FACTORS OF INSERTED GRAPHS

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(Received: 12 January 2006)

Abstract. A k-regular spanning subgraph of G is called a k-factor of G. In this paper we give a necessary and sufficient condition for the inserted graph $I(G)$ of a non-trivial graph G to have a 1-factor. If $d_G(v)$ denotes the degree of the vertex v in G and if $d_G(v) \geq m+1$ for all vertices v in G, where $m \geq 2$ is an even integer then we prove that the inserted graph $I(G)$ of G has a $2m$-factor. Also we prove that if m is a non-negative integer and G is a connected graph with even number of edges such that $d_G(v) \geq m+2$ for all vertices v in G, then the inserted graph $I(G)$ of G has a $(2m+1)$-factor.

AMS Classification: 05C70
Keywords: Box Graph, Inserted Graph, k-factor

1. Introduction

In this paper, by a graph we mean an undirected, finite graph without loops and multiple edges. Let G be a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. If e be an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$. The degree of a vertex v of G is denoted by $d_G(v)$. For a subset A of V_G, $G - A$ denotes the subgraph of G obtained from G by deleting the vertices in A together with the edges incident with them. If A and B are disjoint subset of V_G, then we denote $e_G(A, B)$ the number of edges of G joining A and B. A subset $A \subseteq V_G$ is often identified with subgraph of G induced by A.

A factor of a graph G is a spanning subgraph of G. A k-factor of a graph G is a k-regular spanning subgraph of G. A decomposition of a regular graph G into k-factors is a k-factorization of G. A graph with a k-factorization is k-factorable.

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [1].

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 4, NUMBER 2 (2006) 159-165.
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In §2, we recall some definitions and results which will be used in this paper. Also we give an example of 1-factorization.

Tutte characterize the graphs possessing a 1-factor in [5]. He also characterize those graphs having a spanning subgraphs with prescribed degree sequence in [6]. In [7], Tutte gave a short proof of the factor theorem for finite graphs. Beineke and Plummer have shown that many graphs can not have exactly one 1-factor in [3]. In §3, we give a necessary and sufficient condition for the inserted graph $I(G)$ of G has a 1-factor. More particularly, for a non-trivial connected graph G the inserted graph $I(G)$ has a 1-factor if and only if $|I_G|$ is even.

In §4, we consider k-factors with $k \geq 2$ and give a sufficient condition for the existence of such factors in the inserted graph of a graph G in terms of the degree of vertices of G. We also give an example which shows that the condition $d_G(v) \geq m+2$ can not be weakened to $d_G(v) \geq m+1$.

2. Preliminaries

In this section we recall some definitions and results.

Definition 2.1. [1] *A new graph can be constructed by inserting a new vertex on each edge of a non-trivial graph G, the resulting graph is called Box graph of G, denoted by $B(G)$. For an edge e of G, $\overline{e}$ denote the vertex of $B(G)$ corresponding to the edge e.*

The graph $B(G)$ has the property that, there always exists a one-one correspondence between the vertices and the elements of G such that any two vertices of $B(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. Obviously, $B(G)$ is a bipartite graph whose number of vertices is equal to the number of elements of G. Moreover, if $V_G = \{v_1, v_2, ..., v_n\}$ and $E_G = \{e_1, e_2, ..., e_m\}$ then $V_{B(G)} = \{v_1, v_2, ..., v_n, \overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$.

Definition 2.2. [1] *Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$. Therefore if $E_G = \{e_1, e_2, ..., e_m\}$ then $I_G = V_{I(G)} = \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$.*

Definition 2.3. [4] *For a positive integer k, a k-factor of a graph G is a spanning subgraph H of G such that $d_H(v) = k$, for each vertex v in G.*

Now we state the following well-known criterion for the existence of a k-factor.

Theorem 2.4. [7] *A graph G has a k-factor if and only if*

$$\delta_G(S,T) = k|S| + \sum_{v \in T} d_G(v) - k|T| - e_G(S,T) - h(S,T) \geq 0$$

for any disjoint subsets S, T of V_G, where $h(S,T)$ denotes the number of connected components C of $G-(S \cup T)$ such that $k|C|+e_G(C,T) \equiv 1$ (mod 2) [such components are referred to as odd components]. Furthermore, $\delta_G(S,T) \equiv k|V_G|$ (mod 2) for any disjoints subsets S, T of V_G [whether G has a k-factor or not].

Example 2.5. *The following figure shows a 1-factorization of* K_6.

