

A NOTE ON PYTHAGOREAN APPROACH IN BANACH SPACES

JI GAO

DEPARTMENT OF MATHEMATICS
COMMUNITY COLLEGE OF PHILADELPHIA
PHILADELPHIA, PA 19130-3991, USA E-MAIL: JGAO@CCP.EDU

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ABSTRACT. Let X be a normed or a Banach space and $S(X) = \{x \in X, \|x\| = 1\}$ be the unit sphere of X . Further properties of parameters

$$E(X) = \sup\{\alpha(x), x \in S(X)\},$$

$$e(X) = \inf\{\alpha(x), x \in S(X)\}, \quad F(X) = \sup\{\beta(x), x \in S(X)\},$$

and $f(X) = \inf\{\beta(x), x \in S(X)\}$, where $\alpha(x) = \sup\{\|x+y\|^2 + \|x-y\|^2, y \in S(X)\}$ and $\beta(x) = \inf\{\|x+y\|^2 + \|x-y\|^2, y \in S(X)\}$ are studied. We proved that a Banach space X with $E(X) < 3 + \sqrt{5}$, or $f(X) > \frac{16}{3+\sqrt{5}}$ has uniform normal structure, the existing results are improved.

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1. INTRODUCTION

Let X be a normed linear space, and let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ be the unit sphere and unit ball of X respectively.

The following concepts:

the modulus of convexity, $\delta(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x-y\| \leq \epsilon, x, y \in S(X)\}$ where $0 \leq \epsilon \leq 2[3]$;

the modulus of smoothness, $\rho_X(\epsilon) = \sup\{\frac{\|x+y\| + \|x-y\| - 2}{2}, x \in S(X), \|y\| = \epsilon\}$ where $\epsilon \geq 0[3]$;

the girth and perimeter, $m(X) = \inf\{d(x, -x) : x \in S(X)\}$, and $M(X) = \sup\{d(x, -x) : x \in S(X)\}$, where $d(x, -x)$ is the shortest length on the arcs joining antipodal points $x, -x$ on $S(X)[10]$;

the constants, $J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x \in S(X), y \in S(X)\}$, and $g(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x \in S(X), y \in S(X)\}[6]$;

the concept involving the weakly nearly uniform smoothness,

$R(X) = \sup\{\liminf_{n \rightarrow \infty} \|x_n + x\| : (x_n) \subseteq B(X), x_n \rightarrow_w 0, x \in B(X)\}[7]$, and so

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on were introduced and used to study the geometry of unit spheres and unit balls, and the geometric properties of Banach spaces are obtained.

The Pythagorean theorem describes the shape of unit sphere of Euclidean spaces H by considering the inscribed triangle with two antipodal points x and $-x$ on $S(H)$, and characterizes the Euclidean spaces by the equation $\|x+y\|^2 + \|x-y\|^2 = 4$, for all $x, y \in S(H)$. This idea has been used to study Banach spaces.

In [5], the following quadratic parameters:

$$\begin{aligned} E(X) &= \sup\{\alpha(x), x \in S(X)\}, \quad e(X) = \inf\{\alpha(x), x \in S(X)\}, \\ F(X) &= \sup\{\beta(x), x \in S(X)\}, \quad \text{and } f(X) = \inf\{\beta(x), x \in S(X)\}, \text{ where} \\ \alpha(x) &= \sup\{\|x+y\|^2 + \|x-y\|^2, y \in S(X)\} \text{ and} \\ \beta(x) &= \inf\{\|x+y\|^2 + \|x-y\|^2, y \in S(X)\} \end{aligned}$$

were introduced. The values of these parameters for some classical Banach spaces were computed. And the relationships between these parameters and uniformly nonsquare spaces, and the spaces with uniform normal structure were demonstrated. Among other results, we proved that a normed space X with $E(X) < 8$, or $f(X) > 2$ is uniformly nonsquare and a normed space X with $E(X) < 5$, or $f(X) > \frac{32}{9}$ has uniform normal structure.

In this paper, the values for more normed spaces are computed and the relationships between these parameters and normal structure are improved.

Let X be a Banach space. A mapping $T : X \rightarrow X$ is called a non-expansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in X$. Kirk [9] proved that if a weakly compact convex subset K of X has normal structure then any non-expansive mapping on K has a fixed point.

