

CLASSICAL EQUATIONS OF MOTION IN GRAVITATIONAL AND ELECTROMAGNETIC FIELDS FROM A VARIATIONAL PRINCIPLE

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ABSTRACT. A variational principle which has appeared previously is further developed, made rigorous and applied to a specific Lagrangian. This method produces the equations of motion for a massive, charged particle moving in gravitational and electromagnetic fields. Several interesting consequences of these results are discussed.

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1. INTRODUCTION

There has always been considerable interest in developing equations of motion from variational principles. For example, there are gauge-invariant variational principles [2], and certain types of geometric formulations, such as the Lagrangian and Poisson reduction for electromagnetism [4]. The intention here is to employ various differential geometric ideas to formulate a more rigorous variational principle, which has been discussed before, in particular, in [1]. This is applied to a particular action which is based on a specific model Lagrangian, and it is shown that equations result which have the structure of the equations for a charge in combined gravitational and electromagnetic fields. This treatment carries some of the aspects of these other treatments, but applied in a rigorous formal way.

The idea is to consider the motion of a particle of rest mass m_0 as a trajectory on a manifold M , such that the coordinates of the point particle depend on a particular parameter. The manifold will have a pseudo-riemannian structure associated with it, and this in turn gives rise to a covariant derivative. The basic functional will be defined on a parameter dependent vector field, deformed with respect to a reference vector field by means of this parameter, and will be defined rigorously shortly. The action functional is then extremized with respect to this new parameter, and the condition of the vanishing of this derivative provides the equation of motion.

Although some work along these lines has appeared before, the intention here is to set forth and develop this variational principle from a more rigorous differential geometric point of view. It will be shown that this procedure yields an equation of motion which reduces to the classical Lorentz force law when the components of the metric are all constant, and incorporates what can be interpreted as a gravitational interaction and reference frame dependent forces when the metric is not constant.

Let us begin by defining the mathematical framework that will be required and to define the Lagrangian. Several theorems from differential geometry will be reviewed that will be required to formulate the problem in rigorous terms [3].

Consider a general four-dimensional C^∞ manifold M which is coordinatized in terms of a four-dimensional spacetime variable x by a set of coordinates

$$\mathbf{x} = (x^0, x^1, x^2, x^3).$$

The trajectory of a point particle in the manifold M is a function of a parameter λ such that $\lambda \rightarrow x(\lambda)$ for $\lambda_0 \leq \lambda \leq \lambda_1$. Suppose also that the manifold has a pseudo-riemannian structure $\langle, \rangle$ which is associated with it. The bilinear form $\langle, \rangle$ is referred to as a Riemannian structure if it is positive definite and Lorentzian if it has index one. The Lagrangian will depend explicitly on the derivatives of the coordinates $dx^i/d\lambda$, but will not depend explicitly on the independent parameter λ . Moreover, it will be constructed such that it vanishes when we set $dx^i/d\lambda = 0$ everywhere in the Lagrangian. The Lagrangian $\mathcal{L}$ will have the following structure

$$\mathcal{L} = \langle \frac{dx}{d\lambda}, \frac{dx}{d\lambda} \rangle + \langle A, \frac{dx}{d\lambda} \rangle, \quad (1.1)$$

where

$$\langle \frac{dx}{d\lambda}, \frac{dx}{d\lambda} \rangle = g_{ij}(x) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}, \quad \langle A, \frac{dx}{d\lambda} \rangle = g_{ij}(x) A^i \frac{dx^j}{d\lambda}. \quad (1.2)$$

The first term in (1.1) can be regarded as an energy integral and the second term determined by an additional field A . Thus, the fields $g_{ij}(x)$ and $A_i(x)$ will determine the dynamics of the point particle under a variational principle. In fact, it will be seen from the equations that A plays the role of a vector potential in the final equations. Moreover, the parameter λ carries some physical information. The graph of a world line is a curve in a five-dimensional space, four dimensions for the spacetime and one for λ , which acts like an extra dimension, but different from the Kaluza-Klein approach. The curves in this five-dimensional space are not geodesics of a five-dimensional metric. In effect the parameter λ carries physical information related to rest mass and charge.

2. MATHEMATICAL FORMALISM

The following propositions will be useful [5].

