QUANTUM FLUCATIONS IN EARLY COSMOLOGY

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ABSTRACT. We study the quantum cosmology of a stringlike toy model. We show that fluctuations of the third-quantized universe field become exponentially large as cosmic scale factor approaches zero, and in the course of cosmic expansion quantum fluctuations rapidly decrease.

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1. INTRODUCTION

In the last years, string theory has been seen, as the main consistent candidate for a fundamental theory of gravity [5]. This is based on the assumption that the fundamental objects that describe matter and its interactions are not point-like but one-dimensional.

It is widely thought that quantum gravitational effects become important on scales of the order of the Planck length. Therefore, quantum cosmology is the natural framework which may answer questions related to the initial states of the universe, and one approach to the problem is based on the WDW equation[2] $\hat{H}\psi = 0$. However, since the WDW equation is a second-order hyperbolic differential equation, there is a problem in the interpretation of $\mid \psi \mid^2$ as a probability, because the conserved current associated with it is no necessarily semi-positive definite. An alternative approach to quantum cosmology is the third quantization in analogy with the second quantization of the Klein-Gordon equation[2].

In this paper we study the feature of quantum fluctuations. This problem can be addressed by analyzing the tensor metric perturbations[4] or studying the behavior of fluctuations around the third quantized solutions to the WDW equation[8, 7]. We adopt the second procedure to study a solvable toy string model by means

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of WDW functional equation. The paper is organized as follows. In Sec. II we describe the canonical formalism of our minisuperspace model and we obtain the WDW equation. In Sec. III we study the quantum fluctuations of the universe field by means of the uncertainty relation, and Sec. IV consists of final remarks.

2. The minisuperspace model

We start from the four-dimensional low energy effective action of string theory

$$S=-\frac{1}{\lambda^2}\int_M d^4x\sqrt{-g}\,e^{-\phi}\left[R+\phi^{;\alpha}\phi_{;\alpha}+V(\phi)\right]-\frac{2}{\lambda^2}\int_{\partial M} d^3x\sqrt{h}e^{-\phi}h_{ij}K^{ij}, \tag{1}$$

where $g=det(g_{\mu\nu})$, R is the scalar curvature, λ is the fundamental string length parameter and $V(\phi)$ is a dilaton potential. The scalar curvature R includes the second derivatives of the metric $g_{\mu\nu}$, and since in the canonical quantization formalism, the dynamical piece of the metric is the induced three-dimensional metric $h_{ij}\,(i,j=1,2,3)$ on the boundary of the manifold M over which integration in (1) is performed, we can get rid of these second derivatives terms, by means the second integral in (1) which is a surface term where, h is the determinant of the induced metric and K^{ij} is the second fundamental form of the boundary [2]. In order to study isotropic and homogeneous quantum cosmology, we adopt the Friedmann-Robertson-Walker flat model for the universe, then the total action (1) becomes

$$S=\frac{6V}{\lambda^2}\int dt\,e^{-\phi}\left[a^2\frac{\dot{a}\dot{\phi}}{N}-a\frac{\dot{a}^2}{N}-\frac{a^3\dot{\phi}^2}{6N}-\frac{Na^3}{6}V(\phi)\right], \tag{2}$$

here a is the scale factor of the universe, and the lapse function $N(t)$represents the general time-coordinate transformation freedom. We have introduced the spatial volume V of the homogeneous region we consider, this must be some properly fixed finite constant. The dot denotes time derivative with respect to the cosmic time t. Now, in order to eliminate the term $a^2\dot{a}\dot{\phi}/N$ in action(2), we introduce the independent variable $\beta=a^2\,e^{-\phi}$, then action (2) becomes

$$S=\frac{6V}{\lambda^2}\int dt\,e^{\frac{\phi}{2}}\left[\frac{1}{12N}\beta^{\frac{3}{2}}\dot{\phi}^2-\frac{1}{4N}\beta^{-\frac{1}{2}}\dot{\beta}^2-\frac{N}{6}\beta^{\frac{3}{2}}V(\phi)\right]. \tag{3}$$

The canonical conjugate momenta corresponding to β,ϕ and N are

$$\Pi_\beta=-\frac{3V}{\lambda^2N}\beta^{-\frac{1}{2}}e^{\frac{\phi}{2}}\dot{\beta}\quad,\quad\Pi_\phi=\frac{V}{\lambda^2N}\beta^{\frac{3}{2}}e^{\frac{\phi}{2}}\dot{\phi}\quad,\quad\Pi_N=0. \tag{4}$$

