}$ converges.

In this note we board these problems in a more general context than symbolic spaces. Indeed our results does not capture completely those in [11] (about problem 1), but in the compact case, i.e subshifts with finite alphabet, the result that the zero limit is maximizing is here generalized. The problems 2 and 3 were posed in [11] in the context of subshift with infinite alphabet. In this article these problems are treated, although in the compact case, in a more context that symbolics and even hyperbolic systems.

The following geometric approach will be done herein: In [13] we have considered a finitely generated groups Γ acting on the hyperbolic disc H^2. For a class of such groups is possible the assignation to any point $\xi \in \Lambda$ (the limit set of Γ) a unique reduced sequence in the set S of generators of Γ. This is done by mean of the technique by C. Series [16],[17] to code geodesics in H^2/Γ whose ends are in the limit set of Γ. Any closed geodesic in H^2/Γ has assigned a periodic sequence in the generators of Γ, let us denote by $[e_0e_1...e_{n-1}]$, $e_i \in S$, the periodic block of this sequence. Now is considered the element $\gamma = e_0 \bullet e_1 \bullet \bullet \bullet e_{n-1} \in \Gamma$ (here the dots means the operation in the group), which has length (in the word metric) n, this is denoted by $|\gamma| = n$. Thus we have two quantities: the hyperbolic length of the geodesic g, denoted by $\ell(g)$, and its "word metric" $|g| := |\gamma| = n$. We show here, going along lines of [14], that the ground state, for a certain potential, is related with the "length-word ratio" $\frac{\ell(g)}{|g|}$ of geodesics in H^2/Γ. A free energy, with a partition function defined by a sum over geodesics with word metric n, is introduced to this end. By a result of Series any $\gamma \in \Gamma$ admits an unique decomposition in the generators of this group, in such a way that the there is a method for counting elements in Γ and is obtained a solution for the Dehn word problem. Thus it can defined a free energy from a partition function in which are counted the so called *cone types* in the Cayley graph of Γ, which are finite since by the mentioned result by Series the class of groups considered are automatic. In this way a combinatorial approach can be developed.

1. The results

Before formulating and proving our results we display some preliminary definitions and the main facts about the multifractal formalism of Takens and Verbitski[18]:

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism. The *topological pressure* associated to f and the potential $\varphi : X \to \mathbf{R}$ is the number:

$$P(\varphi) = \sup_{\mu} \left\{ h_{\mu}(f) + \int \varphi d\mu \right\}, \tag{1}$$

where the supreme is taken over all the $f-$invariant Borel measures μ on X, and $h_{\mu}(f)$ is the usual Kolmogorov measure-theoretic entropy of f.

An *equilibrium state* for the potential φ is a measure μ_{φ} for which:

$$P(\varphi) = h_{\mu_{\varphi}}(f) + \int \varphi d\mu_{\varphi}. \tag{2}$$

The set of equilibrium states for a potential φ is denoted by $\mathcal{M}_{\varphi}(X)$.

Definition: A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $\delta > 0$, such that $d(f^n(x), f^n(y)) < \delta$, for any integer n implies $x = y$.

The *specification property* for a map $f : X \to X$ intuitively says that for specified orbit segments a periodic orbit approximating the trajectory can be found. This condition ensures abundance of periodic points and it is a concept introduced by R. Bowen[5]. Formally, a homeomorphism $f : X \to X$ has the s*pecification property* if: $I_1, ..., I_k$ is a finite disjoint collection of integer intervals $I_1, ..., I_k$, then for $\varepsilon > 0$, there is a integer $M(\varepsilon)$ and a function $\Phi : I = \cup I_i \to X$, such that:

i) $dist(I_i, I_j) > M(\varepsilon)$ (Euclidean distance)

ii) $f^{n_1 - n_2}(\Phi(n_1)) = \Phi(n_2)$

iii) $d(f^n(x), \Phi(n)) < \varepsilon$, for some $x : f^m(x) = x$, with $m \geq M(\varepsilon) + length(I)$ and for every $n \in I$.