Proposition 1 : If $(M, \langle, \rangle)$ is a C^∞ pseudo-Riemannian manifold with an affine connection ∇ , then ∇ is compatible with the metric if and only if, for any differentiable curve $c = c(\lambda)$ and any two differentiable vector fields V and W along c ,

$$\frac{d}{d\lambda} \langle V, W \rangle = \langle \frac{DV}{d\lambda}, W \rangle + \langle V, \frac{DW}{d\lambda} \rangle. \quad (2.1)$$

In (2.1), $D/d\lambda$ is the unique law that to each vector field Y along a differentiable curve $c : I \rightarrow M$ associates another vector field $DY/d\lambda$ along c , which is referred

to as the covariant derivative of Y along c . There is also a proposition due to Levi-Civita.

Proposition 2 : Given a pseudo-Riemannian metric $\langle, \rangle$ on a C^∞ differentiable manifold M , there exists a unique affine connection ∇ on Q such that (a) ∇ is symmetric (b) ∇ is compatible with the metric.

This pseudoriemannian structure also allows the definition of the gradient ∇ of a function f by means of the expression

$$\langle \nabla f, V \rangle = df(V), \quad (2.2)$$

for every $V \in T_x M$, and also the gradient of a vector field Y by means of

$$\langle \nabla Y \cdot V, W \rangle = \langle dY(V), W \rangle, \quad V \in T_x M. \quad (2.3)$$

For example, in local coordinates, we write $df = \sum \frac{\partial f}{\partial x^i} dx^i$ and so

$$\nabla f = \sum g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Formally, ∇Y can be regarded as a matrix. Hence the following identity holds

$$\langle \nabla Y \cdot V, W \rangle = \langle V, (\nabla Y)^T \cdot W \rangle, \quad (2.4)$$

where B^T denotes the transpose of B .

Now, let us set up the variational problem explicitly. To do this, some information about geodesics under variation will be required. Suppose we begin with a one-parameter family of smooth curves which are labeled $c_\epsilon : [0, 1] \rightarrow M$, such that $c_\epsilon(\lambda) \in M$ for all $|\epsilon| < \epsilon_0$. Thus ϵ is a parameter of variation. Each c_ϵ is called a variation of the basic curve c_0 , and can be thought to be homotopic to it. Let this family of curves be given by a C^2 -differentiable function $x = x(\epsilon, \lambda)$. Let us denote the tangent vector field along the curve c_ϵ by

$$T(\epsilon, \lambda) = \frac{\partial x}{\partial \lambda}. \quad (2.5)$$

Similarly, let us denote by

$$X(\epsilon, \lambda) = \frac{\partial x}{\partial \epsilon}, \quad (2.6)$$

the tangent vector field along the curve $x(\cdot, \lambda)$ with λ fixed. Now, let us write down the action functional using $\mathcal{L}$ in (1.1), but in terms of the deformed vector fields of (2.5) and (2.6). Then the action is given by

$$S(\epsilon) = \int_{\lambda_0}^{\lambda_1} \langle T(\epsilon, \lambda), T(\epsilon, \lambda) \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle A(x(\epsilon, \lambda)), T(\epsilon, \lambda) \rangle d\lambda. \quad (2.7)$$

Now, the action (2.7) can be differentiated with respect to ϵ , written ∂_ϵ , and using the propositions mentioned above, the derivative is given by

$$\partial_\epsilon S(\epsilon) = 2 \int_{\lambda_0}^{\lambda_1} \langle T, \nabla_X T \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle \nabla_X A, T \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle A, \nabla_X T \rangle d\lambda. \quad (2.8)$$

The connection is torsionless, and from (2.5), (2.6) $[X, T] = 0$, hence using $\nabla_X T = \nabla_T X$ and writing the derivative $\nabla_X A$ as $dA \cdot X$, using (2.3) equation (2.8) becomes

$$\partial_\epsilon S(\epsilon) = 2 \int_{\lambda_0}^{\lambda_1} \langle T, \nabla_T X \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle dA \cdot X, T \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle A, \nabla_T X \rangle d\lambda$$

$$= 2 \int_{\lambda_0}^{\lambda_1} \langle T, \frac{D}{d\lambda} X \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle (\nabla A)^T \cdot T, X \rangle d\lambda + \int_{\lambda_0}^{\lambda_1} \langle A, \frac{D}{d\lambda} X \rangle d\lambda,$$

where we use (2.4), and have written the covariant derivative along the curve. The next step involves an integration by parts of the first and last terms, and presuming X is such that the boundary terms vanish at the endpoints when $\epsilon = 0$. Neglecting these terms and using (2.3) and (2.4), we obtain

$$\partial_\epsilon S(\epsilon) = \int_{\lambda_0}^{\lambda_1} \langle -2 \frac{D}{d\lambda} T + [(\nabla A)^T - \nabla A] \cdot T, X \rangle d\lambda. \quad (2.9)$$

