

COMPLEX TOPOLOGICAL INVARIANTS OF THREE-HYPERBOLIC MANIFOLDS

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ABSTRACT. We derive invariants associated with compact real three-dimensional hyperbolic manifolds which can be combined to form a component of Thurston complex invariant. Explicit formulas for the Chern-Simons invariant of irreducible $U(n)$ -flat connections on hyperbolic fibered manifolds are obtained. We discuss supersymmetry surviving for supergravity solutions involving real hyperbolic factors, and briefly review the determinant line bundle with its connection to the eta invariant, the global anomaly formula, and the vanishing theorems for type $(0, q)$ cohomology of locally symmetric spaces.

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1. INTRODUCTION

In this paper we give an expository overview of various topological invariants associated to the Dirac operator, notably the eta invariant, and their relation to the Chern-Simons functional. In an introductory part we comment on the possible role these invariants might play in three-dimensional quantum gravity, though this part of the paper is mainly concerned with a

short overview of the classification (uniformization) and sum over the topology for low-dimensional cases, following the lines of [27, 68, 24, 16, 19].

The problem of geometries classification is one of the main problem in complex analysis and in mathematics as a whole, and plays a fundamental role in physical models. All curves of genus zero can be uniformized by rational functions, all those of genus one can be uniformized by elliptic functions, and all those of genus more than one can be uniformized by meromorphic functions, defined on proper open subsets of $\mathbb{C}$. This result, due to Klein, Poincaré and Koebe, is one of the deepest achievements in mathematics. A complete solution of the uniformization problem has not yet been obtained, excepted for the one-dimensional complex case. However, there have been essential advances in this problem, which have brought to foundations for topological methods, covering spaces, existence theorems for partial differential equations, existence and distortion theorems for conformal mappings, etc.

In accordance with Klein-Poincaré uniformization theorem, each Riemann surface can be represented (within a conformal equivalence) in the form $\Gamma \backslash \Sigma$, where Σ is one of the three canonical regions, namely the extended plane $\bar{\mathbb{C}}$ (the sphere $\mathbb{S}^2$), the plane $\mathbb{C}$ ($\mathbb{R}^2$), or the disk, and Γ is a discrete group of Möbius automorphisms of Σ acting freely there. Riemann surfaces with such coverings are elliptic, parabolic and hyperbolic type, respectively. This theorem admits a generalization also to surfaces with branching. A different approach to the solution of the uniformization problem was proposed by Koebe. The general uniformization principle of Koebe asserts that if a Riemann surface $\tilde{\Sigma}$ is topologically equivalent to a planar region $\mathcal{P}$, then there also exists a conformal homeomorphism of $\tilde{\Sigma}$ onto $\mathcal{P}$. The same problem of analytic uniformization reduces to the topological problem of finding all the (branched, in general) planar coverings $\tilde{\Sigma} \rightarrow \Sigma$ of a given Riemann surface Σ . The solution of this topological problem is given by the theorem of Maskit. With the help of standard uniformization theorems and decomposition theorems [44], one can construct and describe all the uniformizations of Riemann surfaces by Kleinian groups. Furthermore, by using the quasiconformal mappings, one can obtain an uniformization theorem of more general character¹, namely it is possible to prove that several surfaces can be uniformized simultaneously.

1.1. Classification of three-geometries. For any closed orientable two-dimensional manifold Σ_Γ the following result holds: every conformal structure on Σ_Γ is represented by a constant curvature geometry. The only simply connected manifolds with constant curvature are $\mathbb{S}^2$ or $\mathbb{R}^2$ or $\mathbb{H}^2$ and Σ_Γ can be represented as $\Gamma \backslash \Sigma$, where Γ is a group of isometries.

Let us now turn to the classification of the three-geometries following the presentation of [64, 63]. By a geometry or a geometric structure we mean a pair (M, Γ) , that is a manifold M and a group Γ acting transitively

¹This fact is related to Teichmüller spaces.

on M with compact point stabilizers (following [64] we also propose that the interior of every compact three-manifold has a canonical decomposition into pieces which have geometric structure). Two geometries (M, Γ) and (M', Γ') are equivalent if there is a diffeomorphism of M with M' which throws the action of Γ onto the action of Γ' . In particular, Γ and Γ' must be isomorphic. Let us assume that:

- The manifold M is simply connected. Otherwise it will be sufficient to consider a natural geometry $(\widetilde{M}, \widetilde{\Gamma})$, $\widetilde{M}$ being the universal covering of M and $\widetilde{\Gamma}$ denoting the group of all diffeomorphisms of $\widetilde{M}$ which are lifts of elements of Γ .
- The geometry admits a compact quotient. In another words, there exists a subgroup $\widehat{\Gamma}$ of Γ which acts on M as a covering group and has compact quotient.
- The group Γ is maximal. Otherwise, if $\Gamma \subset \Gamma'$ then any geometry (M, Γ) would be the geometry (M, Γ') at the same time.

Conjecture 1 (W. P. Thurston [64], Section 4). Any maximal, simply connected, three-dimensional geometry admitting a compact quotient is equivalent to one of the geometries (M, Γ) , where M is one of the eight manifolds:

$$\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{SL}(2, \mathbb{R}), Nil, Sol.$$

The group properties and more details of these manifolds may be found in [63]. The first five geometries are familiar objects, so we briefly discuss the last three ones. The group $\widetilde{SL}(2, \mathbb{R})$ is the universal covering of $SL(2, \mathbb{R})$, the three-dimensional Lie group of all 2×2 real matrices with unit determinant. The geometry of Nil is the three-dimensional Lie group of all 3×3 real upper triangular matrices with ordinary matrix multiplication. It is also known as the nilpotent Heisenberg group. The geometry of Sol is the three-dimensional (solvable) group. The manifold modelled on $\widetilde{SL}(2, \mathbb{R})$ or Nil are Seifert fibre spaces and those modelled on Sol are bundles over $\mathbb{S}^1$ with fibers the torus or the Klein bottle.

We also note the recent asserts for Ricci flow on three-manifolds. The Ricci flow with surgery was considered in [34]. An attempt to construct a canonical Ricci flow, defined on largest possible subset of space-time, has been made in [57, 58]. In fact it provides a proof of Poincaré's conjecture (implying Thurston's conjecture) - even though this proof is not yet generally accepted in literature.

1.2. Classification of four-geometries. Unlike the case of compact Riemann surfaces or three-dimensional manifolds, very little is known about the uniformization of N -dimensional manifolds ($N > 3$) by Kleinian groups. Some information on four-geometries from the point of view of homogeneous Riemannian manifolds and Lie groups the reader can find in [38, 10].

The list of Thurston three-geometries has been organized in terms of the compact stabilizers Γ_σ of $\sigma \in M$ isomorphic to $SO(3)$, $SO(2)$ or trivial

Table 1. List of four-geometries

Stabilizer Γ_σ	Manifold M
$SO(4)$	$\mathbb{S}^4, \mathbb{R}^4, \mathbb{H}^4$
$U(2)$	$\mathbb{C}P^2, \mathbb{C}H^2$
$SO(2) \times SO(2)$	$\mathbb{S}^2 \times \mathbb{R}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2$
$SO(3)$	$\mathbb{S}^3 \times \mathbb{R}, \mathbb{H}^3 \times \mathbb{R}$
$SO(2)$	$Nil^3 \times \mathbb{R}, \widetilde{PSL}(2, \mathbb{R}) \times \mathbb{R}, Sol^4$
$\mathbb{S}^1$	F^4
trivial	$Nil^4, Sol_{m,n}^4$ (including $Sol^3 \times \mathbb{R}$), Sol_1^4

group $SO(1)$. The analogue list of four-geometries also can be organized (using only connected groups of isometries) as in Table 1.

Here we have the four irreducible four-dimensional Riemannian symmetric spaces: sphere $\mathbb{S}^4$, hyperbolic space $\mathbb{H}^4$, complex projective space $\mathbb{C}P^2$ and complex hyperbolic space $\mathbb{C}H^2$ (which we may identify with the open unit ball in $\mathbb{C}^2$ with an appropriate metric). The other cases are more specific and for the sake of completeness we shall illustrate them.

The nilpotent Lie group Nil^4 can be presented as the split extension $\mathbb{R}^3 \circ_U \mathbb{R}$ of $\mathbb{R}^3$ by $\mathbb{R}$ (the symbol $\circ$ denotes semidirect product). The quotient $\mathbb{R}$ acts on the subgroup $\mathbb{R}^3$ by means of $U(t) = \exp(tB)$, where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ In the same way, for the soluble Lie groups one}$$

has $Sol_{m,n}^4 = \mathbb{R}^3 \circ_{T_{m,n}} \mathbb{R}$, where $T_{m,n}(t) = \exp(tC_{m,n})$ and $C_{m,n} = \text{diag}(\alpha, \beta, \gamma)$, with the real numbers $\alpha > \beta > \gamma$ and $\alpha + \beta + \gamma = 0$. Furthermore e^α , e^β and e^γ are the roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$, with m, n positive integers. If $m = n$, then $\beta = 0$ and $Sol_{m,n}^4 = Sol^3 \times \mathbb{R}$. In general, if $C_{m,n} \propto C_{m',n'}$, then $Sol_{m,n}^4 \cong Sol_{m',n'}^4$. When $m^2n^2 + 18 = 4(m^3 + n^3) + 27$, one has a new geometry, Sol_0^4 , associated with the group $SO(2)$ of isometries rotating the first two coordinates. The soluble group

$$Sol_1^4, \text{ is most conveniently represented as the matrix group } \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix},$$

with $\alpha, a, b, c \in \mathbb{R}$, $\alpha > 0$. Finally the geometry F^4 , related to the isometry group $\mathbb{R}^2 \circ PSL(2, \mathbb{R})$ ($PSL(2, \mathbb{R}) \equiv SL(2, \mathbb{R})/\{-1, 1\}$) and stabilizer $SO(2)$, is the only geometry which admits no compact model. A connection of these geometries with complex and Kählerian structures (preserved by the stabilizer Γ_σ) can be found in [68].