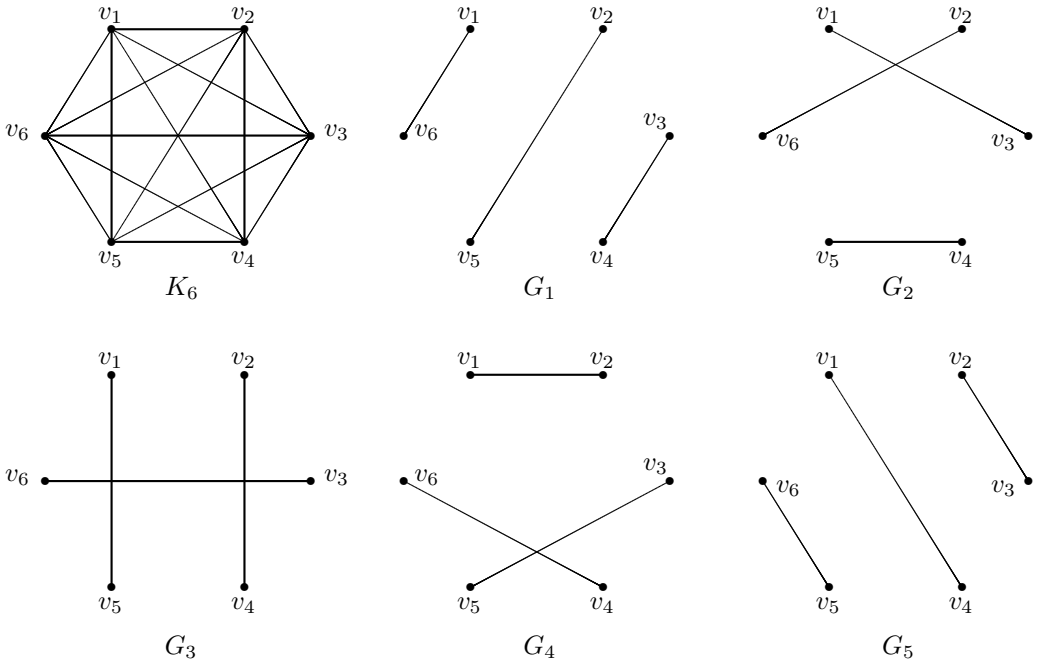


Figure 1

3. Existence of 1-factors in $I(G)$

A 1-factor in a graph H is a spanning 1-regular subgraph of H. It is immediate that a necessary condition for a connected graph H to contain a 1-factor is that H have an even number of vertices. This condition is easily seen not to be sufficient. For example, the complete bipartite graph $K(1, n-1)$, where n is even, does not even contain two non-adjacent edges, much less $\frac{n}{2}$ such edges which would be required of a 1-factor.

Lemma 3.1. *If a non-trivial* G *has no or two odd degree vertices, then* $I(G)$ *contains a 1-factor.*

Proof: If G has no odd degree vertices, implying that G is Eulerian and thus contains an Eulerian circuit, so $I(G)$ has a 1-factor. Again if G contain exactly two odd degree vertices, implying that G contains Eulerian trails connecting the two odd degree vertices, then $I(G)$ contains a 1-factor. In order to see this, one needs only to note that the existence of an Eulerian circuit or Eulerian trail in G implies that the edges of G can be ordered as $e_1, e_2, ..., e_{2k}$, in such a way that consecutive edges in the ordering are adjacent. Now in $I(G)$ the consecutive vertices in the sequence $\overline{e}_1, \overline{e}_2, ..., \overline{e}_{2k}$ are adjacent. This implies that $\overline{e}_1\overline{e}_2, \overline{e}_3\overline{e}_4, ..., \overline{e}_{2k-1}\overline{e}_{2k}$ are edges of $I(G)$ and $I(G)$ has a 1-factor.

Lemma 3.2. *If* G *be a connected graph with* $|V_G| = 2n$*;* $n \geq 1$ *and degree of each vertex of* G *is odd. Then* G *is the edge-disjoint union of* n *trails, such that at most one of these trails has odd length.*

Proof: Suppose G is the edge-disjoint union of open trails $T_1, T_2, ..., T_n$ such that T_i, $i = 1, 2, ..., n$ is a $u_i - v_i$ trail, where $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ are the $2n$ odd degree vertices of G. If at most one T_i has odd length, then of course there is nothing to prove. Suppose this is not the case so that there are at least two trails having odd length.