Thus, if $\partial_\epsilon S(\epsilon)$ is to vanish at $\epsilon = 0$, for all variations which satisfy $X(0, \lambda_0) = X(0, \lambda_1) = 0$, then as X is arbitrary, the following equation for the trajectories on the four-dimensional manifold must hold

$$\frac{D}{d\lambda} x' = \frac{1}{2} [(\nabla A)^T - \nabla A] \dot{x}. \quad (2.10)$$

Here the prime indicates differentiation with respect to λ . Equation (2.10) establishes the equations of motion for the particle. In fact, using the equation of motion (2.10), it can be shown that there is a first integral corresponding to this action, which is given by $\langle x', x' \rangle$. To prove this, differentiate $\langle x', x' \rangle$ with respect to λ

$$\begin{aligned} \frac{d}{d\lambda} \langle x', x' \rangle &= 2 \langle \frac{D}{d\lambda} \frac{dx}{d\lambda}, \frac{dx}{d\lambda} \rangle = \langle [(\nabla A)^T - \nabla A] \cdot \frac{dx}{d\lambda}, \frac{dx}{d\lambda} \rangle \\ &= \langle \frac{dx}{d\lambda}, \nabla A \cdot \frac{dx}{d\lambda} \rangle - \langle \nabla A \cdot \frac{dx}{d\lambda}, \frac{dx}{d\lambda} \rangle = 0. \end{aligned}$$

Therefore, there exists a constant k such that

$$\langle x', x' \rangle = k, \quad (2.11)$$

holds for all λ .

3. REDUCTION TO SPACE MANIFOLD AND EQUATIONS OF MOTION IN REDUCED VARIABLES

Consider now the following procedure. We restrict the development to trajectories such that

$$\frac{dx^0}{d\lambda} \neq 0.$$

This implies that the function $x^0 = x^0(\lambda)$ is an invertible function. Here x^0 is the first coordinate component in the coordinatization of the manifold. With this restriction, we can consider x^0 as the independent variable and express $\lambda = \lambda(x^0)$. The trajectories can be parametrized by x^0 or by a scaled variable written $t = x^0/c$, where c is a constant. In effect, what we can now do is carry out a type of reduction in which the t variable is separated from the other variables in the set. We shall suppose that $U \subset M$ contains a three-dimensional submanifold Σ_s such that each integral line of the Killing vector $K = \partial/\partial t$ intersects Σ_s exactly once. The manifold Σ_s will be referred to as the space manifold. The integral lines of K provide a natural projection $\pi : M \rightarrow \Sigma_s$ corresponding to the quotient map. This provides the justification for defining a global time function $t : U \rightarrow \mathbb{R}$ by setting $t(p)$ equal to the parameter corresponding to $p \in U$ along the integral line of K through p , where we assign $t = 0$ to the intersection of the integral line with Σ_s . Then if $\{x^i\}$ are local coordinates in Σ_s , we can use the integral lines of K to extend them as functions

to the whole of U . The space manifold is a way of keeping track of observers, and of course, a different space manifold could have been employed. These spaces can be related by transformation. Also, there is no special significance to selecting the Killing field K among the set $e^c K$, $c \in \mathbb{R}$ with the same integral curves. There exists a transformation such that the equation of any other space manifold is given by $t = f(x^1, x^2, x^3)$. Then picking this space manifold and the Killing vector field $e^c K$ amounts to taking a new global time function, or in other words, performing a coordinate transformation such that $t' = e^{-c}(t - f)$. If t is interpreted as a time variable, then c would have the dimensions of velocity. If differentiation with respect to t is denoted with a dot, then the constraint which arises from the first integral condition (2.11) after introducing t is given by

$$k = \langle \dot{x}, \dot{x} \rangle = \left(\frac{dt}{d\lambda} \right)^2. \quad (3.1)$$

Physical trajectories correspond to the case in which $k < 0$, the only ones which make physical sense. On dimensional grounds, we could then take $k = -m_0^2 c^2 < 0$, where m_0 would have dimensions of a mass. Substituting this into (3.1), we can obtain an explicit expression for $dt/d\lambda$,

$$\frac{dt}{d\lambda} = \frac{qm_0 c}{\sqrt{-\langle \dot{x}, \dot{x} \rangle}}, \quad (3.2)$$

where $q = \pm 1$ appears when calculating the square root, and reflects the sign of the derivative. The scalar q will play the role of a charge which carries two values when interpreting the final results. The significance of (3.2) is that it can be used to change the variable of integration in the action.