Definition 1. [1]. A bounded convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$ such that $r_{x_0}(H) = \sup\{\|x_0 - y\|, y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\|, x, y \in H\}$ denotes the diameter of H . A Banach space X is said to have normal structure if every bounded, convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\|, y \in K\} < c \cdot (d(K))$.

For a reflexive Banach space X , the normal structure and weak normal structure coincide.

The following is a geometric property of a Banach space without the w-normal structure.

Lemma 1. [4]. *Let X be a Banach space without weak normal structure, then for any $0 < \epsilon < 1$, there exist x and $y \in S(X)$ such that both $\|x + y\|$ and $\|x - y\| \geq \frac{1+\sqrt{5}}{2} - \epsilon$.*

The geometric meaning of the lemma is that if a Banach space X fails to have the w-normal structure then there is a parallelogram with four sides arbitrarily close to $\frac{1+\sqrt{5}}{2}$.

A continuous mapping $x(t)$ from a closed interval $[a, b]$ to a Banach space X is called a curve in X : $C = x(t), a \leq t \leq b$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces.

Definition 2. ([2];[10]). For curve $C = x(t)$, let P stand for a partition $a = t_0 < t_1 < t_2 < \dots < t_i < \dots < t_n = b$ of interval $[a, b]$ and $l(C, P) = \sum_{i=1}^n \|x(t_i) - x(t_{i-1})\|$, where $x_i(t), i = 0, 1, 2, \dots, n$ are called partition points on C , then the length $l(C)$ of curve $C = x(t), a \leq t \leq b$, is defined as the least upper bound of $l(C, P)$ for all possible partitions of $[a, b]$:

$$l(C) = \sup_P \{l(C, P)\}.$$

If $l(C)$ is finite the curve is called rectifiable.

For a normed linear space X , we use X_2 to denote a two dimensional subspace of X , then $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation.

Theorem 1. ([2];[10]). Let X_2 be a two dimensional Banach space, and K_1, K_2 be closed convex subsets of X_2 , and have nonvoid interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \leq l(\partial(K_2))$, where $l(\partial(K_i))$ denote the lengths of the circumferences of $K_i, i = 1, 2$.

For $x \in S(X_2)$, let κ be one of the arcs of $S(X_2)$ from x to $-x$ counterclockwise, and let $g : [0, L] \rightarrow \kappa$ be the standard representation in terms of arc length, where L is the length of κ .

Lemma 2. [6] The functions $\Phi, \Psi : [0, L] \rightarrow [0, 2]$ defined by $\Phi(s) = \|g(s) - x\|$, $\Psi(s) = \|g(s) + x\|$ are continuously increasing, and decreasing respectively.

2. THE VALUES OF PARAMETERS FOR SOME FUNCTIONAL SPACES

Proposition 1. [5] For any Banach space X , $2 \leq f(X) \leq e(X) \leq E(X) \leq 8$, and $2 \leq f(X) \leq F(X) \leq E(X) \leq 8$.

We first list the values of these parameters for some classical Banach linear spaces [5].

Proposition 2. $S(X_2)$ is affinely homeomorphic to a parallelogram if and only if $E(X_2) = 8$, or $f(X_2) = 2$.

Proposition 3. If $S(X_2)$ is affinely homeomorphic to a normal hexagon, then $E(X_2) \geq 5$, and $f(X_2) \leq \frac{32}{9}$.

Proposition 4. : $E(H) = e(X) = F(X) = f(X) = 4$, where H is a Hilbert space.

Theorem 2. Let $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

- (i) $f(l_p) = 2^{1+\frac{2}{q}}, F(l_p) \leq 2^{1+\frac{2}{p}} \leq e(l_p) \leq E(l_p), 1 < p \leq 2;$
- (ii) $f(l_p) \leq F(l_p) \leq 2^{1+\frac{2}{p}} \leq e(l_p), E(l_p) = 2^{1+\frac{2}{q}}, p > 2.$

Theorem 3. Let $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

- (i) $f(L_p[0, 1]) = F(L_p[0, 1]) = 2^{1+\frac{2}{q}},$
 $2^{1+\frac{2}{p}} \leq e(L_p[0, 1]) = E(L_p[0, 1]), 1 < p \leq 2;$
- (ii) $f(L_p[0, 1]) \leq F(L_p[0, 1]) = 2^{1+\frac{2}{p}},$
 $e(L_p[0, 1]) = E(L_p[0, 1]) = 2^{1+\frac{2}{q}}, p > 2.$

Lemma 3. [5] Let $x, y \in S(X)$ with $\|x + y\| = \|x - y\| = a$, and u, v be the normalization of $x + y, x - y$ respectively, then $\|u + v\| = \|u - v\| = \frac{2}{a}$.