If two trails T_j and T_k of odd length possess a common vertex w, then T_j and T_k may be replaced by the trails T_j^* and T_k^* of even length in a manner which we shall now describe. Let $T_j^{'}$ denote a $u_j - w$ subtrail of T_j and let $T_j^{''}$ denote a $w - v_j$ subtrail of T_j. Similarly, denote by $T_k^{'}$ a $u_k - w$ subtrail of T_k and by $T_k^{''}$ the remaining $w - v_k$ subtrail of T_k. The lengths of $T_j^{'}$ and $T_j^{''}$ are of opposite parity, as are the lengths of $T_k^{'}$ and $T_k^{''}$.

If $T_j^{'}$ and $T_k^{'}$ are of the same parity, then the $u_j - u_k$ trail T_j^* composed of $T_j^{'}$ and $T_k^{'}$ has even length. In this case, $T_j^{''}$ and $T_k^{''}$ have the same parity, and the $v_j - v_k$ trail T_k^* composed of $T_j^{''}$ and $T_k^{''}$ also has even length. On the other hand, if $T_j^{'}$ and $T_k^{'}$ have opposite parity, then $T_j^{'}$ and $T_k^{''}$ are of the same parity, and we may obtain trails T_j^* and T_k^* of even length by combining $T_j^{'}$ and $T_k^{''}$ and combining $T_j^{''}$ and $T_k^{'}$, respectively.

We may continue this process until no two trails of odd length remain having a vertex in common. Of course, if at most one remaining trail T_i has odd length, then the proof is complete. However we assume there are trails T_r and T_s of odd length which have no vertex in common.

We define the distance $d(T_r, T_s)$ between T_r and T_s as the minimum of the distance between a vertex of T_r and a vertex of T_s. From our assumption $d(T_r, T_s) \geq 1$. Let w_r and w_s be vertices of T_r and T_s respectively, such that $d(w_r, w_s) = d(T_r, T_s)$, and let P be a $w_r - w_s$ path of length $d(w_r, w_s)$.

Let $w_r w_1$ be the edge of P incident with the vertex w_r. In the current decomposition of G into n trails, assume $w_r w_1$ belongs to the trail T. Necessarily, T has even length. Suppose T is a $u - v$ trail. Let $T^{'}$ be a $u - w_r$ subtrail of T and $T^{''}$ the remaining $w_r - v$ subtrail of T containing $w_r w_1$. Also let $T_r^{'}$ be a $u_r - w_r$ subtrail of T_r and $T_r^{''}$ be the remaining $w_r - v_r$ subtrail of T_r. Either $T_r^{'}$ and $T^{''}$ or $T_r^{''}$ and $T^{''}$ are of opposite parity. Assume, with out loss of generality, that $T_r^{'}$ and $T^{''}$ are of opposite parity. Define T_r^1 to be the trail composed of $T_r^{'}$ and $T^{''}$; the trail T_r^1 has odd length and $d(T_r^1, T_s) < d(T_r, T_s)$. Note that the trail composed of $T_r^{''}$ and $T^{'}$ has even length.

If T_r^1 and T_s have a vertex in common, then as we have already seen, these trails may be replaced by two trails of even length. Otherwise we may repeat the above process with T_r and T_s, replaced by T_r^1 and T_s, obtaining trails T_r^2 and T_s of odd length such that $d(T_r^2, T_s) < d(T_r^1, T_s)$. This process may be continued until trails T_r^* and T_s of odd length are obtained which have a common vertex. No other trail of odd length is altered. The trails T_r^* and T_s may then be replaced by two trails of even length.

We may now repeat this argument as many times as there are pairs of trails having odd length, arriving at a collection of n trails, at most one of which has odd length.

Theorem 3.3. *Let G be a connected graph. Then the inserted graph $I(G)$ of G has a 1-factor if and only if $|I_G|$ is even.*

Proof: The condition $|I_G|$ is even is obviously necessary condition for $I(G)$ to have a 1-factor. For the sufficiency, assume $|I_G|$ is even so that G has an even number of edges. By Lemma 3.1 if G has no or two odd degree vertices, then $I(G)$ contains a 1-factor. Thus we may assume G has exactly $2n$ $(n \geq 1)$ vertices each of them with odd degree. By the Lemma 3.2, G can be expressed as the union of n edge-disjoint trails $T_1, T_2, ..., T_n$ each of which has even length.

