

# CONTINUATION OF PERIODIC ORBITS OF A THIRD ORDER OSCILLATOR UNDER AUTONOMOUS PERTURBATION

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**ABSTRACT.** We consider systems of nonlinear third-order differential equations depending on a small parameter such that the unperturbed system has a three dimension manifold of periodic solutions. We impose some conditions on the system in order to reduce the third order perturbed system to the second one. By using Poincare-Andronov-Hopf theorem we prove the existence of many periodic solutions the perturbed system.

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## 1. INTRODUCTION

Suppose that we have the system

$$(1) \quad x''' + \omega^2 x' = \epsilon F(x, x', x'') , \epsilon$$

where  $\epsilon$  is a real parameter characterizing the strength of the external driving force. The third order differential equations which are of the form (1) usually arise in problems related to energy and acceleration. For  $\epsilon = 0$  the equation (1) has periodic solutions. Although there are some papers about existence of the periodic solutions of a nonlinear third order differential equation, for examples see [1,2,8,9,...]. But finding periodic solutions for this kind of system is not very easy . In 2003 Rabiei and Afsharnejad [8] proved that under some conditions on function  $F(x, \dot{x}, \ddot{x})$  there exist many periodic solutions for system

$$(2) \quad x''' + \omega^2 x' = \epsilon F(x, x', x'')$$

Mehri and Niksirat in [6] considered the nonlinear third-order (2) and obtained some conditions for the existence of a periodic solution for it. They use the fixed point theorem and they impose the boundary values for the equation for finding one periodic solution for the system (2). In this paper we suppose that the function  $F$  is depending on parameter  $\epsilon$  and we have no restriction on the parameter  $\epsilon$ . Also we

omit the condition  $Df(0, 0, 0) = 0$  as it is in paper [8] and present a new condition for the system (1) in order to have many periodic solutions. Let us suppose that the function  $F(x, x', x'', \epsilon)$  is

$$(3) \quad F(x, x', x'', \epsilon) = x' f_x(x, x', \epsilon) + x'' f_{x'}(x, x', \epsilon)$$

where  $f$  is  $C^r$  ( $r \geq 2$ ) and  $f_x, f_{x'}$  are derivation of  $f$  with respect to  $x$  and  $x'$  respectively. We show that if  $f_{x'}(x_0, x'_0, \epsilon_0) \neq 0$ , then (1) has many periodic solutions.

Section 2 of this paper is devoted to some preliminaries. In section 3, we discuss about the existence of fixed points and periodic solutions. In section 4 we give some applications about the existence of periodic solutions and homoclinic orbits.

## 2. PRELIMINARIES

We write the equation (1) as

$$(4) \quad X' = AX + \epsilon G(X, \epsilon)$$

where  $X = (x, y, z) = (x, x', x'') \in R^3$ ,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{pmatrix}, \quad G(X, \epsilon) = \begin{pmatrix} 0 \\ 0 \\ F(x, x', x'', \epsilon) \end{pmatrix}$$

and  $F(x, x', x'', \epsilon)$  as in (3). By using theory of ordinary differential equations it is not difficult to see that

$$\Phi(t) = \begin{pmatrix} 1 & \frac{1}{\omega} \sin \omega t & \frac{1}{\omega^2} (1 - \cos \omega t) \\ 0 & \cos \omega t & \frac{1}{\omega} \sin \omega t \\ 0 & -\omega \sin \omega t & \cos \omega t \end{pmatrix}$$

is a fundamental matrix of the unperturbed system

$$(5) \quad X' = AX$$

So every point of  $R^3$  is on a periodic orbit of (5). By considering (3) we have

$$\frac{d}{dt}(x'' + w^2 x - \epsilon f(x, x', \epsilon)) = x''' + w^2 x' - \epsilon(x' f_x(x, x', \epsilon) + x'' f_{x'}(x, x', \epsilon)) = 0$$

Therefore

$$x'' + w^2 x - \epsilon f(x, x', \epsilon) = k$$

where  $k$  is a real constant. If we put  $k = \epsilon \mu$  then we get

$$x'' + w^2 x - \epsilon(f(x, x', \epsilon) + \mu) = 0$$

or

$$(6) \quad x'' + w^2 x - \epsilon g(x, x', \epsilon, \mu) = 0$$

in which  $g(x, x', \epsilon, \mu) = f(x, x', \epsilon) + \mu$ . Also we can write (6) as

$$(7) \quad \begin{cases} x' = y \\ y' = -w^2 x + \epsilon g(x, y, \epsilon, \mu) \end{cases}$$

Therefore the following lemma is obvious.