Then we consider the flat spaces.

Definition 3. A Banach space X is said to be Flat, if $m(X) = 2$ and it is attained by a curve C on $S(X)$. The curve C is called a girth curve.

Proposition 5. Let X be a flat space, then $f(X) = 2$ and $E(X) = 8$.

Proof. Let C be the girth curve of X , then $l(C) = 2$. Let $x \in C$, from lemma 2 there exists a $y \in C$ such that $\|x + y\| = \|x - y\|$. From theorem 1, $\|x + y\| = \|x - y\| = 1$. So, $f(X) = 2$. From lemma 3, $E(X) = 8$. $\square$

We consider the Banach space of Continuous Functions on $[a, b]$, $C[a, b]$.

Theorem 4. $f(C[a, b]) = 2$, $F(C[a, b]) = 5$ and $e(C[a, b]) = E(C[a, b]) = 8$.

Proof. To show $f(C[a, b]) = 2$, let $x(t) = (t - a)\frac{1}{b-a}$ and $y(t) = (b - t)\frac{1}{b-a}$ where $t \in [a, b]$, then $\|x(t) + y(t)\| = \|x(t) - y(t)\| = 1$. By definition of $f(X)$, $f(C[a, b]) = 2$.

We prove $F(C[a, b]) = 5$.

Let $x(t) \in S(C[a, b])$ with $0 \leq |x(t_1)| = \min |x(t)| = d \leq 1$ where $a \leq t_1 \leq b$. Without loss of generality, assume $a < t_1 < b$. For any $\epsilon > 0$, take δ such that $a \leq t_1 - \delta < t_1 + \delta \leq b$ and if $|t - t_1| \leq \delta$, $|x(t) - x(t_1)| \leq \epsilon$.

Let

$$y(t) = \begin{cases} 0, & t \in [a, t_1 - \delta] \cup [t_1 + \delta, b], \\ y(t_1) = \operatorname{sgn}(x(t_1)), & \text{if } d \neq 0; 1, \text{ if } d = 0 \\ \text{linear function}, & t \in [t_1 - \delta, t_1] \cup [t_1, t_1 + \delta]. \end{cases}$$

Then $y(t) \in S(C[a, b])$, and $\|x(t) - y(t)\| \leq 1 + \epsilon$ and $\|x(t) + y(t)\| \leq 1 + d + \epsilon$. Since ϵ is arbitrary, from definition of $\beta(x(t))$ we have $\beta(x(t)) \leq 1 + (1 + d)^2 \leq 5$.

Since $x(t)$ is arbitrary too, from definition of $F(X)$ we have $F(C[a, b]) \leq 5$.

Let $x(t) = 1$ and $y(t) = \frac{t-a}{b-a}$, we have $\|x(t) + y(t)\| = 2$ and $\|x(t) - y(t)\| = 1$.

So, $F(C[a, b]) = 5$.

We prove $e(C[a, b]) \geq 8$.

Let $x(t) \in S(C[a, b])$ with $|x(t_1)| = 1$ where $a \leq t_1 \leq b$. Without loss of generality, we may assume $a < t_1 < b$. For any $\epsilon > 0$, take δ such that $a \leq t_1 - \delta < t_1 + \delta \leq b$ and if $|t - t_1| \leq \delta$, $|x(t) - x(t_1)| \leq \epsilon$.

Let

$$y(t) = \begin{cases} x(t), & t \in [a, t_1 - \delta] \cup [t_1 + \delta, b], \\ y(t_1) = -x(t_1), & \\ \text{linear function}, & t \in [t_1 - \delta, t_1] \cup [t_1, t_1 + \delta]. \end{cases}$$

Then $y(t) \in S(C[a, b])$, and both $\|x(t) + y(t)\|$ and $\|x(t) - y(t)\| \geq 2 - \epsilon$.

Since ϵ is arbitrary, from definition of $\alpha(x(t))$ we have $\alpha(x(t)) \geq 8$.

Since $x(t)$ is arbitrary too, from definition of $e(X)$ we have $e(C[a, b]) \geq 8$.