1.3. Comments. We conclude this section with some remarks. In two-dimensional quantum gravity, as defined by string theories, it seems necessary to perform the sum over all topologies. As for the closed compact case, there is a complete set, the Euler characteristic $\chi = 2 - 2g$. Thus, the functional integral could be written in the form: $\int [Dg] = \sum_{g=0}^{\infty} \int_{(\text{fixed genus})} [Dg]$.

A necessary first step to implement this to the three-dimensional case is the classification of all possible three-geometries by Kleinian groups. In three-dimensional quantum gravity the partition function for a fixed topology can be computed using the Chern-Simons theory [72, 73, 74, 8]. Such ansatz is an equivalent to a mathematical question: the computation of the corresponding Ray-Singer torsion. If Thurston's conjecture is true, every compact closed three-dimensional manifold can be represented as follows [1]: $\bigcup_{\ell=1}^{\infty} \Gamma_{n_\ell} \backslash G_{n_\ell}$, where $n_\ell \in (1, \dots, 8)$ represents one of the eight geometries, and Γ is the (discrete) isometry group of the corresponding geometry. It has to be noted that gluing the above geometries, characterizing different coupling constants by a complicated set of moduli, is a very difficult task. Perhaps this can be done, however, with a bit of luck.

Important geometric invariants which can be defined are the volume and the Chern-Simons invariants. Thurston suggests to combine these two invariants into a single complex invariant whose absolute value is $\exp((2/\pi)\text{Vol}(M))$ and whose argument is the Chern-Simons invariant of M , $CS(M)$ [64]. Taking into account the Thurston classification of all possible three-geometries this invariant can also be presented in the form

$$(1) \quad \mathfrak{W} = \exp \left\{ \bigcup_{\ell=1}^{\infty} \left(\frac{2}{\pi} \text{Vol}(\Gamma_{n_\ell} \backslash G_{n_\ell}) + 4\pi\sqrt{-1}CS(\Gamma_{n_\ell} \backslash G_{n_\ell}) \right) \right\}.$$

If we make the intuitive requirement that only irreducible manifolds have to be taken into account (see the next Section for supersymmetry surviving arguments in favour of this requirement), then the manifolds modelled on

$$\mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$$

have to be excluded in three-dimensions, while the manifolds modelled on

$$\begin{aligned} \mathbb{S}^2 &\times \mathbb{R}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2, \\ \mathbb{S}^3 &\times \mathbb{R}, \mathbb{H}^3 \times \mathbb{R}, \widetilde{PSL}(2, \mathbb{R}) \times \mathbb{R}, Sol^3 \times \mathbb{R} \end{aligned}$$

have to be neglected in four-dimensions. There is only a finite number of manifolds of the form $\Gamma \backslash \mathbb{R}^N$, $\Gamma \backslash \mathbb{S}^N$ for any N [75]. It seems that in field theory the more important contribution to the vacuum persistence amplitude should be given by the hyperbolic geometry, the other geometries appearing only for a small number of exceptions [10]. Indeed, many three-manifolds are hyperbolic (according to a famous theorem by Thurston [64]). For example, the complement of a knot in $\mathbb{S}^3$ admits a hyperbolic structure unless it is a torus or satellite knot. Moreover, after the Mostow Rigidity Theorem [53], any geometric invariant of a hyperbolic three-manifold is a topological invariant. In this work, we will follow this prescription and concentrate on

compact hyperbolic spaces and topological invariants associated with that geometry.

This paper is organized as follows. In Section 2 we discuss solutions of the eleven-dimensional supergravity, which can be presented by means of direct product of spaces containing real hyperbolic space forms as factors, and the questions of supersymmetry surviving under the orbifolding. In Section 3.1, we recall the definition of the Laplace operator on p -forms, and discuss its relation to the Ray-Singer norm, specially for the case of three-dimensional compact hyperbolic manifolds. This is one of the essential point for our calculation of hyperbolic contribution to the Thurston complex invariant. Spectral functions of hyperbolic geometry are considered in Section 3.2. Using the standard definition for the Dirac bundle and holomorphic eta function we review the results of Millson [49], and Moscovici and Stanton [51, 52] on the spectral Selberg type zeta functions (Shintani functions). The invariant under consideration involves analytic torsion on a hyperbolic three-manifold, which can be expressed by means of Selberg zeta functions associated with twisted eta invariant of Atiyah-Patodi-Singer. Odd dimensional non-compact hyperbolic spaces with cusps are considered in Section 3.3. We note that spectral eta invariants in the case of non-compact spaces with cusps similar to that in the compact case. In Section 4 the explicit formula for $U(n)$ -Chern-Simons invariant (Theorem 4) is obtained. If the three-manifold is a homology three-sphere, every $U(n)$ -representation of the fundamental group is a $SU(n)$ -representation. Thus our main Theorem 4 computes the Chern-Simons invariant for any representation of the fundamental group in the case of hyperbolic fibered homology three-spheres. Brief discussion of the classification of sufficiently large manifolds (Haken class) is contained in Section 4.1. The determinant line bundle with its connection to the eta invariant and the global anomaly formula is considered in Section 4.2. Thus, the topological invariants mentioned above can be combined to the hyperbolic components of Thurston complex invariant. Finally in Section 4.3 we review the vanishing theorems for type $(0, q)$ cohomology of locally symmetric spaces G/K , using results of Williams [70]. If $\mathcal{K}$ is a finite complex, and if the orthogonal representation χ of fundamental group $\pi_1(\mathcal{K})$ is acyclic (i.e. the vector space $H^q(\mathcal{K}; \chi)$ of twisted cohomology classes is zero for all q), then the Ray-Singer torsion is an invariant. We assume that the complexification $G^{\mathbb{C}}$ of G is simply-connected, and take under consideration the vanishing theorems for the case of G -invariant complex structure. We use known interactions between complex manifold theory and the representation theory of compact Lie groups. The classical Cartan-Weil “highest weight” theory parametrizes the irreducible holomorphic modules via Lie algebra data. The Borel-Weil theorem gives a concrete realization of such a module. This theorem asserts, in effect that all finite-dimensional, holomorphic, irreducible modules for a simply-connected complex semi-simple Lie group G are related by the

modules of holomorphic sections of appropriate homogeneous G bundles [13, 14, 66, 65, 69].

2. HYPERBOLIC SPACE FORMS IN M-THEORY

In this Section we discuss solutions of the eleven-dimensional supergravity which can be presented by means of direct product of spaces containing real hyperbolic space forms as factors and supersymmetry surviving under the orbifolding. In eleven-dimensional supergravity the graviton multiplet contains the graviton g_{MN} , the antisymmetric three-form A_{MNK} and the gravitino Ψ_M ($M, N, K, \dots = 0, 1, \dots, 10$). The bosonic part of the supergravity Lagrangian has the form

$$(2) \quad \mathcal{L}_{(\text{boson})} = \frac{\sqrt{g}}{2\kappa_{11}^2} \left(R - \frac{1}{2 \cdot 4!} F_{MNPQ} F^{MNPQ} \right) - \frac{1}{12^4} \varepsilon^{M_1 \dots M_{11}} A_{M_1 M_2 M_3} F_{M_4 \dots M_7} F_{M_8 \dots M_{11}} .$$

A solution to the equations of motions

$$(3) \quad R_{MN} = \frac{1}{12} \left(F_{MPQR} F_N{}^{PQR} - \frac{1}{12} g_{MN} F^2 \right) ,$$

$$(4) \quad \nabla_M F^{MNPQ} = -\frac{1}{1152} \varepsilon^{NPQR_1 \dots R_8} F_{R_1 \dots R_4} F_{R_5 \dots R_8} ,$$

is provided by the Freund-Rubin ansatz for the antisymmetric field strength

$$(5) \quad F_{mnpq} = 6m_0 \varepsilon_{mnpq} \quad \text{for } m, n, \dots = 7, \dots, 10; \quad F_{MNPQ} = 0 \quad \text{otherwise} .$$

By substituting this ansatz into the field equations (3) we get

$$(6) \quad R_{\mu\nu} = -6m_0^2 g_{\mu\nu} , \quad \mu, \nu = 0, \dots, 6, \quad R_{mn} = 12m_0^2 g_{mn} .$$

The requirement of unbroken supersymmetry, i.e., the vanishing of the gravitino transformation

$$(7) \quad \delta\Psi_M = \nabla_M \varepsilon - \frac{1}{288} \left(\Gamma_M{}^{PQRS} - 8\delta_M{}^P \Gamma^{QRS} \right) \varepsilon F_{PQRS} ,$$

for the ansatz (5), is equivalent to the existence of $SO(1, 6)$ and $SO(4)$ Killing spinors θ and η , respectively, which satisfy

$$(8) \quad \nabla_\mu \theta = \pm \frac{1}{2} m_0 \gamma_\mu \theta , \quad \nabla_m \eta = \pm m_0 \gamma_m \eta ,$$

where $\gamma_\mu (\gamma_m)$ are $SO(1, 6)$ ($SO(4)$) γ -matrices. Eq. (6) admits solution of the form $X^7 \times Y^4$ where X^7 and Y^4 are Einstein spaces of negative and positive curvature, respectively. But only those spaces that admit Killing spinors obeying Eq. (8) preserve supersymmetry. The integrability conditions of Eq. (8) are $W_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \theta = 0$, $W_{mnpq} \gamma^{pq} \eta = 0$, where $W_{\mu\nu\rho\sigma}$, W_{mnpq} are the Weyl tensors of X^7 , Y^4 , respectively. Thus, obvious supersymmetric examples for Y^4 include the round four-sphere $\mathbb{S}^4$ and its orbifolds $\Gamma \backslash \mathbb{S}^4$, where Γ is an appropriate discrete group [26]. For the X^7 space one can take the anti de Sitter space AdS_7 , which preserves supersymmetry as well, and leads to the $AdS_7 \times \mathbb{S}^4$ vacuum of eleven-dimensional supergravity.