**Lemma 1:** The curve  $(x(t), x'(t))$  is a (periodic, homoclinic) solution of (6) if and only if  $(x(t), x'(t), x''(t))$  is a (periodic, homoclinic) solution of (1).

**Proof :** The proof of this lemma is same as [8, Theorem 3.1].  $\square$

Since we are going to use the Hopf bifurcation theorem, we give it at the end of this section.

**Theorem 2(Poincare-Andronove-Hopf):** Let  $X' = A(\epsilon)X + G(X, \epsilon) = F(x, \epsilon)$  be a  $C^r, r \geq 3$  planar vector field depending on a scalar parameter  $\epsilon$  such that  $G(0, \epsilon) = 0$  and  $D_X G(0, \epsilon) = 0$  for all sufficiently small  $|\epsilon|$ . Assume that the linear part  $F$ , that is to say  $A(\epsilon)X$ , at 0, has the eigenvalues  $\alpha(\epsilon) \pm i\beta(\epsilon)$  with  $\alpha(0) = 0$  and  $\beta(0) \neq 0$ . If  $\alpha'(0) \neq 0$ , then in any neighborhood  $U$  of origin in  $R^2$  and for any given  $\epsilon_0 > 0$ , there is a  $\bar{\epsilon}$  with  $|\bar{\epsilon}| < \epsilon_0$  such that  $X' = A(\bar{\epsilon})X + G(X, \bar{\epsilon})$  has a nontrivial periodic orbit in  $U$ .

For proof and more information see [5]

### 3. PERIODIC SOLUTIONS

The origin is the only fixed point of the unperturbed system (5). By lemma 1 for studying fixed points of (1) it is enough to find fixed points of (7). In fact we look for the zeroes of  $-w^2x + \epsilon g(x, 0, \epsilon, \mu) = 0$ , so we define

$$\varphi : R^3 \rightarrow R$$

$$\varphi(x, \epsilon, \mu) = -w^2x + \epsilon g(x, 0, \epsilon, \mu)$$

We see that  $\varphi(0, 0, \mu) = 0$  and  $\varphi_x(0, 0, \mu) = -w^2 \neq 0$ , then by Implicit function theorem for each  $\mu$  there exists an open set  $U_\mu$  containing  $(0, \mu)$  and a  $C^1$  map  $x_\mu : U_\mu \rightarrow R$  such that for each  $(\epsilon, \mu) \in U_\mu$ , we have

$$\varphi(x_\mu(\epsilon, \mu), \epsilon, \mu) = 0$$

**Theorem 3:** For each positive real number  $N$ , there exists an open rectangle  $M \subseteq R^2$  containing  $\{0\} \times [-N, N]$  and a  $C^1$  map  $x : M \rightarrow R$  such that  $(x(\epsilon, \mu), 0)$  is a fixed point of (7).

**Proof:** By the above arguments since  $\{0\} \times [-N, N]$  is compact we can find  $-N \leq \mu_1, \dots, \mu_n \leq N$  such that  $\{0\} \times [-N, N] \subseteq \cup U_{\mu_i}$ . Let  $M$  be an open rectangle with  $\{0\} \times [-N, N] \subset M \subset \cup U_{\mu_i}$ . If  $(\epsilon, \mu) \in U_{\mu_i} \cap U_{\mu_j}$ , then by uniqueness of the solution of the Implicit function theorem  $x_{\mu_i}(\epsilon, \mu) = x_{\mu_j}(\epsilon, \mu)$ . This shows that we can define the  $C^1$  map