So, $e(C[a, b]) = E(C[a, b]) = 8$. $\square$

Proposition 6. The Banach space $C[a, b]$ fails to have normal structure.

Proof. Let $W = \{x(t) \in C[a, b] : x(a) = 0, x(b) = 1 \text{ and } 0 \leq x(t) \leq 1\}$, then W is a closed convex subset of $C[a, b]$.

W is a bounded too. In fact, if $x(t)$ and $y(t) \in W$, then $\|x(t) - y(t)\| = \max\{|x(t) - y(t)|\} \leq 1$.

$d(W) = \sup \{\|x(t) - y(t)\|\} = 1$. In fact, let $x_1(t), y_1(t) \in W$ such that $x_1(\frac{1}{2}) = 0$ and $y_1(\frac{1}{2}) = 1$. We have $\|x(t) - y(t)\| = 1$.

But for any $x(t) \in W$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $t - a \leq \delta$, $|x(t)| \leq \epsilon$. Take $y(t) \in W$ such that $y(\delta) = 1$, then $|x(t) - y(t)| \geq 1 - \epsilon$. Therefore $r_{x(t)} = \sup \{|x(t) - y(t)| : y(t) \in W\} \geq 1 - \epsilon$. Let $\epsilon \rightarrow 0$, $r_{x(t)} = d(W) = 1$. Since $x(t) \in W$ is arbitrary, W has no normal structure. $\square$

We now consider the normed space of Continuous Functions on $[a, b]$, $C_A[a, b]$ with norm $\|x(t)\| = \int_a^b |x(t)| dt$, where $x(t) \in C_A[a, b]$.

Theorem 5. $f(C_A[a, b]) = F_A(C[a, b]) = 2$ and $e(C_A[a, b]) = E(C_A[a, b]) = 8$.

Proof. We first prove $F(C_A[a, b]) = 2$.

Let $x(t) \in S(C_A[a, b])$ then $\int_a^b |x(t)| dt = 1$. Let $t_1 \in (a, b)$ such that $\int_a^{t_1} |x(t)| dt = \frac{1}{2}$.

For any $\epsilon > 0$, take δ such that $a \leq t_1 - \delta < t_1 + \delta \leq b$ and $\int_{t_1 - \delta}^{t_1 + \delta} |x(t)| dt < \epsilon$.

Let

$$y(t) = \begin{cases} x(t), & t \in [a, t_1 - \delta], \\ \text{linear function}, & t \in [t_1 - \delta, t_1] \cup [t_1, t_1 + \delta], \\ -x(t), & t \in [t_1 + \delta, b]. \end{cases}$$

Then $1 - \epsilon \leq \|y(t)\| \leq 1 + \epsilon$, and $\|x(t) - y(t)\| \leq 1 + \epsilon$ and $\|x(t) + y(t)\| \leq 1 + \epsilon$.

Since ϵ is arbitrary, from definition of $\beta(x)$ we have $\beta(x(t)) \leq 2$.

Since $x(t)$ is arbitrary too, from definition of $F(X)$ we have $F(C_A[a, b]) \leq 2$.

So, $f(C_A[a, b]) = F(C_A[a, b]) = 2$.

We prove $e(C_A[a, b]) \geq 8$.

Let $x(t) \in S(C_A[a, b])$, for any $\epsilon > 0$, take δ such that $\int_a^{a+\delta} |x(t)| dt \leq \epsilon$.

Let

$$y(t) = \begin{cases} \frac{2}{\delta}, & t \in [a, a + \frac{\delta}{2}], \\ 0, & t \in [a + \frac{\delta}{2} + \frac{\delta\epsilon}{2}, b], \\ \text{linear function}, & t \in [a + \frac{\delta}{2}, a + \frac{\delta}{2} + \frac{\delta\epsilon}{2}]. \end{cases}$$

Then $1 - \epsilon \leq \|y(t)\| \leq 1 + \epsilon$, and both $\|x(t) + y(t)\|$ and $\|x(t) - y(t)\| \geq 2 - \epsilon$.

Since ϵ is arbitrary, from definition of $\alpha(x(t))$ we have $\alpha(x(t)) \geq 8$.

Since $x(t)$ is arbitrary too, from definition of $e(X)$ we have $e(C_A[a, b]) \geq 8$.

So, $e(C_A[a, b]) = E(C_A[a, b]) = 8$. $\square$

3. THE PARAMETERS AND NORMAL STRUCTURE

Theorem 6. A Banach space X with $E(X) < 3 + \sqrt{5}$, or $f(X) > \frac{16}{3+\sqrt{5}}$ has normal structure.