For a potential φ we put

$$S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi\left(f^i(x)\right) \tag{3}$$

which is called the *statistical sum.*

Definition: A potential φ belongs to the class $\nu_f(X)$ if there are constants $\varepsilon, K > 0$ such that

$$d_n(x,y) < \varepsilon \Longrightarrow |S_n(\varphi)(x) - S_n(\varphi)(y)| < K.$$

If $\varphi \in \nu_f(X)$, with f a homeomorphism with the specification property then the topological pressure can be computed by:

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{x \in P_n(f)} \exp(S_n(\varphi)(x)) \right) \text{ ([10], prop. 20.3.3).} \tag{4}$$

where $P_n(f) = \{x : f^n(x) = x\}$, notice that since f is expansive and X compact the set of periodic points is finite for any n.

Definition: A measure μ on X is called a *Gibbs measure, or a Gibbs state,* if for sufficiently small $\varepsilon > 0$, there are constants A_ε, $B_\varepsilon > 0$, such that for any $x \in X$ and for any positive integer n:

$$A_\varepsilon(\exp(S_n(\varphi)(x)) - nP(\varphi)) \le \mu(B_{n,\varepsilon}(x)) \le B_\varepsilon(\exp(S_n(\varphi)(x)) - nP(\varphi)) \tag{5}$$

To a potential $\varphi \in \nu_f(X)$, with f a homeomorphism with the specification property can be associated an unique equilibrium state which is a Gibbs state [10, 15]. This is constructed as the weak limit of the "Gibbs ensembles"

$$\mu_{\varphi,n} = \frac{1}{\widetilde{Z}(f,\varphi,n)} \sum_{x\in P_n(f)} \exp(S_n(\varphi)(x))\,\delta_x, \tag{6}$$

where $\widetilde{Z}(f,\varphi,n) = \sum\limits_{x\in P_n(f)} \exp(S_n(\varphi)(x))$ and δ_x is the unit measure at $\{x\}$.

Let $T(q) = P(q\varphi) - qP(\varphi)$, from the multifractal formalism by Takens and Verbitski[18] if $f : X \to X$ is an expansive homeomorphism with the specification property and $\varphi \in \nu_f(X)$. then the following results hold:

i) the function $T(q)$ is convex and continuously differentiable. This map has a Legendre transform $\mathcal{E}(\alpha) = \inf\limits_{q\in\mathbf{R}} \{q\alpha - T(q)\}$ and $\mathcal{E}(\alpha)$ describes the spectrum of local entropies of f .

ii) If $K_\alpha = \left\{x : h_{\mu_\varphi}(x,f) = \alpha\right\}$, ($\mu_\varphi \neq \mu_{\max}$, the measure maximal entropy), then $\mathcal{E}(\alpha) = h_{top}(f,K_\alpha)$. Here $h_{top}(f,K_\alpha)$ denotes the topological entropy restricted to the set K_α, this is a generalization for non-compact sets of the classical topological entropy[6]. Besides

$$\mathcal{E}(\alpha(q)) = q\alpha(q) + T(q); \qquad \alpha(q) := -T'(q),\ q = \mathcal{E}'(\alpha). \tag{7}$$

Let $\alpha_i = \lim\limits_{q\to\infty} \alpha(q) = \inf\limits_{q\in\mathbf{R}} \{\alpha(q)\}$, $\alpha_s = \lim\limits_{q\to-\infty} \alpha(q) = \sup\limits_{q\in\mathbf{R}} \{\alpha(q)\}$, then $K_\alpha = \emptyset$, if $\alpha \in (\alpha_i, \alpha_s)$, so that the domain of definition of $\mathcal{E}(\alpha)$ is the range of $T'(q)$.

iii) From the above variational relationship between $\mathcal{E}(\alpha)$ and $T(q)$we have that $\mathcal{E}(\alpha)$ is concave and continuously differentiable in (α_i, α_s)

Thus the spectrum of local entropies is described by the family of measures $\{\mu_q\}$ where each μ_q is the Gibbs state associated to the potential $\varphi_q := q\varphi - P(q\varphi) \in \nu_f(X)$, with the parametrization $\alpha(q) := -T'(q)$

Along this section, we shall consider dynamical systems defined by a compact metric space X and homeomorphisms $f : X \to X$ such that:

- $f : X \to X$ is an expansive homeomorphism with the specification property.
- The potentials φ belong to the class $\nu_f(X)$.