There are solutions to Eq. (6) involving hyperbolic spaces which are vacua of eleven-dimensional supergravity, and solve Eqs. (3), (4):

- (i) $AdS_{7-N} \times \mathbb{H}^N \times \mathbb{S}^4$, $N \geq 2$
- (ii) $AdS_3 \times \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{S}^4$
- (iii) $AdS_2 \times \mathbb{H}^2 \times \mathbb{H}^3 \times \mathbb{S}^4$

Let us consider an irreducible rank one symmetric space $X = G/K$ of non-compact type. Thus G will be a connected non-compact simple split rank one Lie group with finite center and $K \subset G$ will be a maximal compact subgroup. Up to local isomorphism we can represent X by the following quotients:

$$(9) \quad \begin{aligned} X = & \quad SO_1(N, 1)/SO(N), \quad SU(N, 1)/U(N), \\ & SP(N, 1)/(SP(N) \times SP(1)), \quad F_{4(-20)}/Spin(9), \end{aligned}$$

where the dimension of the spaces is $N, 2N, 4N, 16$ respectively in these cases. For details on these matters the reader may consult [35]. The spherical harmonic analysis on X is controlled by Harish-Chandra's Plancherel density $\mu(r)$, a function on the real numbers $\mathbb{R}$, computed by Miatello [47, 48, 20], and others, in the rank one case we are considering. The object of interest is the groups $G = SO_1(N, 1)$ ($N \in \mathbb{Z}_+$) and $K = SO(N)$. The corresponding symmetric space of non-compact type is the real hyperbolic space $X = \mathbb{H}^N = SO_1(N, 1)/SO(N)$ of sectional curvature -1 . Its compact dual space is the unit N -sphere.

2.1. Cosets $\Gamma \backslash G/K$ and Killing spinors. We have discussed supergravity solutions involving anti de Sitter and real hyperbolic space factors. Hyperbolic space forms admit Killing spinors [15, 32, 42, 43] but they have infinite volume with respect to the Poincaré metric, and do not seem useful for describing internal spaces in fields compactifications. Let us regard Γ as a discrete subgroup of G acting isometrically on X , and take X_Γ to be quotient space by that action: $X_\Gamma = \Gamma \backslash G/K$. The question of interest is whether space X_Γ admits Killing spinors and preserves supersymmetry.

The hyperbolic manifolds $\mathbb{H}^N$, as factors in solution (i), admit Killing spinors. However, having the space forms (ii), (iii) : $AdS_3 \times \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{S}^4$, $AdS_2 \times \mathbb{H}^2 \times \mathbb{H}^3 \times \mathbb{S}^4$, as the solutions of supergravity theory, one can recognize that the factors $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^3$ cannot leave any unbroken supersymmetry. Indeed, the following result holds:

Proposition 1. (*T. Friedrich [31]*). *A Riemannian spin manifold (M^N, g) admitting a Killing spinor $\psi \neq 0$ with Killing number $\mu \neq 0$ is locally irreducible.*

Proof. Suppose that the locally Riemannian product has the form $M^N = M^K \times M^{N-K}$. Let $\mathcal{X}, \mathcal{Y}$ be vectors tangent to M^K and M^{N-K} respectively, and, therefore, the curvature tensor of the Riemannian manifold (M^N, g)

is trivial. Since ψ is a Killing spinor the following equations hold:

$$\begin{aligned} \nabla_{\mathcal{X}}\psi &= \mu\mathcal{X} \cdot \psi, \\ 4\mu^2 &= [N(N-1)]^{-1}R \text{ at each point of a connected Riemannian} \\ (10) \quad &\text{spin manifold } (M^N, g), \end{aligned}$$

where R is a scalar curvature. Because of (10) we have

$$\begin{aligned} \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\psi &= \mu(\nabla_{\mathcal{X}}\mathcal{Y}) \cdot \psi + \mu^2\mathcal{Y} \cdot \mathcal{X} \cdot \psi \implies \\ (11) \quad &(\nabla_{\mathcal{X}}\nabla_{\mathcal{Y}} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}} - \nabla_{[\mathcal{X}, \mathcal{Y}]})\psi = \mu^2(\mathcal{Y} \cdot \mathcal{X} - \mathcal{X} \cdot \mathcal{Y})\psi. \end{aligned}$$

The curvature tensor $R(\mathcal{X}, \mathcal{Y})$ in the spinor bundle $\mathfrak{S}$ is related to the curvature tensor of the Riemannian manifold (M^N, g) : $R(\mathcal{X}, \mathcal{Y}) = (1/4)\sum_{j=1}^N e_j R(\mathcal{X}, \mathcal{Y})e_j \cdot \psi$, where $\{e_j\}_{j=1}^N$ is a orthogonal basis in the manifold. Therefore Eq. (11) can also be written as

$$(12) \quad \sum_{j=1}^N e_j R(\mathcal{X}, \mathcal{Y})e_j \psi + [N(N-1)]^{-1}R(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})\psi = 0.$$

From Eq. (12) we get $R \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \psi = 0$, and moreover $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal vectors. Since $\mu \neq 0$ ($R \neq 0$) it follows that $\psi = 0$, hence a contradiction. $\square$

We have also the following statement:

Proposition 2. (*T. Friedrich [31]*). *Let (M^N, g) be a connected Riemannian spin manifold and let ψ is a non-trivial Killing spinor with Killing number $\mu \neq 0$. Then (M^N, g) is an Einstein space.*

Proof. The proof easily follows from Proposition 2; indeed (M^N, g) is an Einstein space of scalar curvature given by Eq. (10). $\square$

Spaces with finite volume of fundamental domain can be obtained by forming the coset spaces with topology $\Gamma \backslash \mathbb{H}^N$ where Γ is a discrete subgroup of the isometry group. Let us comment on the supersymmetry of these spaces following the lines of [40, 62]. For non-trivial Γ and finite volume space $\Gamma \backslash \mathbb{H}^N$ it has been shown [40] that for even N supersymmetries are always broken by the identifications. Indeed, the isometry group of $\mathbb{H}^N$ is $SO_1(N, 1)$ and Γ is in general a subgroup of $SO_1(N, 1)$, which may or may not have fixed points. Killing spinors are in the spinorial representation of $SO_1(N-1, 1)$, and if Γ is a subgroup of $SO_1(N-1, 1)$, but it is not a subgroup of $SO_1(N-3, 1)$, then there are no surviving Killing spinors. The later exist if $\Gamma \in SO_1(N-3, 1)$, but in this case $\Gamma \backslash \mathbb{H}^N$ will still be of infinite volume. Therefore, for even N there is no finite volume cosets $\Gamma \backslash \mathbb{H}^N$ with unbroken supersymmetries. On the other hand, for odd N this analysis does not exclude that an appropriate choice of Γ could give a supersymmetric model with finite volume hyperbolic space. For odd N there are two Killing spinors on $\mathbb{H}^N$ in the spinorial representation of $SO_1(N-1, 1)$. These spinors are also Weyl spinors of the isometry group

$SO_1(N, 1)$, so they form an irreducible Dirac spinor of $SO_1(N, 1)$. All supersymmetries are broken if Γ is not a subgroup $SO_1(N - 1, 1)$. If Γ is a subgroup $SO_1(N - 1, 1)$, then half of the supersymmetries survive. A question of interest is whether supersymmetry survives under the orbifolding by the discrete group Γ . Perhaps there are more solutions involving real hyperbolic spaces, where some supersymmetries are unbroken. However the analysis of that problem is complicated and we leave it for other occasion.

3. BUNDLES OVER LOCALLY SYMMETRIC SPACES AND SPECTRAL FUNCTIONS

3.1. The Laplace operator and the Ray-Singer norm. Let us consider an N -dimensional compact real hyperbolic space X_Γ with universal covering $\tilde{X}$ and fundamental group Γ . As before we can represent $\tilde{X}$ as the symmetric space G/K . Then we regard Γ as a discrete subgroup of G acting isometrically on $\tilde{X}$, and we take X_Γ to be the quotient space by that action: $X_\Gamma = \Gamma \backslash \tilde{X} = \Gamma \backslash G/K$. Let τ be an irreducible representation of K on a complex vector space V_τ , and form the induced homogeneous vector bundle $G \times_K V_\tau$ (the fiber product of G with V_τ over K) $\rightarrow \tilde{X}$ over $\tilde{X}$. Restricting the G action to Γ we obtain the quotient bundle $E_\tau = \Gamma \backslash (G \times_K V_\tau) \rightarrow X_\Gamma = \Gamma \backslash \tilde{X}$ over X_Γ . The natural Riemannian structure on $\tilde{X}$ (therefore on X_Γ) induced by the Killing form $(\cdot, \cdot)$ of G gives rise to a connection Laplacian $\mathcal{L}$ on E_τ . If Ω_K denotes the Casimir operator of K – that is $\Omega_K = -\sum y_j^2$, for a basis $\{y_j\}$ of the Lie algebra $\mathfrak{k}_0$ of K , where $(y_j, y_\ell) = -\delta_{j\ell}$, then $\tau(\Omega_K) = \lambda_\tau \mathbb{I}$ for a suitable scalar λ_τ . Moreover for the Casimir operator Ω of G , with Ω operating on smooth sections $\Gamma^\infty E_\tau$ of E_τ one has $\mathcal{L} = \Omega - \lambda_\tau \mathbb{I}$; see Lemma 3.1 of [67]. For $\lambda \geq 0$ let

$$(13) \quad \Gamma^\infty(X, E_\tau)_\lambda = \{s \in \Gamma^\infty E_\tau \mid -\mathcal{L}s = \lambda s\}$$

be the space of eigensections of $\mathcal{L}$ corresponding to λ . Here we note that since X_Γ is compact we can order the spectrum of $-\mathcal{L}$ by taking $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$; $\lim_{j \rightarrow \infty} \lambda_j = \infty$. We shall specialize τ to be the representation $\tau^{(p)}$ of $K = SO(N)$ on $\Lambda^p \mathbb{C}^N$. It will be convenient moreover to work with the normalized Laplacian $\mathcal{L}_p = -c(N)\mathcal{L}$ where $c(N) = 2(N - 1)$. $\mathcal{L}_p$ has spectrum $\{c(N)\lambda_j, m_j\}_{j=0}^\infty$ where the multiplicity m_j of the eigenvalue $c(N)\lambda_j$ is given by

$$(14) \quad m_j = \dim \Gamma^\infty(X, E_{\tau^{(p)}})_{\lambda_j}.$$

If $\mathcal{L}_p$ is a self-adjoint Laplacian on p -forms then the following results hold. There exists $\varepsilon, \delta > 0$ such that for $0 < t < \delta$ the heat kernel expansion for Laplace operators on a compact manifold M is given by

$$(15) \quad \text{Tr}(e^{-t\mathcal{L}_p}) = \sum_{0 \leq \ell \leq \ell_0} a_\ell(\mathcal{L}_p)t^{-\ell} + \mathcal{O}(t^\varepsilon).$$

The zeta function of $\mathfrak{L}_p$ is the Mellin transform

$$(16) \quad \zeta(s|\mathfrak{L}_p) = \widehat{\mathfrak{M}} [\mathrm{Tre}^{-t\mathfrak{L}_p}] = \frac{1}{\Gamma(s)} \int_{\mathbb{R}_+} \mathrm{Tre}^{-t\mathfrak{L}_p} t^{s-1} dt.$$

This function equals $\mathrm{Tr} (\mathfrak{L}_p^{-s})$ for $s > (1/2)\dim M$.