Proof. Since $E(X) < 3 + \sqrt{5}$ implies $E(X) < 8$, X is a uniformly nonsquare hence reflexive [8]. The normal structure and weak normal structure are coincide.

Suppose X does not have normal structure, $\forall \epsilon > 0$ then there exist x and $y \in S(X)$ such that $\min\{\|x + y\|, \|x - y\|\} > \frac{1+\sqrt{5}}{2} - \epsilon$ by lemma1.

Since ϵ can be arbitrarily small, we have $E(X) = \sup\{\|x + y\|^2 + \|x - y\|^2 : x \in S(X), y \in S(X)\} \geq 2(\frac{1+\sqrt{5}}{2})^2 = 3 + \sqrt{5}$. The first assert is approved.

To prove the second assert, by lemma 2 we can take $y' \in S(X)$ such that $\|x + y'\| = \|x - y'\| \geq \min\{\|x + y\|, \|x - y\|\} > \frac{1+\sqrt{5}}{2} - \epsilon$. Let u and v be normalization of $x + y'$ and $x - y'$ respectively, then $u, v \in S(X)$ and $\|u + v\| = \|u - v\| = \frac{2}{\frac{1+\sqrt{5}}{2} - \epsilon} \leq \frac{4}{1+\sqrt{5}} + \epsilon$ by lemma 3. Since ϵ can be arbitrarily small we have $f(X) \leq 2(\frac{4}{1+\sqrt{5}})^2 = \frac{16}{3+\sqrt{5}}$.

□

4. THE PARAMETERS AND UNIFORM NORMAL STRUCTURE

Let F be a filter on an index set I , and let $\{x_i\}_{i \in I}$ be a subset in a Hausdorff topological space X , $\{x_i\}_{i \in I}$ is said to converge to x with respect to F , denote by $\lim_F x_i = x$, if for each neighborhood U of x , $\{i \in I : x_i \in U\} \in F$. A filter U on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $A : A \subseteq I, i_0 \in A$ for some fixed $i_0 \in I$. We will use the fact that if U is an ultrafilter, then (i): for any $A \subseteq I$, either $A \subseteq U$ or $I - A \subseteq U$; (ii): if $\{x_i\}_{i \in I}$ has a cluster point x , then $\lim_U x_i$ exists and equals to x .

Let $\{X_i\}_{i \in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i \in I} \|x_i\| < \infty$.

Definition 4. [11]. Let U be an ultrafilter on I and let $N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_U \|x_i\| = 0\}$. The ultraproduct of $\{X_i\}_{i \in I}$ is the quotient space $l_\infty(I, X_i)/N_U$ equipped with the quotient norm.

We will use $(x_i)_U$ to denote the element of the ultraproduct. It follows from (ii) of the fact, and the definition of quotient norm that

$$(4.1) \quad \|(x_i)_U\| = \lim_U \|x_i\|$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X . For an ultrafilter U on $\mathbb{N}$, we use X_U to denote the ultraproduct.

Theorem 7. For any Banach space X , and for any nontrivial ultrafilter U on $\mathbb{N}$, $K(X_U) = K(X)$, where K stands for either f , F , e , or E .

Proof. We have proved the theorem for $K = f$ and $K = F$ in [5]. We now prove the theorem for $K = E$ which is almost same as the proof of $K = f$ in [5].

Let $x_i = x$ for all $i \in \mathbb{N}$, the mapping $x \rightarrow (x_i)_U = x_U$ is an isometric embedding of X into X_U , we can consider X as a subspace of X_U . By the definition of $E(X)$, we have $E(X_U) \geq E(X)$.

To prove the reverse inequality, we choose a $(X_U)_2 \subseteq X_U, (x_i)_U, (y_i)_U \in S((X_U)_2)$ such that $\|(x_i)_U + (y_i)_U\|^2 + \|(x_i)_U - (y_i)_U\|^2 > E(X_U) - \delta$.

From (ii) of the fact for ultrafilter and 4.1 above, the sets:

$$I = \{i \in \mathbb{N} : \|\|x_i\| - 1\| < \delta\},$$

$$J = \{i \in \mathbb{N} : \|\|y_i\| - 1\| < \delta\}, \text{ and}$$

$$L = \{i \in \mathbb{N} : \|x_i + y_i\|^2 + \|x_i - y_i\|^2 > E(X_U) - \delta\}$$

are all in U . So the intersection $I \cap J \cap L$ is in U too, and is hence not empty.