Theorem 1: Let $\{\mu_q\}_{q\geq 1}$ be the family of measures as before with a weak* accumulation point μ_∞, then μ_∞ is maximizing for the potential $\varphi := \varphi_1 = \varphi - P(\varphi)$.

Proof: If μ_∞ were not maximizing there would exist a measure ν such that $\int(\varphi - P(\varphi))d\nu > \int(\varphi - P(\varphi))d\mu_\infty$. We have, from the properties of the topological pressure, that the map $q \longmapsto P(q\varphi)$ is convex, and from the multifractal formalism of [18] is established that it is strictly convex, i.e.

$P(q\varphi) > P(q_0\varphi) + h(q_0)(q - q_0)$, for any real q_0 and for some number $h(q_0)$, which is taken to be as $\int \varphi d\mu_{q_0}$. The map $q \longmapsto P(q\varphi)$ is differentiable with $\dfrac{dP(q\varphi)}{dq} = \int \varphi d\mu_q$ and due to its convexity is continuously differentiable. The derivative formula is proved from the relationship between equilibrium states and tangent functionals, we sketch in which way does: recall that if the topological pressure is considered as a functional in a Banach space by $P : C(X) \to \mathbf{R}$ then the concept of functional tangent to P at φ is the set of measures μ such that

$P(\varphi + \psi) - P(\varphi) \geq \int \psi d\mu$, for any $\psi \in C(X)$. Let us denote by $t_P(\varphi)$ the set of functional tangents to P at φ. By 1 holds $\mathcal{M}_\varphi(X) \subset t_P(\varphi)$, being the opposite inclusion valid when the entropy map $\mu \longmapsto h_\mu(f)$ is upper semi-continuous. Since the dynamics are expansive this implies that the entropy map is upper semi-continuous[19] and so that within in our framework holds $\mathcal{M}_\varphi(X) = t_P(\varphi)$. Now, by the uniqueness of equilibrium states, the set of tangent functional has an only member, the Gibbs state associated to φ. Besides the functional P is G$\widehat{a}$teaux differentiable. Now, in particular, we have

$\dfrac{P(q\varphi) - P\left(q'\varphi\right)}{q'} \geq \int \varphi d\mu_{q'}$ for $q' > 0$ and

$\dfrac{P(q\varphi) - P\left(q'\varphi\right)}{q'} \leq \int \varphi d\mu_{q'}$ for $q' < 0$. So that, since P is G$\widehat{a}$teaux differentiable is obtained that

$\dfrac{dP(q\varphi)}{dq} = \int \varphi d\mu_q$. Thus if $T(q) = P(q\varphi) - qP(\varphi)$ then $T'(q) = \int \varphi d\mu_q - P(\varphi) \leq 0$.

We have that $\int \varphi d\mu_\infty = \lim\limits_{q\to\infty} \int \varphi d\mu_q$, this is due that μ_∞ is an accumulation point of $\{\mu_q\}$ and X is compact so φ is bounded. By the convexity of $q \longmapsto F(q) := P(q\varphi)$ and because $\int \varphi d\mu_q = F'(q)$ it holds that

$\int \varphi d\mu_q \leq \int \varphi d\mu_\infty$ for q enough large.

Now we have

$\int \varphi d\nu > \int \varphi d\mu_\infty \geq \int \varphi d\mu_q = T'(q) + P(\varphi)$ it is

$\dfrac{d\left(h_\nu(f) + q\int \varphi d\nu\right)}{dq} > T'(q) + P(\varphi)$ and so

$h_\nu(f) + q\int \varphi d\nu > T(q) + qP(\varphi) = P(q\varphi)$ for q enough large, which would contradict the variational definition of topological pressure (c.f. Eq1). Thus μ_∞ is maximizing. ■

Proposition 1: It holds $h_{\mu_\infty}(f) = \lim_{q\to\infty} h_{\mu_q}(f)$.