$$x : M \rightarrow R$$

$$(\epsilon, \mu) \rightarrow x_{\mu_i}(\epsilon, \mu) \quad (\epsilon, \mu) \in U_{\mu_i}$$

such that  $(x(\epsilon, \mu), 0)$  is a fixed point for (7).  $\square$

**Remark :** The linear part of (7) at  $(x(\epsilon, \mu), 0)$  is

$$A(\epsilon, \mu) = \begin{pmatrix} 0 & 1 \\ -\omega^2 + \epsilon g_x & \epsilon g_y \end{pmatrix}$$

where all partial derivatives of  $g$  are computed at  $(x(\epsilon, \mu), 0, \epsilon, \mu)$ . So the eigenvalues of  $A(\epsilon, \mu)$  are

$$(8) \quad \lambda_{1,2} = \frac{\epsilon g_y \pm \sqrt{(\epsilon g_y)^2 + 4(\epsilon g_x - \omega^2)}}{2}$$

In the rectangle  $M$  if we consider small enough values of  $\epsilon$  then  $(\epsilon g_y)^2 + 4(\epsilon g_x - w^2) < 0$ , thus the real part of  $\lambda_{1,2}$  is  $\frac{\epsilon g_y}{2}$ .

**Lemma 4:** Suppose that  $M$  is as in the previous theorem and  $\epsilon$  is small enough. Then the fixed point  $(x(\epsilon, \mu), 0)$  is isolated.

Proof: Define

$$h : R \rightarrow R$$

$$h(x) = -w^2x + \epsilon g(x, 0, \epsilon, \mu)$$

Then  $h(x(\epsilon, \mu)) = 0$  and  $h_x(x(\epsilon, \mu)) = -w^2 + \epsilon g_x = H$ . Now if  $H < 0$  then  $h$  is decreasing in a neighborhood of  $x(\epsilon, \mu)$ . But by the remark we take the values of  $\epsilon$  in  $M$  such that  $H < 0$ . Therefore the fixed point is isolated.  $\square$

Now we use the Hopf bifurcation theorem (Poincare-Andronov-Hopf) in order to prove the existence of periodic solutions for (7). By relation (8) the real and imaginary part of  $\lambda_{1,2}$  are

$$\alpha(\epsilon) = \frac{\epsilon g_y}{2}, \quad \beta(\epsilon) = \frac{\sqrt{4(w^2 - \epsilon g_x) - (\epsilon g_y)^2}}{2}$$

It is easy to see that  $\alpha(0) = 0$ ,  $\beta(0) = w \neq 0$ ,  $\alpha'(0) = \frac{1}{2}g_y(x(0, \mu), 0, 0, \mu)$ . Therefore the following theorem is true.

**Theorem 5:** Consider the system (7), where  $(x(\epsilon, \mu), 0)$  is a fixed point for it. If  $g_y(x(0, \mu), 0, 0, \mu) \neq 0$ , then at  $\epsilon = 0$  the Hopf bifurcation occurs at  $(x(0, \mu), 0)$  for each  $\mu \in [-N, N]$ .

Indeed by verifying the Hopf bifurcation at  $(x(0, \mu), 0)$ , we can see that whether the system (7) has periodic solutions for  $\epsilon > 0$  or  $\epsilon < 0$ . Here we assume that it has periodic solutions for  $\epsilon \in \bar{I}$  ( $\bar{I} = \{\epsilon | \epsilon > 0\}$  or  $\bar{I} = \{\epsilon | \epsilon < 0\}$ ).

**Theorem 6 :** If  $f_{x'}(x(0, \mu), 0, 0) \neq 0$  then for each  $\epsilon \in \bar{I}$ , the system

$$(9) \quad x''' + w^2x' - \epsilon(f_x(x, x', \epsilon)x' + f_{x'}(x, x', \epsilon)x'') = 0$$

has many periodic solutions.

