

PERTURBATION OF CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS

A. FAKHARI AND F.H. GHANE
DEPARTMENT OF MATHEMATICS
FERDOWSI UNIVERSITY OF MASHHAD
P.O.BOX 1159-91775, MASHHAD, IRAN
E-MAIL: GHANE@MATH.UM.AC.IR

(Received 5 November 2004)

ABSTRACT. New results on perturbation of continuum-expansive homeomorphisms on locally connected compact metric spaces are deduced. For a continuum-wise expansive homeomorphism $f : X \rightarrow X$, it is not true that: any small perturbation g of it, has continuum-wise expansive property. Here, for small perturbation g of f , we define an equivalence relation $\sim$ on X and a homeomorphism $\tilde{g}$ on the quotient space $X/\sim$ which is continuum-wise expansive.

AMS Classification: 54H20, 54F15

Keywords: cw-expansive, Perturbation, Quotient space

1. INTRODUCTION

We say a point $x \in X$ is a Lyapunov stable point for a homeomorphism f on a compact metric space X if for any $\epsilon > 0$ there exists a $\delta > 0$ such that, $d(x, y) < \delta$ implies that $\sup_{n \in \mathbb{N}} d(f^n(x), f^n(y)) < \epsilon$. f is expansive, if there exists $\alpha > 0$, such that, for any x and y in X with $x \neq y$, there exists an integer n such that, $d(f^n(x), f^n(y)) > \alpha$. A continuum is a nonempty, compact and connected metric space. In [2], by using of continuum a weaker notion than expansiveness is introduced which is: a homomorphism $f : X \rightarrow X$ is said to be continuum-wise expansive (cw-expansive) if there exists a constant $\alpha > 0$ such that each nontrivial continuum C satisfies $\sup_{n \in \mathbb{Z}} \text{diam } f^n(C) > \alpha$. α is said to be expansivity constant. In [1] an example of cw-expansive homomorphism which is not expansive homeomorphism is presented.

In this paper we are concerned with the perturbations of cw-expansive homeomorphisms on metric space X which is connected, locally connected and compact. Although a homeomorphism g which is sufficiently close to a cw-expansive homeomorphism f is not (in general) cw-expansive, it has some property resembling

cw-expansiveness: for any $\epsilon > 0$, if g is sufficiently close to f , then for any continuum C of X , with $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) > \alpha$, we have $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) > \epsilon$ (see the next lemma).

This allows us, for g sufficiently close to f , to define an equivalent relation on X . The quotient space $\tilde{X}$, given by this relation, endowed with the quotient topology is a compact metric space (see 2.3 below). In this space two point $\tilde{x}$ and $\tilde{y}$ are close, if their continuum C containing $x \in \tilde{x}$ and $y \in \tilde{y}$ and the diameter of its iterations are small enough. Moreover, the map g induces a cw-expansive homeomorphism $\tilde{g}$ on $\tilde{X}$ (see 2.11 below). In [1] Hertz proved that, there is no Lyapanov stable point for cw-expansive homeomorphisms.

Before stating our results precisely, we introduce some other definitions. Let X be a compact metric space, endowed with a metric d , and let $\mathcal{FX}$ composed of all the nonempty closed subsets of X . Moreover let A and B be in $\mathcal{FX}$, put

$$d_H(A, B) = \inf\{\alpha > 0; A \subset U_\alpha(B) \text{ and } B \subset U_\alpha(A)\},$$

where $U_\alpha(\cdot)$ denotes the α -ball measured in d . Then by [3], d_H is a complete metric on $\mathcal{FX}$, called the Hausdorff metric. It is a well known result that: when C_n is a sequence of continua and C_n tends to C in the Hausdorff metric, then C is a continuum too (see [4]). The main goal of this note is studying the perturbation of cw-expansive homeomorphism on a locally connected compact metric space, which is precisely as follows:

Main theorem: *Let f be a cw-expansive homeomorphism on X with expansivity constant $\alpha > 0$. If g is sufficiently close to f , then, there exists an equivalence relation on X such that the quotient space corresponding to it, is a Hausdorff compact space. Also g induce a cw-expansive homeomorphism on it with expansivity constant $\alpha/2$.*

2 PROOF OF RESULTS

In this section we assume that X is a locally connected compact metric space, and f is a cw-expansive homeomorphism on X with the expansivity constant α .