Proof: We have, by the multifractal formalism described above, that $\mathcal{E}(\alpha(q)) = q\alpha(q) + T(q)$; where $\alpha(q) := -T^{'}(q)$ and $(T, \mathcal{E})$ being a Legendre pair. But by the another way

$T(q) + q\alpha(q) = T(q) - q\left[\int \varphi d\mu_q - P(\varphi)\right] =$

$= T(q) + qP(\varphi) - q\int \varphi d\mu_q = P(q\varphi) - q\int \varphi d\mu_q$

Recall that μ_q is the Gibbs state corresponding to the potential $\varphi_q := q\varphi - P(q\varphi)$, i.e. $P(\varphi_q) = 0$. Now

$P(q\varphi) - q\int \varphi d\mu_q = -\int \varphi_q d\mu_q =.\ P(\varphi_q) - \int \varphi_q d\mu_q = h_{\mu_q}(f)$, and so that

$\mathcal{E}(\alpha(q)) = h_{\mu_q}(f)$.

Then due to μ_∞ is maximizing,

$h_{\mu_q}(f) = T(q) + q\alpha(q) = P(q\varphi) - q\int \varphi d\mu_q \geq h_{\mu_\infty}(f) + q\left[\int \varphi d\mu_q - \int \varphi d\mu_q\right] \geq h_{\mu_\infty}(f)$

Thus by the properties of $\mathcal{E}(\alpha(q))$ as a Legendre transform of the free energy $T(q)$, which is decreasing as was pointed out the earlier theorem and is obtained that $\lim_{q\to\infty} h_{\mu_q}(f) \geq h_{\mu_\infty}(f)$.

The opposite inequality follows from this known fact about topological dynamics[19]: For expansive dynamical systems is valid that for any $f-$ invariant measure μ and for any $\varepsilon > 0$ there is a neighborhood U (in the weak topology) of μ such that for every $\nu \in U$ holds:

$h_\nu(f) < h_\mu(f) + \varepsilon$. ■

Next is obtained a variational formula for the entropy of μ_∞.

Theorem 2: It is valid that $h_{\mu_\infty}(f) = \sup h_\nu(f)$, where the supreme is taken over all the φ_{q-}maximizing measures

Proof: If it were not true there would exist a maximizing measure ν such that

$h_\nu(f) = h_{\mu_\infty}(f) + \varepsilon$, for $\varepsilon > 0$ and so, by the above proposition,

$h_\nu(f) = \lim_{q\to\infty} h_{\mu_q}(f) + \varepsilon$. Therefore, for q enough large, it has

$h_\nu(f) - h_{\mu_q}(f) > \varepsilon > 0$. Besides, by maximality,

$\int \varphi d\nu \geq \int \varphi d\mu_q$ and so that

$h_\nu(f) > h_{\mu_q}(f) = T(q) + q\alpha(q) = P(q\varphi) - q\int \varphi d\mu_q$. Thus

$h_\nu(f) + q\int \varphi d\nu - qP(\varphi) > P(q\varphi) + q\left[\int \varphi d\nu - \int \varphi d\mu_q\right] - qP(\varphi) \geq T(q)$, which contradicts the fact that μ_q is an equilibrium state for the potential $\varphi_q = q\varphi - P(q\varphi)$. ■

2. Geometric and combinatorial approaches

In this section we present a connection between zero limit and spectra of geodesics in negative curvature surfaces for a special case. Let Γ be a finitely generated Kleinian group on H^2 (the disc model of the hyperbolic plane), i.e a group which acts discontinuously on H^2. Recall that ξ is a limit point of Γ if and only there exists a point $w \in H^2$ such that the $\Gamma-$orbit $\Gamma(w) = \{\gamma w : \gamma \in \Gamma\}$ accumulates at ξ. The set Λ of limit points is called the *limit set of* Γ. Because Γ acts discontinuously $\Lambda \subset \partial H^2$, besides Λ is a closed subset, so compact, of ∂H^2 and $\Gamma-$invariant; moreover it is the smallest non-empty, $\Gamma-$invariant closed set in ∂H^2 (for more about Kleinian groups and justification of the facts mentioned see e.g. [2]). For certain class of groups Γ, with a finite symmetric sets of generators S, acting on the hyperbolic disc H^2 there is a way of coding geodesics in H^2/Γ due to C. Series[16],