**Proof:** By theorem 5 the system (6) has a periodic solution for each  $\mu \in [-N, N]$ , so lemma 1 implies that there are many periodic solutions for the system (9).  $\square$

**Corollary 7 :** Let

$$h : R^3 \rightarrow R$$

$$(x, y, \epsilon) \rightarrow h(x, y, \epsilon)$$

be a  $C^2$  map such that  $h(0, 0, 0) = 0$  and  $h_y(0, 0, 0) \neq -a$ , where  $a \in R$  is a real constant. Then for each  $\epsilon \in \bar{I}$ , the equation

$$x''' + w^2x' = \epsilon(h_x x' + (h_y + a)x'')$$

has many periodic solutions.

## 4. SOME APPLICATIONS

In this section we consider the system (7) where  $g(x, y, \epsilon, \mu) = f(x, y, \epsilon) + \mu$ . If we suppose that  $f$  is a  $c^r$  function ( $r \geq 2$ ), the Taylor expansion of  $f$  at  $(0, 0, 0)$  up to  $O(3)$  as follows:

$$f(x, y, \epsilon) = a_0 + a_1x + a_2y + a_3\epsilon + a_4x^2 + a_5y^2 + a_6\epsilon^2 + a_7xy + a_8x\epsilon + a_9y\epsilon + O(3)$$

in which:

$$\begin{aligned} a_0 &= f(0, 0, 0), a_1 = f_x(0, 0, 0), a_2 = f_y(0, 0, 0), a_3 = f_\epsilon(0, 0, 0), \\ a_4 &= f_{xx}(0, 0, 0), a_5 = f_{yy}(0, 0, 0), a_6 = f_{\epsilon\epsilon}(0, 0, 0), a_7 = 2f_{xy}(0, 0, 0), \\ a_8 &= 2f_{x\epsilon}(0, 0, 0), a_9 = 2f_{y\epsilon}(0, 0, 0). \end{aligned}$$

We can suppose that  $f(0, 0, 0) = 0$  (otherwise put  $h(x, y, \epsilon) = f(x, y, \epsilon) - f(0, 0, 0)$ ). So by truncating the Taylor expansion at  $O(3)$ , the system (7) has the following form:

$$(10) \quad \begin{cases} x' = y \\ y' = -w^2x + \epsilon(a_1x + a_2y + a_3\epsilon + a_4x^2 + a_5y^2 + a_6\epsilon^2 + a_7xy + a_8x\epsilon + a_9y\epsilon + \mu) \end{cases}$$

**Lemma 8 :** If  $a_2 \neq 0$ , then the equation

$$(11) \quad x''' + w^2x' - \epsilon(a_1 + 2a_4x + a_7x' + a_8\epsilon)x' - \epsilon(a_2 + 2a_5x' + a_7x + a_9\epsilon)x'' = 0$$

for each  $\epsilon \in \bar{I}$  has many periodic solutions.

**Proof:** If  $a_2 \neq 0$  then  $f_{x'}(x(0, \mu), 0, 0) \neq 0$ , so the theorem 6 implies that the system (11) has many periodic solutions.  $\square$

**Example 1:** Put in the system (10) for  $1 \leq i \leq 9, i \neq 2, 4, a_i = 0, a_2 = b, a_4 = a$ . So we have

$$(12) \quad \begin{cases} x' = g_1(x, y, \epsilon, \mu) \\ y' = g_2(x, y, \epsilon, \mu) \end{cases}$$

where  $g_1(x, y, \epsilon, \mu) = y$  and  $g_2(x, y, \epsilon, \mu) = -w^2x + \epsilon(by + ax^2 + \mu)$ . It is easy to see that

$$A = \left( \frac{w^2 + \sqrt{w^4 - 4\epsilon^2 a \mu}}{2\epsilon a}, 0 \right), B = \left( \frac{w^2 - \sqrt{w^4 - 4\epsilon^2 a \mu}}{2\epsilon a}, 0 \right)$$

are the fixed points of (12). The eigenvalues of  $D(g_1, g_2)$  at A and B are

$$\lambda_{1,2}^A = \frac{b\epsilon \pm \sqrt{(b\epsilon)^2 + 4\sqrt{w^4 - 4\epsilon^2 a \mu}}}{2}, \quad \lambda_{1,2}^B = \frac{b\epsilon \pm \sqrt{(b\epsilon)^2 - 4\sqrt{w^4 - 4\epsilon^2 a \mu}}}{2}$$