Let $i \in I \cap J \cap L$, we have $||x_i|| - 1| < \delta$, $||y_i|| - 1| < \delta$, and $||x_i + y_i||^2 + ||x_i - y_i||^2 > E(X_U) - \delta$. Let $x'_i, y'_i \in S(X_2)$ be the normalization of x_i, y_i respectively. Since $||x'_i - x_i||$ and $||y'_i - y_i|| \leq \delta$, then $||x'_i + y'_i|| \geq ||x_i + y_i|| - 2\delta$, $||x'_i - y'_i|| \geq ||x_i - y_i|| - 2\delta$, and therefore $||x'_i + y'_i||^2 + ||x'_i - y'_i||^2 \geq ||x_i + y_i||^2 + ||x_i - y_i||^2 - 18\delta > E(X_U) - 19\delta$. Hence $E(X) > E(X_U) - 19\delta$. Since δ can be arbitrarily small, $E(X) \geq E(X_U)$.

We then prove this theorem for $K = e$.

For any $\delta > 0$, from definition of $e(X)$, there exists a $x \in S(X)$ such that $||x + y||^2 + ||x - y||^2 < e(X) + \delta$ for all $y \in S(X)$.

Let $x_i = x$, for all $i \in N$, then $(x_i)_U \in S((X_U))$. For any $(y_i)_U \in X_U$, from definition of the norm in X_U , we may assume that $y_i \in S(X)$, for all $i \in N$. Therefore $||(x_i)_U + (y_i)_U||^2 + ||(x_i)_U - (y_i)_U||^2 = ||(x + y_i)_U||^2 + ||(x - y_i)_U||^2 = (||x + y_i||^2 + ||x - y_i||^2)_U < e(X) + \delta$. Therefore $e(X_U) < e(X) + \delta$.

Since δ can be arbitrarily small, we proved $e(X_U) \leq e(X)$.

To prove the reverse inequality, for any $\delta > 0$, from definition of $e(X_U)$ there exists a $(x_i)_U \in S((X_U))$ such that $||(x_i)_U + (y_i)_U||^2 + ||(x_i)_U - (y_i)_U||^2 < e(X_U) + \delta$ for all $(y_i)_U \in S((X_U))$. From definition of the norm in X_U , we may assume $x_i, y_i \in S(X)$, for all $i \in N$.

Let $L = \{i \in N : ||x_i + y_i||^2 + ||x_i - y_i||^2 < e(X_U) + \delta\}$ for all $(y_i)_U \in S((X_U))$, then L is in U , and is hence not empty. Let $j \in L$, from (ii) of the fact for ultrafilter and 4.1 above, we have $||x_j + y_j||^2 + ||x_j - y_j||^2 < e(X_U) + \delta$ for all $(y_i)_U \in S((X_U))$. From definition of $e(X)$, we have $e(X) < e(X_U) + \delta$. Since δ can be arbitrarily small, $e(X) \leq e(X_U)$.

Finally $e(X) = e(X_U)$. □

Theorem 8. *A Banach space X with $E(X) < 3 + \sqrt{5}$ has uniform normal structure.*

Proof. The idea of the proof is same as the proof of theorem 4.4 in [6]. Suppose $E(X) < 3 + \sqrt{5}$, but X does not have uniform normal structure. We can find a sequence $\{C_n\}$ of bounded closed convex subsets of X such that for each n , $0 \in C_n$, $d(C_n) = 1$, and $rad(C_n) = inf\{sup\{||x - y||, y \in C_n\}, x \in C_n\} > 1 - \frac{1}{n}$. Let U be any nontrivial ultrafilter on N , and let $C = \{(x_n)_U : x_n \in C_n, n \in N\}$ then C is a nonempty bounded closed convex subset of X_U . It follows from the properties of C_n above that $d(C) = rad(C) = 1$, so X_U does not have normal structure. On the other hand, from theorem 7, $E(X_U) = E(X) < 3 + \sqrt{5}$. This contradicts theorem 6, and X must have uniform normal structure. □

Similarly we can prove the following theorem:

Theorem 9. *A Banach space with $f(X) < \frac{16}{3 + \sqrt{5}}$ has uniform normal structure.*

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