[17]. The coding is obtained by mean of the introduction of the so called *boundary hyperbolic maps,* which are one-dimensional maps $f : \partial H^2 \to \partial H^2$ or more precisely $f : \Lambda \to \Lambda$. The limit set of Γ can be identified with a finite type subshift $\sum$ constituted by infinite reduced sequences in elements of S and in particular to any point in the limit set is associated an infinite reduced word in the generators of Γ. Moreover there is a bijection $p : \sum \to \Lambda$, which conjugates the Bernoulli shift $\sigma : \sum \to \sum$ with f, and we said with the alphabet for the shift agreeing with the systems of generators for the group. Within this class fall the free groups generated by side pairing isometries in a polygon inscribed in H^2 such that the quotient space does not have punctures, or more generally generated by side paring isometries in a fundamental region R with all its vertices in ∂H^2. For more details about conditions for the group belong to mentioned class [16], [17], [13]. It should pointed out that is not possible, for more general groups that the alphabet for the symbolic space $\sum$ agree with the set of generators of the group. More specifically there is a large class of Kleinian groups for which the situation is the following: there is a continuous surjective map $p : \sum \to \Lambda$, which conjugates the Bernoulli shift $\sigma : \sum \to \sum$ with f. The relation between the alphabet for $\sum$ and the set of generators for Γ is finite-to-one.

We consider so for our analysis groups generated by isometries in the sides of a fundamental region R for which the assignation of a reduced sequence to any element of the limit set be possible and such that the alphabet for the corresponding subshift be identified with the generators. Thus the geodesics in H^2 can be codified in the following way: let g a hyperbolic geodesic with ends $\xi, \eta \in \Lambda$ with $\xi \leftrightarrow (e_0 e_1 ...)$ and $\eta \leftrightarrow (f_0 f_1 ...)$, then to the geodesic g is assigned the bi-infinite sequence $\eta^{-1} * \xi$, where this means the concatenation of the elements in the sequence for ξ with the "inverted" sequence of η, i.e. $\eta^{-1} * \xi := (...f_1^{-1} f_0^{-1} e_0 e_1 ...)$. An important result from this Series theory is: if R is a fundamental region in H^2 then there is a bijection between the set of geodesics in H^2 with ends in Λ and which intersect R and the set of bi-infinite reduced sequences in generators of Γ, this allows to enumerate the closed geodesics. Let us consider the negatively curved surface $M = H^2/\Gamma$, now to any closed geodesic in M is assigned a periodic sequence in elements of S, so we can denote $g \leftrightarrow [e_0 e_1 ... e_{n-1}]$, where $[e_0 e_1 ... e_{n-1}]$ is the periodic block of the sequence of period n. In this way we can say that the geodesic g has "length metric" n, denoting this by $|g| = n$. This terminology is due that to g can be associated, as mentioned in the introduction, the element $\gamma = e_0 \bullet e_1 \bullet \bullet \bullet e_{n-1} \in \Gamma$ which has length metric, for groups, equal n (here are used these dots to distinguish the operation in the group with the codifying sequence).

Let us denote by $\ell(g)$ the hyperbolic metric of the geodesic g in M, an interesting issue may be to obtain information about the ratio $\dfrac{\ell(g)}{|g|}$ for closed geodesics in the surface M. To this end, following [14], it can be introduced a "partition function"

$$Z_n(q) = \sum_{|g|=n} \exp\left(q\ell(g)\right) \tag{8}$$

and the consequent "free energy"