Thus, the action can be written in terms of an integral with respect to t as follows

$$S = -qm_0 c \int_{t(\lambda_0)}^{t(\lambda_1)} \sqrt{-\langle \dot{x}, \dot{x} \rangle} dt + \int_{t(\lambda_0)}^{t(\lambda_1)} \langle A(x), \dot{x} \rangle dt. \quad (3.3)$$

The limits of integration in (3.3) can be redefined by setting $t_0 = \min \{t(\lambda_0), t(\lambda_1)\}$ and $t_1 = \max \{t(\lambda_0), t(\lambda_1)\}$. Another sign will appear depending on whether the limits of integration have to be interchanged so the larger limit will be above. Since $q = 1$ when $dt/d\lambda > 0$ and $q = -1$ if $dt/d\lambda < 0$, the action can be written in the final form

$$S = -m_0 c \int_{t_0}^{t_1} \sqrt{-\langle \dot{x}, \dot{x} \rangle} dt + q \int_{t_0}^{t_1} \langle A(x), \dot{x} \rangle dt. \quad (3.4)$$

Next, the bilinear form is expanded out to exhibit the part of the metric restricted to Σ_s explicitly. Thus, writing $t = x^0/c$ and $y = (x^1, x^2, x^3)$ as the coordinates on Σ_s , the bilinear form which appears in the action (3.4) takes the form

$$\langle dx|dx \rangle = (dy|dy) + c(\mathbf{C}(t, y)|dy) - c^2 g_{00}(t, y) dt^2. \quad (3.5)$$

The quantities in bold will have the structure of three vectors. In particular $\mathbf{C}$ has the form

$$\mathbf{C} = 2(g_{01}, g_{02}, g_{03}), \quad (3.6)$$

and the metric on Σ_s is written as

$$(dy|dy) = \sum_{i,j=1}^3 g_{ij}(t, y) dy^i dy^j. \quad (3.7)$$

The components of $\mathbf{C}$ are three of the off-diagonal components of the metric tensor. In equation (3.7), we take the bilinear form $(\cdot|\cdot)$ and the function $g_{00}(t, y)$ to be positive so that $(\cdot|\cdot)$ is a Riemannian structure, which corresponds to the physical case. This in effect provides justification for identifying the time variable the way we did. This variable pertains to the g_{00} component of the metric which carries the negative sign. Of course, there is a reference frame dependence with regard to $\mathbf{C}$ in that another reference frame could be taken in which all the elements of $\mathbf{C}$ vanish. This might be thought of as a type of metric gauge dependence. Using (3.7), we can write (3.2) in the following form

$$\frac{dt}{d\lambda} = \frac{qm_0}{\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})}}.$$

By virtue of the structure of the metric, we can similarly break up the four-vector A into a term which is multiplied by minus under the Lorentz structure and a remaining part which can be referred to as the space part and write $A = (\varphi, \mathbf{A})$. It can be imagined that this quantity is transformed to another frame of reference in which the components take different values, or imposing other conditions on the components of A . These could be thought of as gauge conditions, and there is an analogy in this sense with the vector $\mathbf{C}$. Adopting these conventions, the action takes the following form

$$S = -m_0 c^2 \int_{t_0}^{t_1} \sqrt{g_{00}(t, y) - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})} dt + q \int_{t_0}^{t_1} \left[\frac{1}{c}(\mathbf{A}|\dot{y}) - \varphi(t|y) \right] dt. \quad (3.8)$$

We carry out the same procedure in the space manifold and obtain the equation of motion in a more familiar form. The covariant derivative with respect to the bilinear form $(\cdot|\cdot)$ on Σ_s will be denoted as D/dt , and since the form is required to be positive definite, this is the covariant derivative with respect to this form. If the metric is diagonal and constant, the usual derivative is recovered.