Hence, it is possible to order the edges of T_1, say $x_1, x_2, ..., x_{2t}$ in such a way that consecutive edges in the sequence are adjacent. Hence every two consecutive vertices in the sequence $\overline{x}_1, \overline{x}_2, ..., \overline{x}_{2t}$ are adjacent. Thus the edges $\overline{x}_1\overline{x}_2, \overline{x}_3\overline{x}_4, ..., \overline{x}_{2t-1}\overline{x}_{2t}$ produce a 1-factor F_1 in $I(T_1)$. In a completely analogous manner, we can obtain a 1-factor F_i, $i = 1, 2, ..., n$ for $I(T_i)$. Then $\bigcup_{i=1}^{n} F_i$ is a 1-factor of $I(G)$.

Corollary 3.4. *The inserted graph $I(G)$ of a graph G has a 1-factor if and only if every component of $I(G)$ has even number of vertices.*

4. Factors of $I(G)$

Theorem 4.1. *Let G be a graph and $m \geq 2$ be an even integer, suppose $d_G(v) \geq m + 1$ for all vertices v in G, then the inserted graph $I(G)$ of G has a $2m$-factor.*

Proof: Let G be a graph with $d_G(v) \geq m + 1; \forall v \in V_G$, where $m \geq 2$ be an even integer. Let S and T be subsets of I_G such that $S \cap T = \phi$. For convenience, we set $U = I(G) - (S \cup T)$ and $h = h(S, T)$. We denote the sets of edges incident with $v \in V_G$ contained in S, T and U by S_v, T_v and U_v respectively. We clearly have the following

$$|S_v \cup T_v \cup U_v| = d_G(v); \forall v \in V_G, \text{.....(1)}$$

$$\sum_{v \in V_G} |S_v| = 2 \mid S|, \sum_{v \in V_G} |T_v| = 2|T|, \sum_{v \in V_G} |U_v| = 2|U|, \text{.....(2)}$$

$$|S \cup T \cup U| = \sum_{v \in V_G} \frac{d_G(v)}{2} = |E_G|, \text{.....(3)}$$

$$\sum_{x \in T} d_{I(G)}(x) = \sum_{v \in V_G} \{d_G(v) - 1\}|T_v|, \text{.....(4)}$$

$$e_{I(G)}(S, T) = \sum_{v \in V_G} |S_v||T_v| \text{.....(5)}$$

Inserting (2), (4) and (5) into Theorem 2.4 with $k = 2m$, we obtain

$$\delta_{I(G)}(S, T) = \sum_{v \in V_G} \{d_G(v) - m - 1\}|T_v| + m|S_v| - |T_v||S_v| - h.$$

Now we set

$$\lambda(v) = \{d_G(v) - m - 1\}|T_v| + m|S_v| - |T_v||S_v|, \forall v \in V_G.$$

It is easy to show that $\lambda(v) \geq 0; \forall v \in V_G$. For each component X of U, let R_X be the set of vertices $v \in V_G$ such that $X \cap U_v \neq \phi$. Then for any two distinct components X and Y we have $R_X \cap R_Y = \phi$.

So in order to prove $\delta_{I(G)}(S,T) \geq 0$, it is sufficient to show that for each odd component X of U, R_X contains a vertex v with $\lambda(v) \geq 1$. Now let X be an odd component of U. By the definition of an odd component,

$$e_{I(G)}(X,T) = \sum_{v \in R_X} |U_v||T_v|$$

is odd; and we can therefore choose a vertex $v \in R_X$ for which $|U_v||T_v|$ is odd. For this v, we have either $|S_v| \neq 0$ or $d_G(v) \geq m+2$. For if $|S_v| = 0$ and $d_G(v) = m+1$ then as $m+1$ is odd, one of $|U_v|$ or $|T_v|$ is forced to be even. Since we also have $|U_v| \neq 0$ and $|T_v| \neq 0$, it is now a matter of simple calculation to verify that the desired inequality $\lambda(v) \geq 1$ holds. Thus $\delta_{I(G)}(S,T) \geq 0$, and the proof is complete.

As an immediate corollary to this theorem, we get

Corollary 4.2. *Let G be a graph and $m \geq 1$ be an odd integer and suppose that $d_G(v) \geq m+2; \forall v \in V_G$, then the inserted graph $I(G)$ of G has a $2m$-factor.*

Theorem 4.3. *Let G be a connected graph with even number of edges , m be a non-negative integer and suppose that $d_G(v) \geq m+2; \forall v \in V_G$, then the inserted graph $I(G)$ of G has a $(2m+1)$-factor.*