Lemma 2.1: *For any $\epsilon > 0$, there exists a neighborhood $\mathcal{U}_\epsilon(f)$ of f such that any $g \in \mathcal{U}_\epsilon(f)$ has the following property: if C is any continuum in X with*

$$\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) > \alpha$$

then $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) > \epsilon$.

Proof: Suppose by contraction that the lemma is not true. Then there exists a sequence of homeomorphisms $\{g_k\}$ converging to f and a sequence C_k of continuum in X such that $\sup_{n \in \mathbb{Z}} \text{diam } g_k^n(C_k) \leq \alpha$ and for some sequence $\{m_k\}$ of integers, $\text{diam } g_k^{m_k}(C_k) > \epsilon$. Without loss of generality, we can assume that $m_k = 0$.

Take a subsequence $\{C_{k_n}\}$, such that $C_{k_n} \rightarrow C$. C is continuum in X and furthermore, $\text{diam } C > \epsilon$. Also, $\sup_{n \in \mathbb{Z}} \text{diam } f^n(C) \leq \alpha$, which is contraction. $\square$

Now let $\epsilon' < \alpha/2$ and $g \in \mathcal{U}_{\epsilon'}(f)$. Define the relation $\sim$ on M as the following form:

" $x \sim_g y$ if and only if, there exists a continuum in X , which contains x and y and also, $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) \leq \epsilon$."

Now if $x \sim_g y$ and $y \sim_g z$, then there are two continuum C_1 and C_2 in X such that $x, y \in C_1$, $y, z \in C_2$, $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C_1) \leq \epsilon$ and $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C_2) \leq \epsilon$. Put $C = C_1 \cup C_2$. C contains x and z and furthermore $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) \leq 2\epsilon < \alpha$ and thus by lemma 2.1, $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) \leq \epsilon$.

So $\sim_g$ define an equivalence relation on X . Let $[x]$ be the equivalence class of x , and $\tilde{X}_g = X / \sim_g$. moreover let $p_g : X \rightarrow \tilde{X}_g$ be the canonical projection. $\tilde{X}_g$ with the quotient topology is a compact connected and locally connected topological space.

Notation: When there is no ambiguity, we put $\sim = \sim_g$, $\tilde{X} = \tilde{X}_g$ and $p = p_g$.

Lemma 2.2: The canonical projection $p : X \rightarrow \tilde{X}$ is closed. In particular, given an open set U the one define by $V = \{x \in U; [x] \subseteq U\}$ is open.

Proof: Equivalently we show that, if $F \subseteq X$ is closed, then the set

$$G = p^{-1}(p(F)) = \{x \in X; [x] \cap F \neq \emptyset\}$$

is closed too. Let $\{x_m\}$ be a sequence in G and $x_m \rightarrow x$. Then we show that $x \in G$. For a given m , since $x_m \in G$, so there exists $y_m \in F$, such that $[y_m] = [x_m]$. By taking a subsequence (if it is necessary) assume that $y_m \rightarrow y \in F$. We claim that $[x] = [y]$. For a given m , let C_m be a continuum in X which contains x_m and y_m and furthermore $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C_m) \leq \epsilon$. Taking a subsequence (if it is necessary) assume that $C_m \rightarrow C$. C is continuum, $x, y \in C$ and $\sup_{n \in \mathbb{Z}} \text{diam } g^n(C) \leq \epsilon$. Thus $x \sim y$. $\square$

Lemma 2.3: The topological space $\tilde{X}$ is a Hausdorff space.