The last equation $\Pi_N=0$ is a primary constraint. The total Hamiltonian of the system is constructed as

$$H_T=\Pi_N\dot{N}+NH, \tag{5}$$

where

$$\begin{aligned} H &= \dot{\beta}\Pi_\beta+\dot{\phi}\Pi_\phi-L,\\ &= \frac{N\lambda^2}{6V}e^{-\frac{\phi}{2}}\left[-\beta^{\frac{1}{2}}\Pi_\beta^2+3\beta^{-\frac{3}{2}}\Pi_\phi^2+\frac{V}{\lambda^2}\beta^{\frac{3}{2}}e^{\phi}V(\phi)\right]. \end{aligned} \tag{6}$$

Now, in order to make quantum theory of the universe (following the Dirac quantization), we introduce the wave function of the universe ψ. The constraint equations must be imposed as restrictions on the states

$$\Pi_N\psi=0\quad,\quad H\psi=0\quad, \tag{7}$$

from first equation we conclude that wave function ψ does not depend on lapse function N. The second equation is the WDW equation, which can be obtained by means of canonical quantization of $H = 0$ (secondary constraint), i.e., the momenta in (4) are converted into operators, $\Pi_\beta^2 \to -\frac{\partial^2}{\partial\beta^2}$ and $\Pi_\phi^2 \to -\frac{\partial^2}{\partial\phi^2}$. Therefore, the WDW equation is

$$\left[\beta^2\frac{\partial^2}{\partial\beta^2} - 3\frac{\partial^2}{\partial\phi^2} + \frac{6V^2}{\lambda^4}\beta^3 e^{\phi}V(\phi)\right]\psi(\beta,\phi) = 0. \tag{8}$$

3. Quantum fluctuations

In this section we will calculate the Heisenberg uncertainty relation in order to study the fluctuations of the third quantized universe at early times ($a(t) \to 0$), and also at latter times ($a(t) \to \infty$). Our starting point is the WDW equation (8). We use the Schrödinger picture in the field representation. Thus we consider the operator $\psi(\gamma,\phi)$ as the time independent c-number field $\psi(\phi)$ and use a functional derivative to represent the momentum operator

$$\Pi(\phi) = -i\frac{\delta}{\delta\psi(\phi)}. \tag{9}$$

Then, the Hamiltonian operator is given by

$$H = \frac{1}{2}\int d\phi\left[-\frac{\delta^2}{\delta\psi(\phi)^2} + 3\frac{\partial^2\psi}{\partial\phi^2} - \frac{6V_0V^2}{\lambda^4}\beta^3\right], \tag{10}$$

we take β as time and ϕ as space, then the amplitude to find an instantaneous field configuration $\psi(\phi)$ at time β on the corresponding space-like hypersurface $\beta = const.$ in superspace, is obtained by solving the Schrödinger functional equation

$$i\frac{\partial\Psi[\psi,\beta]}{\partial\beta} = \Psi[\psi,\beta]. \tag{11}$$

In order to get coherent states of the universe we will solve the Eq. (11) by using the Gaussian ansatz[6]

$$\begin{aligned}\Psi[\beta,\phi] = C\exp\Big\{ & - \frac{1}{2}\int d\phi\, d\phi'[D(\beta,\phi,\phi') + i\, I(\beta,\phi,\phi')] \\ & [\psi(\phi)\eta(\beta,\phi)][\psi(\phi') - \eta(\beta,\phi')] \\ & + i\int d\phi\, P(\beta,\phi)[\psi(\phi) - \eta(\beta,\phi)] \Big\},\end{aligned} \tag{12}$$

the real functions $D(\beta,\phi',\phi)$, $I(\beta,\phi',\phi)$, $P(\beta,\phi)$, and $\eta(\beta,\phi)$ have to be determined from Eq. (11), C is a normalization of the wave function. We can take $D(\beta,\phi,\phi') = D(\beta,\phi-\phi')$ and $I(\beta,\phi,\phi') = I(\beta,\phi-\phi')$, since there is translation invariance with respect to ϕ. Then, to simplify the analysis, we employ the "momentum representation" for the operators

$$\psi(\phi) = \frac{1}{\sqrt{2\pi}}\int dp\, e^{-ip\phi}\psi(p), \tag{13}$$

$$\frac{\delta}{\delta\psi(\phi)} = \frac{1}{\sqrt{2\pi}}\int dp\, e^{ip\phi}\frac{\delta}{\delta\psi(p)}, \tag{14}$$

$$D(\beta,\phi,\phi') = \frac{1}{2\pi}\int dp\, e^{-ip(\phi-\phi')}D(\beta,p), \tag{15}$$