Let χ be an orthogonal representation of $\pi_1(M)$. Using the Hodge decomposition, the vector space $H(M; \chi)$ of twisted cohomology classes can be embedded into $\Omega(M; \chi)$ as the space of harmonic forms. This embedding induces a norm $|\cdot|^{RS}$ on the determinant line $\det H(M; \chi)$. The Ray-Singer norm $\|\cdot\|^{RS}$ on $\det H(M; \chi)$ is defined by [60]

$$(17) \quad \|\cdot\|^{RS} \stackrel{def}{=} |\cdot|^{RS} \prod_{p=0}^{\dim M} \left[\exp \left(-\frac{d}{ds} \zeta(s|\mathfrak{L}_p)|_{s=0} \right) \right]^{(-1)^p p/2},$$

where the zeta function $\zeta(s|\mathfrak{L}_p)$ of the Laplacian acting on the space of p -forms orthogonal to the harmonic forms has been used. For a closed connected orientable smooth manifold of odd dimension and for Euler structure $\eta \in \mathrm{Eul}(M)$ the Ray-Singer norm of its cohomological torsion $\tau_{an}(M; \eta) = \tau_{an}(M) \in \det H(M; \chi)$ is equal to the positive square root of the absolute value of the monodromy of χ along the characteristic class $c(\eta) \in H^1(M)$ [25]: $\|\tau_{an}(M)\|^{RS} = |\det_\chi c(\eta)|^{1/2}$. In the special case where the flat bundle χ is acyclic ($H(M; \chi) = 0$) we have

$$(18) \quad [\tau_{an}(X)]^2 = |\det_\chi c(\eta)| \prod_{p=0}^{\dim M} \left[\exp \left(-\frac{d}{ds} \zeta(s|\mathfrak{L}_p)|_{s=0} \right) \right]^{(-1)^{p+1} p}.$$

For a closed oriented hyperbolic three-manifolds of the form $X_\Gamma = \Gamma \backslash \mathbb{H}^3$ and for acyclic χ the L^2 -analytic torsion gets the form [30, 17, 18]: $[\tau_{an}(X_\Gamma)]^2 = \mathfrak{R}_\chi(0)$, where $\mathfrak{R}_\chi(s)$ is the Ruelle function. A Ruelle type zeta function for $\Re s$ large can be defined as the product over prime closed geodesics γ of factors $\det(I - \xi(\gamma)e^{-s\ell(\gamma)})$, where $\ell(\gamma)$ is the length of γ , and can be continued meromorphically to the entire complex plane $\mathbb{C}$ [22]. The function $\mathfrak{R}_\chi(s)$ is an alternating product of more complicate factors, each of which is a Selberg zeta function $Z_{\Gamma,p}(s, \chi)$. The relation between the Ruelle and Selberg functions is:

$$(19) \quad \mathfrak{R}_\chi(s) = \prod_{p=0}^{\dim M - 1} Z_{\Gamma,p}(p + s, \chi)^{(-1)^p}.$$

The Ruelle function associated with closed oriented hyperbolic three-manifold X_Γ has the form: $\mathfrak{R}_\chi(s) = Z_{\Gamma,0}(s, \chi)Z_{\Gamma,2}(2 + s, \chi)/Z_{\Gamma,1}(1 + s, \chi)$.

3.2. The spectral functions of hyperbolic geometry. We recall some results on the eta invariant of a self-adjoint elliptic differential operator acting on a compact manifold. For details we refer the reader to [3, 4, 5] where the eta invariant was introduced in connection with the index theorem for a manifold with boundary. One can attach the eta invariant to any operator of Dirac type on a compact Riemannian manifold of odd dimension. Dirac operators on even dimensional manifolds have symmetric spectra and, therefore, trivial eta invariants.

To define a spectral invariant which measures the asymmetry of the spectrum $\text{Spec}(\mathfrak{D})$ of an operator $\mathfrak{D}$, one starts with the following formula: The holomorphic function

$$(20) \quad \eta(s, \mathfrak{D}) \stackrel{\text{def}}{=} \sum_{\lambda \in \text{Spec } \mathfrak{D} \setminus \{0\}} \text{sgn}(\lambda) |\lambda|^{-s} = \text{Tr} \left(\mathfrak{D} (\mathfrak{D}^2)^{-(s+1)/2} \right),$$

is well defined for all $\Re s \gg 0$ and extends to a meromorphic function on $\mathbb{C}$. Indeed, from the asymptotic behaviour of the heat operator at $t = 0$ [12],

$$(21) \quad \text{Tr} \left(\mathfrak{D} e^{-t\mathfrak{D}^2} \right) = \mathcal{O}(t^{1/2}),$$

and from the identity

$$(22) \quad \eta(s, \mathfrak{D}) = \frac{1}{\Gamma((s+1)/2)} \int_{\mathbb{R}_+} \text{Tr} \left(\mathfrak{D} e^{-t\mathfrak{D}^2} \right) t^{(s-1)/2} dt,$$

it follows that $\eta(s, \mathfrak{D})$ admits a meromorphic extension to the whole s -plane, with at most simple poles at $s = (\dim M - k)/(\text{ord } \mathfrak{D})$ ($k \in \mathbb{Z}_+$) and locally computable residues. It has been established that point $s = 0$ is not a pole, which makes it possible to define the eta invariant of $\mathfrak{D}$ by $\eta(0, \mathfrak{D})$. It also follows directly that $\eta(0, -\mathfrak{D}) = -\eta(0, \mathfrak{D})$ and $\eta(0, \lambda \mathfrak{D}) = \eta(0, \mathfrak{D})$, $\forall \lambda > 0$. Because $\mathfrak{D}^+$ is isomorphic to $\mathfrak{D}^-$, we have $\eta(s, \mathfrak{D}^+) = \eta(s, \mathfrak{D}^-) = \eta(s, \mathfrak{D})/2$. An important case of such an operator is the even part of the tangential signature operator, $\mathfrak{B}$, acting on the even forms of the given manifold. The eta invariant associated to the operator $\mathfrak{B}$ is given by $\eta_M(0) = \eta(0, \mathfrak{B})$, and is called the eta invariant of M [51, 52].

Let X_Γ be a compact oriented $(4m-1)$ -dimensional Riemannian manifold of constant negative curvature. A remarkable formula relating $\eta(s, \mathfrak{B})$, to the closed geodesics on X_Γ has been derived by Millson [49]. More explicitly, Millson proved the following result for a Selberg type (Shintani) zeta function.

Definition 1 (J. J. Millson [49]). Define a zeta function by the following series which is absolutely convergent for $\Re s > 0$,

$$(23) \quad \log Z(s, \mathfrak{B}) \stackrel{\text{def}}{=} \sum_{[\gamma] \neq 1} \frac{\text{Tr} \tau_\gamma^+ - \text{Tr} \tau_\gamma^-}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)},$$

where $[\gamma]$ runs over the nontrivial conjugacy classes in the fundamental group $\Gamma = \pi_1(X_\Gamma)$, $\ell(\gamma)$ is the length of the closed geodesic c_γ (with multiplicity $m(\gamma)$) in the free homotopy class corresponding to $[\gamma]$, $P_h(\gamma)$ is the restriction of the linear Poincaré map $P(\gamma) = d\Phi_1$ at $(c_\gamma, \dot{c}_\gamma) \in TX_\Gamma$ to the directions normal to the geodesic flow Φ_t and $\tau_\gamma^\pm$ is parallel translation around c_γ on $\Lambda_\gamma^\pm = \pm\sqrt{-1}$ eigenspace of $\sigma_{\mathfrak{B}}(\dot{c}_\gamma)$ ($\sigma_{\mathfrak{B}}$ denoting the principal symbol of $\mathfrak{B}$). Then $Z(s, \mathfrak{B})$ admits a meromorphic continuation to the entire complex plane, which in particular is holomorphic at 0. Moreover, $\log Z(0, \mathfrak{B}) = \sqrt{-1}\pi\eta(0, \mathfrak{B})$.

In fact $Z(s, \mathfrak{B})$ satisfies the functional equation

$$(24) \quad Z(s, \mathfrak{B})Z(-s, \mathfrak{B}) = e^{2\pi\sqrt{-1}\eta(s, \mathfrak{B})}.$$

Millson's formulae have been extended by Moscovici and Stanton [51] to Dirac type operators (acting in non-positively curved locally symmetric manifolds), even with additional coefficients in locally flat bundles.