Proof: Let $\tilde{x}, \tilde{y} \in \tilde{X}$, and $\tilde{x} \neq \tilde{y}$. $p^{-1}(\tilde{x})$ and $p^{-1}(\tilde{y})$ are compact disjoint subset of X and so there are two disjoint open sets U_1 and U_2 which contain $p^{-1}(\tilde{x})$ and $p^{-1}(\tilde{y})$ respectively. Now let $V_i = \{x \in U; [x] \subseteq U_i\}$, $i = 1, 2$. From the previous lemma, $p(V_1)$, and $p(V_2)$ are disjoint open subsets of $\tilde{X}$ and they contain $\tilde{x}$ and $\tilde{y}$ respectively. $\square$

Since $\tilde{X}$ is a compact topological and Hausdorff space, so by Urysohn metrization theorem [3], it is metrizable. In the following we try to construct it's metric exactly and study the action of some homeomorphism which is related to g . First we define a suitable metric on $\tilde{X}$ and then we prove that, this metric induces the topology same as quotient topology on $\tilde{X}$.

In order to do this, we recall that a subset B of $X \times X$ is said to be symmetric, if $(x, y) \in B$, implies that $(y, x) \in B$. For two subset B and C of $X \times X$ define

$$B \circ C = \{(x, y) \in X \times X; \text{there exists some } z \in X \text{ with } (x, z) \in B \text{ and } (z, y) \in C\}$$

Clearly $\circ$ is a associative relation.

We use from the following nice lemma to construct the metric on $\tilde{X}$. Proof of this lemma can be found in [3].

Lemma 2.4: Let X be a set and let $\{B_n\}$ be symmetric subsets of $X \times X$ such that each B_n contain the diagonal of $X \times X$ and furthermore, $B_{n+1} \circ B_{n+1} \circ B_{n+1} \subseteq B_n$, for any positive integer n . Then there exists a pseudo-metric δ on $X \times X$ such that the following holds for any positive integer n :

$$B_n \subseteq \{(x, y) \in X \times X; \delta(x, y) < 2^{-n}\} \subseteq B_{n-1}$$

In the following we assert the uniform expansiveness of cw-expansive homeomorphisms.

Lemma 2.5: If F is any cw-continuum expansive homeomorphism then f is uniformly cw-expansive, that is, for each $\eta > 0$, there exists positive integer N_η , such that, for any continuum C of X with $\text{diam}(C) \geq \eta$, we have, $\max_{|n| \leq N_\eta} \text{diam } f^n(C) > \alpha$.

Proof: By contraction, let for some $\eta > 0$, there exists a sequence $\{C_m\}$ of continuums and a sequence $\{N_m\}$ of positive integers such that $\text{diam}(C_m) \geq \eta$ and

$$\max_{|n| \leq N_m} \text{diam } f^n(C_m) \leq \alpha$$

Taking subsequence (if it is necessary), with $C_m \rightarrow C$. Then C is a continuum in X . Furthermore $\text{diam}(C) \geq \eta$. But, $\sup_{n \in \mathbb{Z}} \text{diam } f^n(C) \leq \alpha$, which is contraction. $\square$

Note: From now on, we assume that $\epsilon \leq \alpha/3$.

Lemma 2.6: There exists a neighborhood $\mathcal{U}_1$ of f such that, any $g \in \mathcal{U}_1$, has the following property:

For any integer j , and any continuum C of X ,

$$\max_{|p| \leq j + N_\epsilon} \text{diam } g^p(C) < \alpha$$

implies that, $\max_{|p| < j} \text{diam } g^p(C) < \alpha/3$.