$$T(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(q) . \tag{9}$$

Next we briefly explain how this free energy is compared with that defined in the more general situation of the above section. Recall that any closed geodesic in M is in correspondence with a periodic sequence in the generators of Γ and so a periodic orbit for the shift $\sigma : \sum \to \sum$ (recall also that the alphabet of $\sum$ agrees with the set of generators of Γ). By the bijective conjugation between f and the shift the periodic points of f are in correspondence with the periodic orbits of σ, i.e periodic sequences in the generators of Γ. Thus to count closed geodesic, with a word metric n, is essentially the same of counting $n-$periodic orbits of the shift and consequently periodic points of f with period n. Now we can related the map of the Eq.9 with the topological pressure of certain potential, i.e. $T(q)$ will result a free energy of the type considered in sec.1. Such a potential φ is something like an analogous to the time return function for the symbolic modelling of geodesic flows, i.e. those maps Φ such that the periodic orbits of the flow have period equal to the statistical sum $S_n(\Phi)(z)$, where z is a $n-$periodic point of Poincaré maps defined on transverse sections to the flow. By the Series method and the already commented correspondence periodic points of f / periodic sequences in S / closed geodesics in M there is a map $\varphi : \Lambda \to \mathbf{R}$ such that if $\xi \in P_n(f)$ in correspondence with a closed geodesic g with $|g| = n$, then $S_n(\varphi)(\xi) = \ell(g)$.

Therefore the free energy corresponding to the partition function given by 8 is equivalent to that defined from:

$$\sum_{\xi : f^n \xi = \xi} \exp\left(q S_n(\varphi)(\xi)\right). \tag{10}$$

So that $T(q) = P(q\varphi)$, i.e. like that defined in the general case of sec.1 except for the term $qP(\varphi)$.

This approach is somewhat more general than [14] which is based on the coding of geodesic flows in negatively curved surfaces. Indeed , the coding technique by Series generalizes the Markov partitions method for obtaining a symbolic modelling of a geodesic flow in negatively curved manifolds. In fact, a "cross-section" $\mathbf{T}$ is defined by considering the set of unitary tangent vectors to geodesics, which intersect R, based in the point in which the first geodesic intersects R. This is in particular a cross section for the geodesic flow in SM (the unitary tangent bundle). A time-return map $\varphi : \mathbf{T} \to \mathbf{R}$ is defined as $\varphi(v) = ev$, where e is defined in the following way: the sides of R can be labelled with the name of elements of S according they are identified to form the quotient M, now e will be the label of the side across which the lifting of the geodesic determined by v leaves R. In a similar way can be defined a map on geodesics, denoted also by φ as $\varphi(g) = e^{-1}g$, where e is the same of above.

We will situate in the following special context: the dynamics are implemented by the boundary hyperbolic maps and the potential is given by the time return function for the process above described. In the appendix we display the definition of the boundary hyperbolic maps as well as we mention some of their connections with multifractal analysis. These maps have the required, in the general case, dynamical properties of expansiveness and specification [16],[17],[12].

Theorem 3: If μ_∞ is the zero limit for sequence $\{q\varphi\}_{q\geq 1}$ then $\int \varphi d\mu_\infty = \sup\left\{ \frac{\ell(g)}{|g|} : \ g \text{ closed geodesic in } M \right\}$

Proof: Let $T(q) = P(q\varphi)$ and $\alpha(q) = -T^{'}(q) = -\int \varphi d\mu_q$. By the variational definition of the pressure (Eq.1) we have

$T(q) = h_{\mu_q}(f) - q\alpha(q) \geq h_{\mu}(f) - q\int \varphi d\mu$, for any $f-$invariant μ and with f a boundary hyperbolic map. Then

$\alpha(q) \geq \dfrac{h_{\mu}(f) - h_{\mu_q}(f)}{q} - \int \varphi d\mu$, by the classical entropy variational principle the term $h_{\mu}(f) - h_{\mu_q}(f)$ remains bounded, for any q, by twice times the topological entropy of the map. Now since $\alpha(q)$ is decreasing (see theorem 1 in the part of convexity) and $\lim_{q\to\infty} \alpha(q) = \inf_{q\in\mathbf{R}} \{\alpha(q)\}$ (c.f. below eq.7) obtains

$\lim_{q\to\infty} \alpha(q) = -\sup_{\mu} \left\{\int \varphi d\mu\right\} = -\sup_{g} \left\{\dfrac{\ell(g)}{|g|}\right\}$ and so that

$\lim_{q\to\infty} \int \varphi d\mu_{\infty} = \int \varphi d\mu_{\infty} = \sup \left\{\dfrac{\ell(g)}{|g|}\right\}$. ■

A similar procedure can be done for the case $q \to -\infty$ to obtain $\lim_{q\to-\infty} \int \varphi d\mu_{\infty} = \inf \left\{\dfrac{\ell(g)}{|g|}\right\}$. Although this limit is not of interest in our context of zero temperature, the result of the above theorem and this one may be interesting in connection with multifractal analysis because the limits of $\alpha(q)$ for $q \to \pm\infty$, give the interval in which the map $\mathcal{E}(\alpha)$, which describes the multifractal spectrum, is defined. Now we have a more explicit computation of such an interval in the special case treated in this section.