(16) $$I(\beta,\phi,\phi') = \frac{1}{2\pi}\int dp\, e^{-ip(\phi-\phi')} I(\beta,p).$$

The inner product of two functionals $\psi_1[\psi,\beta]$ and $\psi_2[\psi,\beta]$ is defined by

(17) $$<\Psi_1 \mid \Psi_2>_\beta = \int d\psi \Psi_1[\psi,\beta]\Psi_2^*[\psi,\beta].$$

We shall calculate Heiserberg's uncertainty relation. From equation (17) and equation (12) , the dispersion of ψ is given by

(18) $$\begin{aligned}(\Delta\psi)^2 &\equiv <\psi^2>_\beta - <\psi>_\beta^2, \\ &= \frac{1}{2D(\beta,p)},\end{aligned}$$

and the dispersion of Π is

(19) $$\begin{aligned}(\Delta\Pi)^2 &\equiv <\Pi^2>_\beta - <\Pi>_\beta^2, \\ &= \frac{D(\beta,p)}{2} + \frac{I^2(\beta,p)}{2D(\beta,p)},\end{aligned}$$

thus, the Heisenberg's uncertainty relation is

(20) $$(\Delta\psi)^2(\Delta\Pi)^2 = \frac{1}{4}\left\{1 + \left[\frac{I(\beta,p)}{D(\beta,p)}\right]^2\right\}.$$

Next we have to evaluate the right-hand side of this equation. In order to obtain $I(\beta,p)$ and $D(\beta,p)$, we substitute the ansatz (12) into equation (11), with the relation

(21) $$A(\beta,p) = D(\beta,p) + iI(\beta,p),$$

then we obtain the equation of motion for $A(\beta,p)$, given by

(22) $$-i\frac{dA}{d\beta} = -A^2 + 3p^2 + \frac{6V_0V^2}{\lambda^4}\beta.$$

We can express $A(\beta,p)$ as

(23) $$A = -i\frac{d}{d\beta}\ln u,$$

where u is any solution of the Fourier transformed classical equation (8) with respect to u

(24) $$\left[\frac{\partial^2}{\partial\beta^2} + 3\frac{p^2}{\beta^2} + \frac{6V_0V^2}{\lambda^4}\beta\right]u(\beta,p) = 0.$$

First, we analyze the behavior of u for large scales ($a \to \infty$), then the second term $3p^2/\beta^2$ in the wave equation (24) can be neglected, and its solution in terms of Airy functions is

(25) $$u(\beta,p) = \mathrm{Ai}(z) + \omega\mathrm{Bi}(z),$$

where $z = -\left(\frac{6V_0V^2}{\lambda^4}\right)^{\frac{1}{3}}\beta$, and ω is a complex constant. Substituting (25) into (23), we obtain that

(26) $$A(\beta,p) = i\left(\frac{6V_0V^2}{\lambda^4}\right)^{\frac{1}{3}}\frac{\mathrm{Ai}'(z) + \omega\mathrm{Bi}'(z)}{\mathrm{Ai}(z) + \omega\mathrm{Bi}(z)},$$

where the prime denotes differentiation with respect to β, also we have

$$D(\beta,p) = \mathrm{Re}A(\beta,p) = \left(\frac{6V_0V^2}{\lambda^4}\right)^{\frac{1}{3}} \frac{\mathrm{Im}\,\omega}{\pi \mid \mathrm{Ai}(z)+\omega\mathrm{Bi}(z)\mid^2}, \tag{27}$$

and

$$I(\beta,p) = -\mathrm{Im}A(\beta,p) = \left(\frac{6V_0V^2}{\lambda^4}\right)^{\frac{1}{3}} \frac{\mathrm{Ai}'\mathrm{Ai}+\mid\omega\mid^2\mathrm{Bi}'\mathrm{Bi}+(\mathrm{Re}\,\omega)(\mathrm{Bi}'\mathrm{Ai}+\mathrm{Ai}'\mathrm{Bi})}{\mid \mathrm{Ai}+\omega\mathrm{Bi}\mid^2}, \tag{28}$$

therefore the uncertainty relation becomes

$$(\Delta\psi)^2(\Delta\Pi)^2 = \frac{1}{4}\left\{1+\frac{\pi^2}{(\mathrm{Im}\omega)^2}\left[\mathrm{Ai}'\mathrm{Ai}+\mid\omega\mid^2\mathrm{Bi}'\mathrm{Bi}+(\mathrm{Re}\,\omega)(\mathrm{Bi}'\mathrm{Ai}+\mathrm{Ai}'\mathrm{Bi})\right]^2\right\}. \tag{29}$$