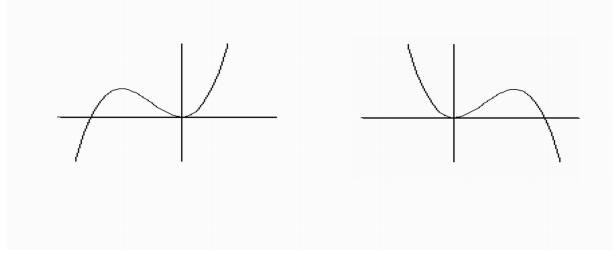
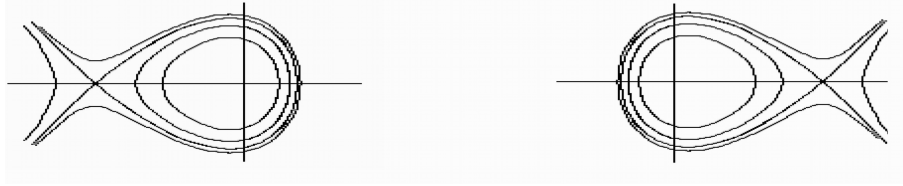
Hence for small values of  $\epsilon$ , A is a saddle point and B is the point at which the Hopf bifurcation occurs.

**Example 2:** We consider 10 with  $b = \mu = 0$  and  $a \neq 0$ , i.e.

$$(13) \quad \begin{cases} x' = y \\ y' = -w^2x + \epsilon ax^2 = p(x) \end{cases}$$

The equations (13) is Hamiltonian with potential energy

$$P(x) = - \int p(x) dx = \frac{w^2 x^2}{2} - \frac{\epsilon a x^3}{3}$$

FIGURE 1. The left one for  $\epsilon a < 0$  and the right one for  $\epsilon a > 0$ FIGURE 2. The left one for  $\epsilon a < 0$  and the right one for  $\epsilon a > 0$ 

and Hamiltonian function  $H(x, y) = \frac{y^2}{2} + \frac{w^2 x^2}{2} - \frac{\epsilon a x^3}{3}$ .

The potential function  $P(x)$  has one local maximum ( $x = \frac{w^2}{\epsilon a}$ ) and one local minimum ( $x = 0$ ). See figure 1.

The fixed points of (13) which are correspond to these extrema of  $P(x)$  are  $(0, 0)$ ,  $(\frac{w^2}{\epsilon a}, 0)$ . By Theorem 3 of page 155 in [7],  $(0, 0)$  is a center and  $(\frac{w^2}{\epsilon a}, 0)$  is a saddle for (13). Note that

$$P(\frac{w^2}{\epsilon a}) = P(-\frac{w^2}{2\epsilon a}) = \frac{w^6}{6\epsilon^2 a^2}$$

so for  $x = \frac{w^2}{\epsilon a}$  or  $x = -\frac{w^2}{2\epsilon a}$ ,  $y$  must be zero for  $(x, y)$  lie on the level curve  $H^{-1}(\frac{w^6}{6\epsilon^2 a^2})$ . Also since  $H(x, -y) = H(x, y)$ , then for  $-\frac{w^2}{2\epsilon a} < x < \frac{w^2}{\epsilon a}$  there are two points  $\pm y$  on  $H^{-1}(\frac{w^6}{6\epsilon^2 a^2})$ . Then the level curve of the energy is shown in the figure 2.

Hence  $(\frac{w^2}{\epsilon a}, 0)$  is a homoclinic point and the homoclinic orbit is filled up with the periodic solutions. Now we apply these results to the system (1).

**Lemma 9:** For each  $\epsilon \in R$ , the equations

$$(14) \quad \begin{cases} x' = y \\ y' = z \\ z' = -w^2x + 2\epsilon axy \end{cases}$$

has a homoclinic orbit and many periodic solutions.

**Proof:** By lemma 1 and example 2 the proof is obvious.  $\square$

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