Let $\mathfrak{D}$ denote a generalized Dirac operator associated to a locally homogeneous Clifford bundle over compact oriented odd dimensional locally symmetric space X_Γ , whose simply connected cover $\tilde{X}$ is a symmetric space of noncompact type. The fixed point set of the geodesic flow, acting on the unit sphere bundle T^1X_Γ , is a disjoint union of submanifolds $X_{\Gamma, \gamma}$. These submanifolds are parametrized by the nontrivial conjugacy classes $[\gamma] \neq 1$ in Γ . By $\mathcal{E}_1(\Gamma)$ we denote the set of those conjugacy classes $[\gamma]$ for which $X_{\Gamma, \gamma}$ has the property that the Euclidean de Rham factor of $\tilde{X}_\gamma$ is one-dimensional. A bundle $C\tilde{X}_\gamma$ over $\tilde{X}_\gamma$, the “central” bundle, is determined by the eigenvalues of absolute value 1 of the linear Poincaré map $P(\gamma)$. The parallel translation around c_γ gives rise to an orthogonal transformation $\tilde{\tau}_\gamma$ of $C\tilde{X}_\gamma$; $T\tilde{X}_\gamma \subset C\tilde{X}_\gamma$ and we let $N\tilde{X}_\gamma$ denote the orthogonal component of $T\tilde{X}_\gamma$ in $C\tilde{X}_\gamma$. The tangent bundle $T\tilde{X}_\gamma$ corresponds to the eigenvalue 1 of $\tilde{\tau}_\gamma$ and $N\tilde{X}_\gamma$ decomposes as [51]

$$(25) \quad N\tilde{X}_\gamma = N\tilde{X}_\gamma(-1) \oplus \sum_{0 < \theta < \pi} N\tilde{X}_\gamma(\theta)$$

according to the other values $-1, \exp(\pm\sqrt{-1}\theta)$ ($0 < \theta < \pi$). The restriction to $X_{\Gamma, \gamma}$ of the exterior bundle can be pushed down to a vector bundle $\tilde{\Lambda}_\gamma$ over $\tilde{X}_\gamma$ which splits into a subbundle $\tilde{\Lambda}_\gamma^\pm$ corresponding to the eigenvalue $\pm\sqrt{-1}$ of the symbol $\mathfrak{D}$. Thus we obtain a $\tilde{\tau}_\gamma$ -equivariant complex $\tilde{\sigma}_\gamma^\mathfrak{D} : \tilde{\Lambda}_\gamma^+ \rightarrow \tilde{\Lambda}_\gamma^-$ over $T\tilde{X}_\gamma$ and a class $[\tilde{\sigma}_\gamma^\mathfrak{D}] \in K_{\tilde{\tau}_\gamma}(T\tilde{X})$, the $\tilde{\tau}_\gamma$ -equivariant K -theory group of $T\tilde{X}_\gamma$. The cohomology class can be formed as in [2] (Section 3), $\text{ch}(\tilde{\sigma}_\gamma^\mathfrak{D}(\tilde{\tau}_\gamma)) \in H^+(T\tilde{X}_\gamma; \mathbb{C})$. We now present the main results.

Theorem 1. (*H. Moscovici and R. J. Stanton [51], Theorem 6.3*). *The following function can be defined, initially for $\Re(s^2) \gg 0$, by the formula*

$$(26) \quad \log Z(s, \mathfrak{D}) \stackrel{\text{def}}{=} \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} (-1)^q \frac{L(\gamma, \mathfrak{D})}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)},$$

where $q = (1/2)\dim N\tilde{X}_\gamma$ is a integer and independent of γ . The Lefschetz number $L(\gamma, \mathfrak{D})$ is given by (see, for example, [37] or [51], Eq. (5.5)):

$$(27) \quad L(\gamma, \mathfrak{D}) = \left\{ \frac{\text{ch}(\tilde{\sigma}_\gamma^{\mathfrak{D}}(\tilde{\tau}_\gamma)) \mathfrak{R}(N\tilde{X}_\gamma(-1)) \prod_{0 < \theta < \pi} \mathfrak{S}^\theta(N\tilde{X}_\gamma(\theta)) \mathfrak{T}(\tilde{X}_\gamma)}{\det(I - \tilde{\tau}_\gamma|N\tilde{X}_\gamma)} \right\} [T\tilde{X}_\gamma].$$

The Lefschetz formula (27) has been given using the stable characteristic classes $\mathfrak{R}$, $\mathfrak{S}^\theta$ and $\mathfrak{T}$ defined in [2] (Theorem 3.9).

Furthermore $\log Z(s, \mathfrak{D})$ has a meromorphic continuation to $\mathbb{C}$ given by the identity

$$(28) \quad \log Z(s, \mathfrak{D}) = \log \det' \left(\frac{\mathfrak{D} - \sqrt{-1}s}{\mathfrak{D} + \sqrt{-1}s} \right) + \sqrt{-1}\pi\eta(s, \mathfrak{D}),$$

where $s \in \sqrt{-1}\text{Spec}'(\mathfrak{D}) \setminus (\text{Spec}(\mathfrak{D}) - \{0\})$, and $Z(s, \mathfrak{D})$ satisfies the functional equation

$$(29) \quad Z(s, \mathfrak{D})Z(-s, \mathfrak{D}) = e^{2\pi\sqrt{-1}\eta(s, \mathfrak{D})}.$$

Let as before $\tilde{X}$ denote a simply connected cover of X_Γ , which is a symmetric space of noncompact type, and let $\tilde{E}$ denote the pull-back to $\tilde{X}$ for any vector bundle E over X_Γ . We restrict ourselves to bundles which will be considered satisfy a local homogeneity condition. Namely, a vector bundle E over X_Γ is $\mathcal{G}$ -locally homogeneous, for some Lie group $\mathcal{G}$, if there is a smooth action of $\mathcal{G}$ on E which is linear on the fibers and covers the action of $\mathcal{G}$ on $\tilde{X}$. Standard constructions from linear algebra applied to any $\mathcal{G}$ -locally homogeneous E give in a natural way corresponding $\mathcal{G}$ -locally homogeneous vector bundles. In particular, all bundles $TX_\Gamma, \mathbb{C}\ell(X_\Gamma), \text{End } E \simeq E^* \otimes E$ are $\mathcal{G}$ -locally homogeneous [51]. We shall require also that all constructions associated with $\mathcal{G}$ -locally homogeneous bundles are $\mathcal{G}$ -equivariant. Let $\mathfrak{D}$ denote a generalized Dirac operator associated to a locally homogeneous bundle E over X_Γ . We shall require $\mathcal{G}$ -equivariance for $\tilde{\nabla}$ the lift of ∇ to $\tilde{E}$ and therefore the corresponding Dirac operator is then $\mathcal{G}$ -invariant.

Suppose now that $\chi : \Gamma \rightarrow U(F)$ is an unitary representation of Γ on F . The Hermitian vector bundle $\mathbb{F} = \tilde{X} \times_\Gamma F$ over X_Γ inherits a flat connection from the trivial connection on $\tilde{X} \times F$. If $\mathfrak{D} : C^\infty(X, V) \rightarrow C^\infty(X, V)$ is a differential operator acting on the sections of the vector bundle V , then $\mathfrak{D}$ extends canonically to a differential operator $\mathfrak{D}_\chi : C^\infty(X, V \otimes \mathbb{F}) \rightarrow$

$C^\infty(X, V \otimes \mathbb{F})$, uniquely characterized by the property that $\mathfrak{D}_\chi$ is locally isomorphic to $\mathfrak{D} \otimes \dots \otimes \mathfrak{D}$ ($\dim F$ times). We specialize to the case of locally homogeneous Dirac operators $\mathfrak{D} : C^\infty(X_\Gamma, \mathbb{E}) \rightarrow C^\infty(X_\Gamma, \mathbb{E})$ in order to construct a generalized operator $\mathfrak{D}_\chi$, acting on spinors with coefficients in χ (see for detail [51]). One can repeat the arguments of the previous discussion to construct a twisted zeta function $Z(s, \mathfrak{D}_\chi)$. The main results can be stated as follows.

Theorem 2. (*H. Moscovici and J. J. Stanton [51], Section 7*). *There exists a zeta function $Z(s, \mathfrak{D}_\chi)$, meromorphic on $\mathbb{C}$, given for $\Re(s^2) \gg 0$ by the formula*

$$(30) \quad \log Z(s, \mathfrak{D}_\chi) \stackrel{\text{def}}{=} \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} (-1)^q \text{Tr} \chi(\gamma) \frac{L(\gamma, \mathfrak{D})}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)};$$

moreover, one has

$$(31) \quad \log Z(0, \mathfrak{D}_\chi) = \sqrt{-1} \pi \eta(0, \mathfrak{D}_\chi).$$

3.3. Cusp forms. A few notes on the result of previous Section are now in order. This result is based on the use of the Selberg trace formula. In fact the expanding $\text{Tr}(\mathfrak{D}e^{-t\mathfrak{D}})$ as a series of orbital integrals associated to the conjugacy classes $[\gamma]$ in Γ has been considered. Each orbital integral, over a necessary semisimple orbit, can be in turn expressed in terms of the noncommutative Fourier transform of the odd heat kernel, along the tempered unitary dual of G , the group of isometries of the symmetric space $\tilde{X}$.

The results of [51] on the case of compact locally symmetric spaces of higher ranks have been extended to the odd dimensional non-compact spaces with cusps in [55]. More precisely, taking into account the fixed Iwasawa decomposition $G = KAN$, consider a Γ -cuspidal minimal parabolic subgroup G_P of G with the Langlands decomposition $G_P = BAN$, B being the centralizer of A in K . Let us consider a family of functions $\mathcal{K}_t$ over $G = \text{Spin}(2k+1, 1)$, which is given by taking the local trace for the integral kernel $\exp(-t\mathfrak{D}^2)$ (or $\mathfrak{D} \exp(-t\mathfrak{D}^2)$). The Selberg trace formula applied to the scalar kernel function $\mathcal{K}_t$ holds [55]:

$$(32) \quad \sum_{\sigma=\sigma_\pm} \sum_{\lambda_k \in \sigma_p^\pm} \hat{\mathcal{K}}_t(\sigma, \sqrt{-1}\lambda_k) - \frac{\sqrt{-1}}{4\pi} \int_{\mathbb{R}} \text{Tr} \left(S_\Gamma(\sigma_\pm, -s) \frac{d}{ds} S_\Gamma(\sigma_+, s) \pi_\Gamma(\sigma_+, s)(\mathcal{K}_t) \right) ds = I_\Gamma(\mathcal{K}_t) + H_\Gamma(\mathcal{K}_t) + U_\Gamma(\mathcal{K}_t),$$

where $\sigma_p := \sigma_p^+ \cup \sigma_p^-$ gives the point spectrum of $\mathfrak{D}$, $S_\Gamma(\sigma_+, \sqrt{-1}\lambda)$ is the intertwining operator and $I_\Gamma(\mathcal{K}_t)$, $H_\Gamma(\mathcal{K}_t)$, $U_\Gamma(\mathcal{K}_t)$ are the identity, hyperbolic and unipotent orbital integrals. If $\mathcal{K}_t$ is given by $\mathfrak{D}e^{-t\mathfrak{D}^2}$, then $I_\Gamma(\mathcal{K}_t) = 0$ by the Fourier transformation of $\mathcal{K}_t$. The analysis of the unipotent orbital integral $U_\Gamma(\mathcal{K}_t)$ gives the following result [9, 55]: All of the

unipotent terms vanish in the Selberg trace formula applied to the odd kernel function $\mathcal{K}_t$ given by $\mathfrak{D}e^{-t\mathfrak{D}^2}$. It means that one can obtain spectral invariants in the case of cusps similar to invariants of smooth compact odd dimensional manifolds.