For large scales ($a \to \infty$), the above uncertainty relation is given by

$$\begin{aligned}(\Delta\psi)^2(\Delta\Pi)^2 &\simeq \frac{1}{4}\left\{1+\frac{\pi^2}{(\mathrm{Im}\,\omega)^2}\left[(\mid\omega\mid^2-1)\sin\left(2z+\frac{\pi}{2}\right)\right.\right.\\ &\left.\left. - 2(\mathrm{Re}\,\omega)\cos\left(2z+\frac{\pi}{2}\right)\right]^2\right\},\end{aligned} \tag{30}$$

for $\omega = \pm i$, we get that

$$(\Delta\psi)^2(\Delta\Pi)^2 = \frac{1}{4}, \tag{31}$$

This means that the quantum fluctuations of the third-quantized string universe field are bounded at a finite value during the universe expansion $a \to \infty$. On the other hand, if we choose $\omega \neq \pm i$, we have that the expression among brackets of equation (30) is oscillatory and therefore the uncertainty relation is bounded.

Now we study the behavior of u for small scales $a \to 0$. The solution to equation (24) for flat model (where we neglected the term $6V_0V^2\beta^3/\lambda^4$ associated to the potential) is

$$u(\beta,p) = \sqrt{\beta}\left[J_\nu\left(\frac{2\sqrt{3}V}{\lambda^2}\beta\right)+\delta Y_\nu\left(\frac{2\sqrt{3}V}{\lambda^2}\beta\right)\right], \tag{32}$$

where J_ν and Y_ν are modified Bessel functions, δ is a complex constant, and $\nu = \frac{1}{2}\sqrt{1+12p^2}$. Substituting the general solution (32) into equation (23), we obtain

$$D = \mathrm{Re}A = \frac{4\sqrt{3}V\,\mathrm{Im}\,\delta}{\pi\lambda^2\beta\mid J_\nu+\delta Y_\nu\mid^2}, \tag{33}$$

and

$$\begin{aligned}(34) &= \mathrm{Im}A\\ &= -\frac{(2\beta)^{-1}\mid J_\nu+\delta Y_\nu\mid^2+\frac{2\sqrt{3}V}{\lambda^2}[Y_\nu'Y_\nu+\mid\delta\mid^2 J_\nu'J_\nu+\mathrm{Re}\,\delta(J_\nu'Y_\nu+Y_\nu'J_\nu)]}{\mid J_\nu+\delta Y_\nu\mid^2},\end{aligned}$$

the prime denotes differentiation with respect to β. Then, the uncertainty relation is

$$(\Delta\psi)^2(\Delta\Pi)^2 = \frac{1}{4}\left\{1+\Omega^2\right\}, \tag{35}$$

where

(36)
$$\Omega = \frac{\pi\beta}{2\,\mathrm{Im}\,\delta}\left[\frac{\lambda^2}{4\sqrt{3}V\beta}\left(J_\nu+\delta Y_\nu\right)^2 + Y_\nu' Y_\nu + \mid\delta\mid^2 J_\nu' J_\nu + \mathrm{Re}\,\delta(J_\nu' Y_\nu + Y_\nu' J_\nu)\right],$$

for small scales ($a \to 0$) and $\nu^2 > 0$, the dominant asymptotic behavior[1] of equation (35) is

$$(\Delta\psi_\alpha)^2(\Delta\Pi_\alpha)^2 = \frac{1}{4}\left\{1+\Theta\beta^{-4|\nu|}\right\}, \tag{37}$$

where Θ is some positive constant. This means that the fluctuation of the third-quantized universe field becomes large for small scales $a \to 0$.

4. Final remarks

In this paper we have investigated some issues of quantum cosmology of a solvable minisuperspace model in string theory. The result is that, for a string flat toy model with dilaton potential $V(\phi) = V_0 e^{-\phi}$, through the Heisenberg's uncertainty relation we found that fluctuations of the third-quantized universe field rapidly decrease in the course of cosmic expansion. Also, we found an exponentially large dominance of quantum effects for small scales. These results shown the importance of the third quantization theory in the study of the earliest stage of the evolution of the universe.

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