Proof: Let G be a connected graph with even number of edges , m be a non-negative integer and suppose that $d_G(v) \geq m+2; \forall v \in V_G$. Also let S and T be subsets of I_G, and define U, S_v, T_v, U_v and h as in Theorem 4.2. Further set,

$$\lambda(v) = \{d_G(v) - m - \frac{3}{2}\}|T_v| + (m + \frac{1}{2})|S_v| - |T_v||S_v|, \forall v \in V_G.$$

Then we have

$$\delta_{I(G)}(S,T) = \sum_{v \in V_G} \lambda(v) - h.$$

In view of the last assertion of Theorem 2 it is sufficient to show that $\delta_{I(G)}(S,T) \geq -1$. Let ω denote the number of components of $U = I(G) - (S \cup T)$. Also let $Y_1, ..., Y_\omega$ be the components of U and for each Y_i, let P_i be the set of vertices $v \in V_G$ such that $Y_i \cap U_v \neq \phi \ \ (1 \leq i \leq \omega)$. Moreover we set

$$Q = V_G - \bigcup_{1 \leq i \leq \omega} P_i = \{u_1, ..., u_l\} \ \ where \ \ l = |Q|.$$

Thus $P_1, ..., P_\omega, \{u_1\}, ..., \{u_l\}$ are the vertex set of the components of $G - (S \cup T)$. Since G is connected, this means $|S \cup T| \geq \omega + l - 1$. Now by calculation, we obtain

$$\lambda(v) \geq \frac{|S_v| + |T_v|}{2} \ \ if \ \ v \in \bigcup_{1 \leq i \leq \omega} P_i,$$

and

$$\lambda(v) \geq \frac{|S_v| + |T_v|}{2} - 1 \ \ if \ \ v \in Q.$$

Hence we get

$$\delta_{I(G)}(S,T) \geq \sum_{v \in V_G} \lambda(v) - \omega \geq \sum_{v \in V_G} \frac{|S_v| + |T_v|}{2} - l - \omega$$
$$= |S \cup T| - (l + \omega) \geq (l + \omega - 1) + (l + \omega) = -1.$$

This completes the proof.

Remark 4.4. *Those results are best possible in the sense that the conditions on the minimum degree can not be weakened any further. This is clear for Theorem 4.1 and Theorem 4.3. As for Corollary 4.2, the following example shows that the condition* $d_G(v) \geq m+2$ *can not be weakened to* $d_G(v) \geq m+1$.

Example 4.5. *Let* G *be an* $(m+1)$*-regular graph with at least* $(m+2)$ *independent edges and* $M = \{u_1v_1, ..., u_{m+2}v_{m+2}\}$ *be the set of independent edges of* G*, where* m *is an odd integer. Now we add two new vertices* x *and* y *to* V_G*, and join* x *to* u_i *and* y *to* v_i*, where* $1 \leq i \leq m+2$*. We consider the new graph* G' *with vertex set* $V_G \cup \{x, y\}$ *and with edge set* $(E_G - M) \cup \{xu_i, yv_i; 1 \leq i \leq m+2\}$*. Note that in* $I(G')$ *all vertices other than* $\overline{xu}_i$ *and the* $\overline{yv}_i$ *have degree exactly 2m. So if* $I(G')$ *had a* $2m$*-factor* F*, then the* $(m+2)$ *vertices* $\overline{xu}_i$ *would induce an* m*-regular subgraph in* F *(and the same would be true of the* $\overline{yv}_i$*). But this is impossible because* m *and* $m+2$ *are odd. Figure 2 shows that the easiest example with* $m = 1$.

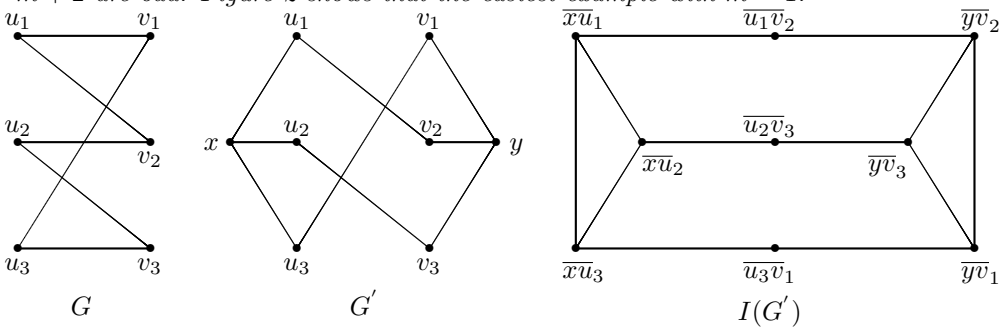


Figure 2

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