4. CHERN-SIMONS INVARIANT AND THE ANOMALY FORMULA

A gauge Chern-Simons theory is interesting both for its mathematical novelty and for its applications for certain planar condensed matter phenomena (such as the fractional quantum Hall effect) and for nonabelian gauge models in field theory. The Chern-Simons functional has been actively studied in low dimensional topology in connection with Floer homology [28], where it was used as a Morse function.

In the case of $SU(2)$ -gauge the formula for the Chern-Simons invariant for some classes of three-manifolds including the Seifert fibered manifolds was derived in [39, 41, 6]. The invariants have been obtained by cutting a three-manifold into pieces for which the invariants can be computed. Such a method has been applied for the case of $SU(n)$ -Chern-Simons invariants of Seifert fibered three-manifolds in [54]. Note the relation of the method, developed in [39, 41, 6, 54], to the prequantum line bundle over the moduli space of flat connections on a Riemannian surface. For the geometrical aspect of Jones-Witten theory this point of view is quite natural. In [61] a quantum invariant for an oriented three-manifold has been defined using a representation theory of quantum group. It is believed that this invariant coincides with Witten's invariant. Thus, the Witten's path integral approach suggests the asymptotic behavior of quantum invariant, which includes the Chern-Simons invariant, the eta-invariant, the Ray-Singer torsion, etc. Therefore it is important to have an explicit formula for the invariant. The goal of this section is to present such a formula for the $U(n)$ -Chern-Simons invariant of an irreducible flat connection on the real compact hyperbolic three-manifolds.

The Chern-Simons functional $CS(X_\Gamma)$ as a function on a space of connections on a trivial principal bundle over a compact oriented three-manifold X_Γ is given by

$$(33) \quad CS(X_\Gamma) = \frac{1}{8\pi^2} \int_{X_\Gamma} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

Let, as in previous Sections, X_Γ be a compact locally symmetric Riemannian manifold with negative sectional curvature. Its universal covering $\tilde{X} \rightarrow X_\Gamma$ is a Riemannian symmetric space. The group of orientation preserving isometries G of $\tilde{X}$ is a connected semisimple Lie group of real rank one and $\tilde{X} = G/K$, where K is a maximal compact subgroup of G . The fundamental group of X_Γ acts by covering transformations on $\tilde{X}$ and gives rise to a discrete, co-compact subgroup $\Gamma \subset \tilde{G}$. If G is a linear connected finite covering of $\tilde{G}$, the embedding $\Gamma \hookrightarrow \tilde{G}$ lifts to an embedding $\Gamma \hookrightarrow G$. Let $\mathcal{P} = X_\Gamma \otimes \mathfrak{G}$ be a principal bundle over X_Γ with the gauge group

$\mathfrak{G} = U(n)$ and let $U_{X_\Gamma} = \Omega^1(X_\Gamma; \mathfrak{g})$ be the space of all connections on $\mathcal{P}$; this space is an affine space of one-forms on X_Γ with values in the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. The value of the function $CS(X_\Gamma) \equiv CS(A)$ on the space of connections U_{X_Γ} , $A \in U_{X_\Gamma}$, at a critical point can be regarded as a topological invariant of a pair (X_Γ, χ) , where χ is an orthogonal representation of the fundamental group Γ .

Let $\mathfrak{A}_{X_\Gamma} = \{A \in \mathfrak{A}_{X_\Gamma} | F_A = dA + A \wedge A = 0\}$ be the space of flat connections on $\mathcal{P}$. An important formula related to the integrand in (33) is

$$(34) \quad d\text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = \text{Tr} (F_A \wedge F_A).$$

Eq. (34) gives another approach to the Chern-Simons invariant. Indeed, let M be an oriented four-manifold with boundary $\partial M = X_\Gamma$. One can extend $\mathcal{P}$ to a trivial $\mathfrak{G}$ -bundle over M ; then Stokes' theorem gives

$$(35) \quad CS(A) = \frac{1}{8\pi^2} \int_M \text{Tr} (F_{\tilde{A}} \wedge F_{\tilde{A}}),$$

where $\tilde{A}$ is any extension of A over M . Eq. (35) can be viewed as a generalization of the Chern-Simons invariant to the case in which $\mathcal{P}$ is a non-trivial $U(n)$ -bundle over X_Γ . In fact we shall show that the Chern-Simons functional is well-defined modulo $(\mathbb{Z}/2)$ in the case of a $U(n)$ -connection (see also [46]).

4.1. The $U(n)$ -Chern-Simons invariant. Let $\mathbb{E}_\chi = \tilde{X} \otimes_\chi \mathbb{C}^n$ be a flat vector bundle, and let $\tilde{\mathcal{P}}$ be a principal $U(n)$ -bundle over M which is an extension of $\mathcal{P}$. Suppose that χ is any one-dimensional representation of Γ factors through a representation of $H^1(X; \mathbb{Z})$. It can be shown that for a unitary representation $\chi : \Gamma \rightarrow U(n)$, the corresponding flat vector bundle $\mathbb{E}_\chi$ is topologically trivial ($\mathbb{E}_\chi \cong X \otimes \mathbb{C}^n$) if and only if $\det \chi|_{\text{Tor}^1} : \text{Tor}^1 \rightarrow U(1)$ is the trivial representation. Here Tor^1 is the torsion part of $H^1(M; \mathbb{Z})$ and $\det \chi$ is a one-dimensional representation of Γ defined by $\det \chi(\gamma) := \det (\chi(\gamma))$, for $\gamma \in \Gamma$. For any representation $\chi : \Gamma \rightarrow U(n)$ one can construct a vector bundle $\tilde{\mathbb{E}}_\chi$ over a certain four-manifold M with boundary $\partial M = \Gamma \backslash \mathbb{H}^3$ which is an extension of a flat vector bundle $\mathbb{E}_\chi$ over $\Gamma \backslash \mathbb{H}^3$. Let $\tilde{A}_\chi$ be any extension of a flat connection A_χ corresponding to χ .

Theorem 3. (*M. F. Atiyah, V. K. Patodi and I. M. Singer [3, 4, 5]*).
The Dirac index is given by

$$(36) \quad \text{Index} \left(D_{\tilde{A}_\chi} \right) = \int_M \text{ch}(\tilde{\mathbb{E}}_\chi) \hat{A}(M) - \frac{1}{2} (\eta(0, \mathfrak{D}_\chi) + h(0, \mathfrak{D}_\chi)),$$

where $h(0, \mathfrak{D}_\chi)$ is the dimension of the space of harmonic spinors on X_Γ ($h(0, \mathfrak{D}_\chi) = \dim \text{Ker } \mathfrak{D}_\chi = \text{multiplicity of the } 0\text{-eigenvalue of } \mathfrak{D}_\chi \text{ acting on } X_\Gamma$); $\mathfrak{D}_\chi$ is a Dirac operator on X_Γ acting on spinors with coefficients in χ .

The Chern-Simons invariant of $X_\Gamma = \Gamma \backslash \mathbb{H}^3$ can be derived from Eq. (36). It is given by the main theorem:

Theorem 4. *The following formula for the $U(n)$ -Chern-Simons invariant of an irreducible flat connection on the real hyperbolic three-manifolds holds:*

$$(37) \quad CS_{U(n)}(A_\chi) = \frac{1}{2\pi\sqrt{-1}} \log \left[\frac{Z(0, \mathfrak{D})^{\dim \chi}}{Z(0, \mathfrak{D}_\chi)} \right] \quad \text{modulo } (\mathbb{Z}/2).$$

Proof. For a closed manifold we have

$$(38) \quad \text{ch}_2(\tilde{\mathbb{E}}_\chi) = -(1/8\pi^2) \text{Tr} \left(F_{\tilde{A}_\chi} \wedge F_{\tilde{A}_\chi} \right),$$

where $\text{ch}_2(\tilde{\mathbb{E}}_\chi)$ is the second Chern character of $\tilde{\mathbb{E}}_\chi$, which is expressed in terms of the first and second Chern classes: $\text{ch}_2(\tilde{\mathbb{E}}_\chi) = (1/2)c_1(\tilde{\mathbb{E}}_\chi)^2 - c_2(\tilde{\mathbb{E}}_\chi)$. The Chern character and the $\hat{A}$ -genus, the usual polynomial related to Riemannian curvature Ω , are given by

$$(39) \quad \begin{aligned} \text{ch}(\tilde{\mathbb{E}}_\chi) &= \text{rank } \tilde{\mathbb{E}}_\chi + c_1(\tilde{\mathbb{E}}_\chi) + \text{ch}_2(\tilde{\mathbb{E}}_\chi) = \dim \chi + c_1(\tilde{\mathbb{E}}_\chi) + \text{ch}_2(\tilde{\mathbb{E}}_\chi), \\ \hat{A}(\Omega^M) &= 1 - \frac{1}{24} p_1(\Omega^M). \end{aligned}$$

Here $p_1(\Omega^M)$ is the first Pontriagin class, Ω^M is the Riemannian curvature of M . Thus we have

$$(40) \quad \begin{aligned} \text{ch}(\tilde{\mathbb{E}}_\chi) \hat{A}(\Omega^M) &= \left(\dim \chi + c_1(\tilde{\mathbb{E}}_\chi) + \text{ch}_2(\tilde{\mathbb{E}}_\chi) \right) \left(1 - \frac{1}{24} p_1(\Omega^M) \right) \\ &= \dim \chi + c_1(\tilde{\mathbb{E}}_\chi) + \text{ch}_2(\tilde{\mathbb{E}}_\chi) - \frac{\dim \chi}{24} p_1(\Omega^M). \end{aligned}$$