From the partition function 9 can be defined a zeta function in the sense of the Ruelle thermodynamic formalism:

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{Z_n(q)}{n} z^n, \tag{11}$$

which is analytic in $|z| < \exp(T(q))$. The usefulness of a such a maps is, for instance, to study the phenomenon of phase-transitions, i.e. the differentiability or not of the free energy. We could have a "thermodynamic-geometric" interface . If $q = 0$ the zeta function leads to the formal series $\sum_{n=0}^{\infty} \dfrac{card\{g : |g| = n\}}{n} z^n$. For the relationship, commented at the beginning of this section, between closed geodesics in H^2/Γ and elements of Γ this series is equivalent to the so called *generating function of the group* $\sum_{n=0}^{\infty} \dfrac{card\{\gamma : |\gamma| = n\}}{n} z^n$. This series plays an important role in combinatorial of groups and interesting related results about it were proved in [9]. Again we point out that the case $q = 0$ is not within the zero limit problems, but in a more general context presents the mentioned interest.

We finish with a combinatorial version of the above analysis, we consider a partition function obtained counted the so called cone types in the Cayley graph of the group. Next we shall explain how is this done as well as we will introduce the necessary background. Recall that the Cayley graph for a group Γ, with respect to a finite set of generators S, is the graph whose vertices are the elements of Γ and such that there is an edge between γ_1 and γ_2 if and only if $\gamma_1\gamma_2^{-1} \in S$. The Cayley graph allows to define geometric properties for a group being not this possible from the word metric, which takes only integer values. In fact the Cayley graph $\mathcal{G}(\Gamma, S)$ can be endowed with a length metric by making any edge isometric to a real interval of length 1, and in this way $\mathcal{G}(\Gamma, S)$ becomes geodesic space: a geodesic is the shortest path from one vertex to the other. An order in $\mathcal{G}$ is defined as follows

$\gamma_1 \leq \gamma_2$ if there is a geodesic from the vertex corresponding to the identity to γ_2 which passes first by γ_1. The *cone* in γ is the set $[\gamma, \infty) = \left\{\gamma' : \gamma \leq \gamma'\right\}$ and let us consider the following equivalence relationship in $\mathcal{G}$

$\gamma_1 \sim \gamma_2$ if and only if $[\gamma_1, \infty)$ is taken to $[\gamma_2, \infty)$ by a homeomorphism which preserves the order. The equivalence classes are named as the *cone types.*

By a result derived from the Series coding theory each $\gamma \in \Gamma$ admits an unique representation as a word $e_0 e_1 ... e_{n-1}$,which is called the canonical form relative to f. This leads to a way of counting and enumerating elements of Γ, which gives a solution for the Dehn problem. This means that if Γ is a group with a finite set of generators Γ considered in our analysis then any word w in elements of S and S^{-1} representing the identity as element of the free group on S (a relator) can be shortened by mean of relator with length word metric $\leq N$, for some integer N. A process of this nature was primarily carried out by Dehn for fundamental groups of negatively curved surfaces with genus ≥ 2. Groups for which such an algorithm for shorting words exits are cases of automatic groups and Cannon [8] proved that for this kind of groups, which are hyperbolics in the Gromov sense, there are finitely many cone types.

Thus we can define a partition function counting cone types instead of geodesics or elements of Γ. By the mentioned Cannon results an automatic group can be understood in finite terms and in practice could be easier to compute the partition function, or in particular the generating function of the group, summing over cone types. In this way can be followed a similar scheme than the presented at the beginning of the section but with a combinatorial approach.