Now, take a variation of the basic curve in the space manifold variables and denote

$$Y(\epsilon, t) = \frac{\partial y}{\partial t}, \quad (3.9)$$

the tangent vector field along the curve $\tilde{c}_\epsilon$ in Σ_s , and by

$$Z(\epsilon, t) = \frac{\partial y}{\partial \epsilon}, \quad (3.10)$$

the tangent vector field along the curve $y(\cdot|t)$ such that t is fixed. The deformed action in terms of these vector fields takes the form

$$S(\epsilon) = -m_0 c^2 \int_{t_0}^{t_1} \sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)} dt + q \int_{t_0}^{t_1} \left[\frac{1}{c}(\mathbf{A}|Y) - \varphi(y, t) \right] dt. \quad (3.11)$$

Differentiate (3.11) with respect to ϵ to obtain

$$\partial_\epsilon S(\epsilon) = m_0 c^2 \int_{t_0}^{t_1} \left[\frac{(Y|\nabla_Z Y)}{c^2 \sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} - \frac{(\nabla g_{00}|Z)}{2 \sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} \right] dt$$

$$\begin{aligned}
 & + \frac{(\mathbf{C}|\nabla_z Y)}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} + \frac{(\nabla \mathbf{C} \cdot Z|Y)}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}}] dt \\
 & + q \int_{t_0}^{t_1} \left[\frac{1}{c}(\mathbf{A}|\nabla_z Y) + \frac{1}{c}(\nabla A \cdot Z|Y) - (\nabla \varphi|Z) \right] dt.
 \end{aligned}$$

Since $\mathbf{C}$ and $\mathbf{A}$ are structurally the same after writing the action in this way, they are treated the same way with respect to differentiation. Proceeding in exactly the same manner as before, we calculate that if the boundary terms can be neglected upon integration by parts, we have

$$\begin{aligned}
 \partial_\epsilon S(\epsilon) = & \int_{t_0}^{t_1} \left[\left(-\frac{D}{dt} \frac{m_0 Y}{\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} - \frac{m_0 c^2 \nabla g_{00}}{2\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} \right. \right. \\
 & - \frac{D}{dt} \frac{\mathbf{C}}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} + \frac{(\nabla \mathbf{C})^T \cdot Y}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|Y) - \frac{1}{c^2}(Y|Y)}} \\
 & \left. \left. - \frac{q}{c} \frac{D}{dt} \mathbf{A} + \frac{q}{c} (\nabla \mathbf{A})^T Y - q \nabla \varphi|Z \right) \right] dt.
 \end{aligned}$$

Therefore, the vanishing of $\partial_\epsilon S(\epsilon)$ at $\epsilon = 0$ yields the following equation of motion

$$\begin{aligned}
 & \frac{D}{dt} \frac{m_0 \dot{y}}{\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})}} = - \frac{m_0 c^2 \nabla g_{00}}{2\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})}} \\
 & - \frac{D}{dt} \frac{\mathbf{C}}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})}} + \frac{(\nabla \mathbf{C})^T \cdot \dot{y}}{2c\sqrt{g_{00} - \frac{1}{c}(\mathbf{C}|\dot{y}) - \frac{1}{c^2}(\dot{y}|\dot{y})}} \\
 & - \frac{q}{c} \frac{D}{dt} \mathbf{A} + \frac{q}{c} (\nabla \mathbf{A})^T \cdot \dot{y} - q \nabla \varphi.
 \end{aligned} \tag{3.12}$$

This is the most general equation of motion that results from nonzero $\mathbf{C}$, $\mathbf{A}$ and function φ under this type of variation.

Of course, the components of the metric can be interpreted as the components of the metric tensor of a manifold in the sense of general relativity. The terms which depend on the vector $\mathbf{C}$ in (3.12) clearly represent forces which depend on the velocity and frame of reference. These terms can be eliminated by a transformation to a new system in which $\mathbf{C} = 0$, since this depends only on three of the components of the metric tensor. Of course, we can transform in such a way that the vector $\mathbf{A} = 0$ as well; however, let us simply consider taking $\mathbf{C} = 0$. In this event, we can now make a correspondence with more well known equations, and in fact, the components of A can be identified with the electromagnetic field. Putting $\mathbf{C} = 0$ and considering the case in which the derivative of $\mathbf{A}$ can be expressed as

$$\frac{D}{dt} \mathbf{A} = \frac{\partial \mathbf{A}}{\partial t} + \nabla \mathbf{A} \cdot \dot{y}.$$