The integral over the four manifold M takes the form

$$(41) \quad \begin{aligned} \int_M \text{ch}(\tilde{\mathbb{E}}_\chi) \hat{A}(\Omega^M) &= \int_M \text{ch}_2(\tilde{\mathbb{E}}_\chi) - \frac{\dim \chi}{24} \int_M p_1(\Omega^M) \\ &= -\frac{1}{8\pi^2} \int_M \text{Tr} \left(F_{\tilde{A}_\chi} \wedge F_{\tilde{A}_\chi} \right) - \frac{\dim \chi}{24} \int_M p_1(\Omega^M), \end{aligned}$$

and we have

$$(42) \quad \begin{aligned} \text{Index} \left(D_{\tilde{A}_\chi} \right) &= -\frac{1}{8\pi^2} \int_M \text{Tr} \left(F_{\tilde{A}_\chi} \wedge F_{\tilde{A}_\chi} \right) \\ &\quad - \frac{\dim \chi}{24} \int_M p_1(\Omega^M) - \frac{1}{2} (\eta(0, \mathfrak{D}_\chi) + h(0, \mathfrak{D}_\chi)). \end{aligned}$$

For a trivial representation χ_0 one can take a trivial flat connection $\tilde{A} \equiv \tilde{A}_{\chi_0}$; then $F_{\tilde{A}_{\chi_0}} = 0$ and for this choice we get

$$(43) \quad \text{Index} \left(D_{\tilde{A}_{\chi_0}} \right) = -\frac{1}{24} \int_M p_1(\Omega^M) - \frac{1}{2} (\eta(0, \mathfrak{D}) + h(0, \mathfrak{D})).$$

Using Eqs. (42) and (43) one can obtain

$$(44) \quad \begin{aligned} \text{Index} \left(D_{\tilde{A}_\chi} \right) - \dim \chi \text{Index} \left(D_{\tilde{A}_{\chi_0}} \right) &= -\frac{1}{8\pi^2} \int_M \text{Tr} \left(F_{\tilde{A}_\chi} \wedge F_{\tilde{A}_\chi} \right) \\ &- \frac{1}{2} (\eta(0, \mathfrak{D}_\chi) - \dim \chi \eta(0, \mathfrak{D})) \quad \text{modulo}(\mathbb{Z}/2), \end{aligned}$$

and

$$(45) \quad CS_{U(n)}(A_\chi) \equiv \frac{1}{2} (\dim \chi \eta(0, \mathfrak{D}) - \eta(0, \mathfrak{D}_\chi)) \quad \text{modulo}(\mathbb{Z}/2).$$

Now we are able to use the formulas (29) and (31) for the eta invariant in Eq. (45) and get the final result (37). $\square$

The classical factor of the complex invariant becomes

$$(46) \quad e^{4\pi\sqrt{-1}CS_{U(n)}(A_\chi)} = \left[\frac{Z(0, \mathfrak{D})^{\dim \chi}}{Z(0, \mathfrak{D}_\chi)} \right].$$

The Witten's invariant defined by the partition function associated with a Chern-Simons gauge theory has the form $W(k) = \int [DA] e^{\sqrt{-1}kCS(A)}$, $k \in \mathbb{Z}$. The function $W(k)$ can be evaluated in terms of L^2 -analytic torsion and a Selberg type function. The result is [18]:

$$(47) \quad W(k) = \left(\frac{\pi}{k} \right)^{\zeta(0, |\mathfrak{D}|)/2} Z(0, \mathfrak{D})^{-1/4} [\tau_{an}(X_\Gamma)]^{1/2} [\text{Vol}(\Gamma \backslash G)]^{-\dim H^0(\nabla)/2}.$$

The analytic torsion for manifold X_Γ has been calculated (in the presence of non-vanishing Betti numbers $b_i \equiv b_i(X_\Gamma)$ in [17, 18], and it is given by

$$(48) \quad [\tau_{an}(X_\Gamma)]^2 = \frac{(b_1 - b_0)! [Z_{\Gamma,0}(2, b_0)]^2}{[b_0!]^2 Z_{\Gamma,1}(1, b_1 - b_0)} \exp \left(-\frac{1}{3\pi} \text{Vol}(\Gamma \backslash G) \right).$$

Remark. Going back to the problem of uniformization, we note here the particular task of classification of manifolds of a given class. A solution of this task lies in producing of two algorithms. First of them, the numeration algorithm, must enumerate, possibly with repetitions, the manifolds of a given class. The second one, the recognition algorithm, applied to any two given manifolds of the class in question, must tell us whether or not they are homeomorphic. The joint action of these algorithms leads to the appearance of an infinite list which contains all manifolds of the given class without duplication, and therefore the manifolds can be described algorithmically. Such a solution of the classification problem can be regarded as satisfactory only up to a first approximation (in a weak sense), usually mathematicians require more.

In general, hyperbolic manifolds have not been completely classified and therefore a systematic computation is not yet possible. However it is not the case for sufficiently large manifolds [33], which give an essential contribution to the torsion (48). There is a class of three-dimensional sufficiently large hyperbolic manifolds which admits arbitrary large value of

$b_1(M) = \text{rank}_{\mathbb{Z}} H_1(M; \mathbb{Z})$. Sufficiently large manifold contains a surface Σ whereas $\pi_1(\Sigma)$ is finite and $\pi_1(\Sigma) \subset \pi_1(M)$. It is known that any three-manifold can be triangulated, and hence can be partitioned into handles. Since the existence and uniqueness of a decomposition of an orientable manifold as the sum of simple orientable parts have been established (see [50, 36]), the question of homeomorphy can be considered only for irreducible manifolds. The method proposed by Haken [33] permits to describe all normal surfaces of a three-manifold M which has been partitioned on handles previously. The Haken's theory of normal surfaces was further verified for the procedure of geometric summation of surfaces. As a result, the classification theorem says:

Theorem 5. (*S. V. Matveev [45], the main theorem*).

- *There exists an algorithm for enumerating all of the Haken manifolds.*
- *There exists an algorithm for recognizing homeomorphy of the Haken manifolds.*

4.2. The determinant line bundle. The exponentiated eta invariant (46) naturally takes values in the determinant line of the boundary. The differential geometry of determinant line bundles has been developed in [59] in a special case and in [11, 12] in general. In papers [21, 29] the results on eta invariant were used to reprove the holonomy formula for determinant line bundles, known as Witten's global anomaly formula [72].

We begin with some conventions which apply throughout. As before, results presented in this Section hold for any Dirac operator on a $\text{Spin}^{\mathbb{C}}$ manifold coupled to a vector bundle with connection (manifolds are assumed spin). On a closed odd dimensional manifold M the Dirac operator $\mathfrak{D}$ is self-adjoint and has discrete spectrum $\text{Spec}(\mathfrak{D})$. At $s = 0$ the eta function $\eta(s, \mathfrak{D})$ is regular and we set

$$(49) \quad \tau_M = \exp(\pi\sqrt{-1}(\eta(0, \mathfrak{D}) + \dim \text{Ker } \mathfrak{D})) \in \mathbb{C}.$$

The eta invariant is discontinuous in general but the general theory shows that τ_M varies smoothly in families ($|\tau_M| = 1$). The invariant (49) is defined and we have $\tau_M \in \text{Det}_{\partial M}^{-1}$, where $\text{Det}_{\partial M}$ is the determinant line of the Dirac operator $\mathfrak{D}$ on the boundary: $\text{Det}_{\partial M} = (\text{Det Ker}^- \mathfrak{D})_{\partial M} \otimes (\text{Det Ker}^+ \mathfrak{D})_{\partial M}^{-1}$.

Let a family of Riemannian manifolds be a smooth fiber bundle: $M \rightarrow W$ together with a metric on the relative tangent bundle $T(M/W)$ which endowed with spin structure. Following notations of [11, 21] we denote such a fiber as M/W . Also we assume that the Riemannian metrics on the fibers are products near boundary. The determinant lines carry the Quillen metric and a canonical connection ∇ [11] and the exponentiated eta invariant is a smooth section $\tau_{M/W} : W \rightarrow \text{Det}_{\partial M/W}^{-1}$. We mention here two basic results on this invariant, namely variation and curvature formulas.

Theorem 6. (*X. Dai and D. S. Freed [21], Theorem 1.9*). *The covariant derivative of the exponentiated eta invariant is*

$$(50) \quad \nabla \tau_{M/W} = 2\pi\sqrt{-1} \left[\int_{M/W} \widehat{A}(\Omega^{M/W}) \right]_{(1)} \cdot \tau_{M/W},$$

where $\Omega^{M/W}$ is the Riemannian curvature of $M \rightarrow W$, and simbol (1) denotes the one-form piece of the differential form.

Note that for a family of closed manifolds this is a result of Atiyah-Patodi-Singer (see formula (36) of Sec. 3.1). Let the fibers M/W be closed, and let $E \rightarrow W$ be a real vector bundle. Then the determinant line bundle $\text{Det}_{M/W} \mathfrak{D}(E)$ is well-defined as a smooth line bundle (it carries a canonical metric and connection) [11]. The complex Dirac operator for the odd dimensional fibers M/W is self-adjoint and there is a geometrical invariant $\xi_{M/W}(E) : W \rightarrow \mathbb{R}/\mathbb{Z}$ defined by Atiyah-Patodi-Singer, $2\xi_{M/W} = (\eta(0, \mathfrak{D}) + \dim \text{Ker } \mathfrak{D})_{M/W}$. We have $\tau_{M/W}(E) = \exp(2\pi\sqrt{-1}\xi_{M/W}) : Z \rightarrow U \subset \mathbb{C}$.

Theorem 7. (*J. M. Bismut and D. S. Freed [12], Theorem 1.21*). The two-form curvature of the determinant line bundle is

$$(51) \quad \Omega^{\text{Det}_{M/W} \mathfrak{D}(E)} = \left[2\pi\sqrt{-1} \int_{M/W} \widehat{A}(\Omega^{M/W}) \text{ch}(\Omega^E) \right]_{(2)} \in \Omega^2(W),$$

where $\Omega^{M/W}$ and Ω^E are the curvature forms.