3. Appendix

In this appendix we display the definitions of the boundary hyperbolic maps and to enlarge the background we add some comments about facts mentioned in the precedent section. As before let Γ be finitely generated a Kleinian group on H^2 and Λ its limit set. Let us consider a fundamental region which will be a polygon R inscribed in H^2, as we commented earlier a group acting by side pairing edges of R is a free group. The Series's general construction [16],[17] of the boundary maps $f : \partial H^2 \to \partial H^2$ is as follows:

Let $e \in S$ (the set of generators for Γ) and each side of R is labelled by elements of S, by $M(e)$ is denoted the half-plane in H^2 determined by the extension of the side $C(e)$ (the side labelled by $C(e)$) which does not contain R to ∂H^2. In each intersection $M(e_i) \cap M(e_j)$ $(\{e_i, e_j\} \subset \Gamma_1)$, it chooses arbitrary $e_k \in \{e_i, e_j\}$ and defines

$f \mid_{M(e_i) \cap M(e_j)} (x) = e_k^{-1} x$, whereas in the remaining part of $M(e_i)$ the map is defined by $f = e_i^{-1}$. Any triple of half-planes has non-empty intersection [7].

The following notation is used:

$B(e_i) := \left\{x \in M(e_i) : f(x) = e_i^{-1} x\right\}$

$I(e_i) := \partial H^2 \cap \overline{B(e_i)}$

$B(e_0e_1...e_n) := \bigcap_{l=0}^{n} f^{-l}(B(e_l))$

$I(e_0e_1...e_n) := \partial H^2 \cap \overline{B(e_0e_1...e_n)} = \bigcap_{l=0}^{n} f^{-l}(I(e_l))$.

For the groups Γ considered in this article, as we pointed out earlier, to each $\xi \in \Lambda$ is assigned a sequence $(e_0e_1...)$, $e_i \in S$, in the following way: if $A(e)$ is the cut-off in ∂H^2 of $C(e)$, let $A = \cup\{A(e) : e \in S\}$,now the assignation is done accordingly $f^i x \in A(e_i)$. In this case $\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(A)$ and there exist the commented bijection between a symbolic space which consist in reduced sequences in S and Λ. The "cutting-sequence" of a geodesic g in H^2 is defined as the sequence $e_0e_1...e_n$, $e_i \in S$ obtained in the following way: the different images of R under the $\Gamma-$action can be considered and so the labels of the sides of R are carried to the corresponding ones in the images. Now the cutting sequence assigned to the geodesic is the sequence of side-labels met by g. If ξ is an end point of a geodesic beginning in O, then the boundary expansion of ξ and the cutting sequence of g are the same and the set of points with infinite boundary expansions are precisely the limit set Λ [16],[17]. From this it can be seen that if ξ has the boundary expansion $e_0e_1...e_n...$, then $e_0e_1...e_n(O)$ converges to ξ as $n \to \infty$. For a free group generated by two side pairing isometries this can checked by the combinatorial interpretation: the Cayley graph associated to a group Γ, relative to a particular set of generators S, the sequence $e_0e_1...e_n$ corresponds to the path in the graph joining $O, .e_0(O), ..., e_0e_1...e_n(O)$, and the ends of the graph are identified with a finite type subshift (with alphabet S) which is in turn identified with the Λ.

In [13] we have used a multifractal formalism to study growth rates of Kleeinian groups. For this was defined a family $\varphi_n(\xi) := d_h(O, e_0e_1...e_n(O))$, $\xi \in \Lambda$, (d_h is the hyperbolic metric) and considered the level sets $K_\alpha := \left\{\xi : \lim_{n\to\infty} \frac{\varphi_n(\xi)}{n} = \alpha\right\}$. The study of the grow rate was done in terms of a description of the multifractal entropy spectrum $\mathcal{E}(\alpha) := h_{top}\left(f, \left\{\xi \in \Lambda : \lim_{n\to\infty} \frac{\varphi_n(\xi)}{n} = \alpha\right\}\right)$. In an earlier paper [12] we connected geometrical constructions from limit sets of Fuchsian groups with multifractal analysis by modifying the definition of a local entropy by Takens and Verbitski.

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