The equation of motion takes the form

$$\frac{D}{dt} \left(\frac{m_0}{\sqrt{g_{00} - \frac{1}{c^2}(\dot{y}|\dot{y})}} \dot{y} \right) =$$

$$-\frac{m_0 c^2}{2\sqrt{g_{00} - \frac{1}{c^2}(\dot{y}|\dot{y})}} \nabla g_{00} + q(-\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}) - \frac{q}{c} [\nabla \mathbf{A} - (\nabla \mathbf{A})^T] \cdot \dot{y}. \quad (3.13)$$

The last term in (3.13) can be rewritten in a more familiar form

$$[\nabla \mathbf{A} - (\nabla \mathbf{A})^T] \cdot \dot{y} = (\nabla \times \mathbf{A}) \times \dot{y}.$$

The equation of motion (3.13) then takes the form

$$\frac{D}{dt} \left(\frac{m_0}{\sqrt{g_{00} - \frac{1}{c^2}(\dot{y}|\dot{y})}} \dot{y} \right) = -\frac{m_0 c^2}{2\sqrt{g_{00} - \frac{1}{c^2}(\dot{y}|\dot{y})}} \nabla g_{00} + q(-\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}) + \frac{q}{c} \dot{y} \times (\nabla \times \mathbf{A}). \quad (3.14)$$

In this form, we can now define two new vectors $(\mathbf{E}, \mathbf{B})$ and scalar m as

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \frac{1}{c} \nabla \times \mathbf{A}, \quad m = \frac{m_0}{\sqrt{g_{00} - \frac{1}{c^2}(\dot{y}|\dot{y})}}. \quad (3.15)$$

In terms of these new quantities, the Euler-Lagrange equation for this system (3.14) takes the more familiar form

$$\frac{D}{dt}(m\dot{y}) = -\frac{1}{2} m c^2 \nabla g_{00} + q\mathbf{E} + q\dot{y} \times \mathbf{B}. \quad (3.16)$$

When t is interpreted as a time variable, (3.16) can be regarded as the equation of motion for a material point particle such that it moves in combined gravitational, electric and magnetic fields described by g_{00} and $(\mathbf{E}, \mathbf{B})$, respectively. In fact, when g_{00} is a constant, the usual Lorentz force law for a charge is obtained, where m acts as the mass, and q plays the role of the electric charge of the particle.

4. DISCUSSION AND CONCLUSIONS

There are several other interesting aspects of the result (3.16) to mention at this point in conclusion. First, the component of the metric tensor g_{00} appears directly in the definition of m in (3.15). On the left-hand side of (3.16), m should be regarded as an inertial mass, in the Newton's second law sense, and on the right hand side, m plays the role of a gravitational mass of the particle. When $\mathbf{E}$ and $\mathbf{B}$ are set to zero, m_0 can be divided out from both sides of (3.16). Therefore, motion in the manifold with nontrivial metric component g_{00} is completely independent of m_0 . This prediction is a direct consequence of this formulation, that is, the structure of the Lagrangian and variational process. The functional relation between m_0 and m also appears in the Lorentz like structure and also is a consequence of the form of the Lagrangian, the principle of special relativity has not been directly invoked. In [6], it is shown that an equation of motion can be developed such that there is a direct coupling between a spin tensor to the gravitational field strength, the Riemann curvature. This generates a gravitational Lorentz force determined by the spin tensor instead of by a scalar charge as in ordinary electromagnetism. Analogously here, a direct functional correspondence between the gravitational field g_{00} and m in (3.15) results. The quantity m in (3.15) depends on the component of the metric g_{00} in this case.

REFERENCES

- [1] Benci V. and Fortunato D., A New Variational Principle for the Fundamental Equations of Classical Physics, *Foundations of Physics*, **28**, (1998) 333-352.
- [2] Chruscinski D. and Kijowski J., Equations of Motion from Field Equations and a gauge-invariant variational principle for the motion of charged particles, *J. of Geometry and Physics* **20**, (1996) 393-403.
- [3] de Wit B. and Smith J., *Field theory in particle physics*, (North-Holland, Amsterdam, 1986).
- [4] López M. C. and Marsden J. E., Some Remarks on Lagrangian and Poisson reduction for field theories, *J. of Geometry and Physics* **48**, (2003) 52-83.
- [5] Oliva W. M., *Geometric Mechanics*, Springer Verlag, Berlin Heidelberg, 2002.
- [6] van Holten J. W., On the Electrodynamics of Spinning Particles, *Nucl. Phys. B* **356**, (1991) 3-26.