Let $M \rightarrow \mathbb{S}^1$ be a loop of manifolds in this geometric setup. A metric and spin structure on M could be induced by a metric and boundary spin structure on $\mathbb{S}^1$. The holonomy of the determinant line bundles around the loop takes the form

$$(52) \quad \text{hol Det } \mathfrak{D}^{M/\mathbb{S}^1}(E) = \text{a-lim } \tau_M^{-1}(E),$$

where a-lim is the adiabatic limit, i.e. the limit as the metric on $\mathbb{S}^1$ blows up ($\varepsilon \rightarrow 0 : g_{\mathbb{S}^1}^1 \rightarrow g_{\mathbb{S}^1}^1 \varepsilon^{-2}$). For the flat determinant line bundles no adiabatic limit is required. As we pointed out, Eq. (52) is the global anomaly formula [71, 11].

4.3. Vanishing theorems for type $(0, q)$ cohomology. Note that it is difficult to compute a topological invariant $\xi_{X/Z}(\mathbb{E})$ directly unless the metric has some special symmetry. In Section 4.1 we derived the Chern-Simons topological invariant related to real compact hyperbolic spaces (see also [16, 19]). For a suitable geometry, the semiclassical approximation for the Chern-Simons partition function leads to a series of C^∞ -invariants, associated with the triplets $\{M; F; \chi\}$ [7], where M is a smooth homology three-sphere, F a homology class of framings of M , and χ an acyclic conjugacy class of orthogonal representations of the fundamental group $\pi_1(M)$. In addition, the cohomology $H(M; \text{Ad } \chi)$ of M with respect to the local system related to $\text{Ad } \chi$ vanishes.

In this Section we review vanishing theorems for type $(0, q)$ cohomology of locally symmetric spaces following the lines of paper [70]. If χ is acyclic, i.e. the vector space $H^q(M; \chi)$ of twisted cohomology is zero, then the Ray-Singer norm (18) is topological invariant: It does not depend on the choice of metric on M and χ , used in the construction.

Vanishing theorems for cohomologies associated with locally symmetric spaces can also be formulated for a more general case of complex manifold.² Indeed let the quotient G/K admit a G -invariant complex structure. We assume that the complexification $G^{\mathbb{C}}$ of G is simply-connected. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ denote the complexification of the (real) Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{h}_0$ of G, K, T , respectively, where a Cartan subgroup T of G we choose such that $T \subset K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. Let $\mathfrak{p}$ denote the complexification of $\mathfrak{p}_0$, let Δ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, and Δ_n (Δ_k) denote the set of non-compact (compact) roots. Therefore, $\alpha \in \Delta$ is in Δ_n (or Δ_k) if and only if the corresponding (one-dimensional) root space $\mathfrak{g}_\alpha \subset \mathfrak{p}$ (or $\mathfrak{g}_\alpha \subset \mathfrak{k}$). If $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is a splitting of the complex tangent space at the origin in G/K into holomorphic and anti-holomorphic tangent vector $\mathfrak{p}^+, \mathfrak{p}^-$ respectively, then for a system Δ^+ of positive roots compatible with the complex structure on G/K we have

$$(53) \quad \mathfrak{p}^\pm = \sum_{\alpha \in \Delta^+ \cap \Delta_n} \mathfrak{g}_{\pm\alpha}.$$

Let us fix a discrete subgroup Γ of G such that Γ acts freely on G/K and such that the quotient $\mathbb{X}_\Gamma = \Gamma \backslash G/K$ is compact locally symmetric Hermitian domain. For any finite-dimensional irreducible representation τ of K on a complex vector space, one has associated to τ a sheaf $\theta_\tau \rightarrow \mathbb{X}_\Gamma$ over $\mathbb{X}_\Gamma$. Indeed, let $\mathbb{E}_\tau \rightarrow G/K$ be the induced homogeneous C^∞ vector bundle over G/K associated to the principal C^∞ fibration $K \rightarrow G \rightarrow G/K$. As is known, $\mathbb{E}_\tau$ has a holomorphic structure and we obtain a presheaf by assigning to each open set U in $\mathbb{X}_\Gamma$ the abelian group of Γ -invariant holomorphic sections of $\mathbb{E}_\tau$ on the inverse image $\tilde{U}$ of U in G/K . Let $H^q(\mathbb{X}_\Gamma; \theta_\tau)$ be the q -th cohomology space of $\mathbb{X}_\Gamma$ with coefficients in θ_τ (θ_τ is the sheaf generated by the presheaf).

The main result obtained by Williams in the paper [70] is Theorem 2.3 (in the proof of the Theorem the application of Parthasarathy's unitarizability criteria for highest weight modules [56] has been used). This result governing the vanishing of the space $H^q(\mathbb{X}; \theta_\tau)$ for τ whose highest weight Λ relative to $\Delta_k^+ = \Delta_k \cap \Delta^+$ belongs to the set

$$\mathfrak{F}_0 = \{\Lambda \in \mathfrak{h}^* | \Lambda \text{ is integral},$$

$$(54) \quad (\Lambda + \delta, \alpha) \neq 0 \text{ for all } \alpha \in \Delta, \quad (\Lambda + \delta, \alpha) > 0 \text{ for all } \alpha \in \Delta_k^+ \}.$$

²If M is a complex manifold (or smooth manifold (C^∞ -manifold), or topological space) then $\mathbb{E} \rightarrow M$ is the induced complex (or smooth, or continuous) vector bundles. We write $H^{p,q}(M; \mathbb{E}) \simeq H^{0,q}(M; \Lambda^{p,0} M \otimes \mathbb{E})$ holonomic vector bundles $\Lambda^{p,0} M \rightarrow M$ (see [70] for detail).

The system of positive roots defined by the regular element $\Lambda + \delta$, $\delta = (1/2) \sum_{\alpha \in \Delta^+} \alpha$, is also compatible with a G -invariant complex structure on G/K , $(,)$ denotes the Killing form and the integrality of Λ means that $2(\Lambda, \alpha)/(\alpha, \alpha)$ is an integer for every α in Δ .

Starting from a general result, let us give some notations. Let τ_Λ be the finite-dimensional irreducible representation of K on a complex vector space V with highest weight Λ , relative to Δ_k^+ . Since $\Lambda + \delta$ is regular, a system of positive roots is

$$(55) \quad P^{(\Lambda)} = \{\alpha \in \Lambda \mid (\Lambda + \delta, \alpha) > 0\}.$$

We assume that every non-compact root α in $P^{(\Lambda)}$ is totally positive; i.e. $\alpha + \beta \in P_n^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_n$ for every $\beta \in \Delta_k$ such that $\alpha + \beta \in \Delta$. There exists a G -invariant complex structure on G/K such that the space of holomorphic tangent vectors at the origin is given by $\sum_{\alpha \in P_n^{(\Lambda)}} \mathfrak{g}_\alpha$. We shall write $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$ if $Q \subset \Delta$. Let $P_k^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_k$ and like in [70] let

$$(56) \quad \begin{aligned} Q_\Lambda &= \{\alpha \in \Delta_n^+ = \Delta^+ \cap \Delta_n \mid (\Lambda + \delta, \alpha) > 0\}, \\ \tilde{Q}_\Lambda &= \Delta_n^+ - Q_\Lambda, \quad 2\delta^{(\Lambda)} = \langle P^{(\Lambda)} \rangle, \\ 2\delta_n^{(\Lambda)} &= \langle P_n^{(\Lambda)} \rangle, \quad 2\delta_k^{(\Lambda)} = \langle P_k^{(\Lambda)} \rangle. \end{aligned}$$

Finally let $|S|$ denote the cardinality of a set S . We are ready to formulate a general result:

Theorem 8. (*F. L. Williams [70], Theorem 2.3*). *Let $\Lambda \in \mathfrak{F}_0$ as above and assume that every non-compact root in $P^{(\Lambda)}$ is totally positive. Suppose $H^q(\Gamma \backslash G/K; \theta_{\tau_\Lambda}) \neq 0$. Then there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha$ with $\mathfrak{u}$ = the nilpotent radical of θ and $\mathfrak{m}$ = the reductive part of θ such that*

- If $\theta_{u,n}$ is the set of non-compact roots in $\mathfrak{u}$ then
 $q = 2|\theta_{u,n} \cap Q_\Lambda| + |\tilde{Q}_\Lambda| - |\theta_{u,n}|.$
- $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every root α in $\mathfrak{m}$. In particular, if $A_\Lambda = \{\alpha \in Q_\Lambda \cup -\tilde{Q}_\Lambda \mid (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\}$, then $|A_\Lambda| \leq |\theta_{u,n}| = 2|\theta_{u,n} \cap Q_\Lambda| + |\tilde{Q}_\Lambda| - q.$

Let us consider some applications of Theorem 2.3. for the classical results on the vanishing of $H^q(\Gamma \backslash G/K; \theta_{\tau_\Lambda})$. We consider the two extreme cases of $\Delta \in \mathfrak{F}_0$:

- (i) $(\Lambda + \delta, \alpha) > 0$ for every α in Δ_n^+ (i.e. Λ is Δ^+ dominant) and
- (ii) $(\Lambda + \delta, \alpha) < 0$ for every α in Δ_n^+ .

In both cases every noncompact root in $P^{(\Lambda)}$ is totally positive ($P^{(\Lambda)} = \Delta^+$ and $P^{(\Lambda)} = \Delta_k^+ \cup -\Delta_n^+$ in cases (i) and (ii) respectively) and therefore Theorem 2.3. is applicable. Suppose now that the Hermitian symmetric space G/K is irreducible (i.e. G is a simple Lie group).

If G is simple and Λ is Δ^+ dominant integral, then $H^q(\Gamma \backslash G/K; \theta_{\tau\Lambda})$ vanishes unless q belongs to the set $\{|\theta_{n,m}||\theta \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha\}$ corresponding to G . Let G is simple and Λ is integral, Δ_k^+ dominant, and satisfies $(\Lambda + \delta, \alpha) < 0$ for every α in Δ_n^+ . Then $H^q(\Gamma \backslash G/K; \theta_{\tau\Lambda}) = 0$ unless $n - q$ belongs to the set $\{|\theta_{u,n}||\theta \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha\}$ corresponding to G . Here $n = |\Delta_n^+| = \dim_{\mathbb{C}} G/K$.

We refer the reader to the original paper [70] where more examples of the vanishing $(0, q)$ cohomology of locally symmetric spaces can be found.

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