Proof: By contraction let the statement is not true. Then there exists a sequence $\{g_n\}$ converging to f such that, for every positive integer n , there exist a continuum c_n and positive integer j_n , such that, $\max_{|p| \leq j_n + N_\epsilon} \text{diam } g_n^p(C_n) < \alpha$ and for some p_n with $p_n < j_n$,

$$\text{diam } g^{p_n}(C_n) \geq \alpha/3 \quad (1).$$

Taking a subsequence (if it is necessary) such that $g^{p_n}(C_n) \rightarrow C$. Then C is a continuum in X and by (1), $\text{diam}(C) \geq \alpha/3 \geq \epsilon$. Now for any integer p with $|p| < N_\epsilon$, we have $|p + p_n| < j_n + N_\epsilon$, and we can conclude that

$$\text{diam } f^p(C) = \lim_{n \rightarrow \infty} \text{diam } g^{p+p_n}(C_n) \geq \alpha.$$

This is a contradiction to previous lemma. $\square$

For $g \in \mathcal{U}_\epsilon \cap \mathcal{U}_1$. We define the following subsets of $X \times X$.

$B_0 = \{(x, y) \in X \times X; \text{there exists a continuum } C \text{ contain } x \text{ and } y \text{ with } \text{diam}(C) < \alpha\}$
and by induction, let B_n be the set of all $(x, y) \in X \times X$, such that there exists a continuum C contain x and y with $\max_{|p| < n N_\epsilon} \text{diam } g^p(C) < \alpha$.

Lemma 2.7: For the above family, we have $B_n \circ B_n \circ B_n \subseteq B_{n-1}$, for any positive

integer n .

Proof: Let $(x, y) \in B_n \circ B_n \circ B_n$. Then there are two point z_1 and z_2 in X , such that $(x, z_1), (z_1, z_2)$ and (z_2, y) are in B_n . Therefore there are three continuums C_1, C_2 and C_3 of X such that $x, z_1 \in C_1, z_1, z_2 \in C_2$ and $z_2, y \in C_3$. Now if we use the previous lemma by $j = (n-1)N_\epsilon$, we conclude that

$$\max_{|p| < (n-1)N_\epsilon} \text{diam } g^p(C_i) < \alpha/3, \quad i = 1, 2, 3.$$

Let $C = C_1 \cup C_2 \cup C_3$. Then $\max_{|p| < (n-1)N_\epsilon} \text{diam } g^p(C) < \alpha$. Therefore $(x, y) \in B_{n-1}$. $\square$

Corollary 2.8: *There exists a pseudo-metric δ on X such that, for any positive integer n ,*

$$B_n \subseteq \{(x, y) \in X \times X; \delta(x, y) < \alpha 2^{-n}\} \subseteq B_{n-1}.$$

Proof: Notice that two points x and y are in the same equivalence class if and only if $\delta(x, y) = 0$. Thus without any ambiguity, we can define a metric $\tilde{\delta}$ on $\tilde{X}$ as follow:

$$\tilde{\delta}(\tilde{x}, \tilde{y}) = \delta(x, y), \quad x \in p^{-1}(\tilde{x}), \quad y \in p^{-1}(\tilde{y}).$$

Lemma 2.9: *The metric $\tilde{\delta}$, induces the same topology as the quotient topology of $\tilde{X}$.*

Proof: First, let U be any open subset in metric topology, then we show that U is open in the quotient topology. For this goal, by the definition of the quotient topology, it is enough to show that $p : X \rightarrow (\tilde{X}, \tilde{\delta})$ is continuous. Let $\{x_n\}$ be a sequence in X which is converging to $x \in X$. By local connectedness of X , for any positive integer m , there exists a positive integer N such that, $(x_n, x) \in B_m$, for any $n \geq N$. Thus by the definition,

$$\tilde{\delta}(p(x_n), p(x)) \rightarrow 0.$$

So p is a continuous map.

Conversely let U be an open set in $\tilde{X}$ with the quotient topology. Let $\tilde{x} \in U$. For any $z \in p^{-1}(\tilde{x})$, let $V \subseteq p^{-1}(U)$ be any compact connected neighborhood of z and choose positive integer $n(z)$ such that, if $y \in X$ is a point with the property $(x, y) \in B_{n(z)}$ then, $y \in V \subseteq p^{-1}(U)$. From the compactness of $p^{-1}(\tilde{x})$, we deduce that: there is some positive integer N such that, for any $z \in p^{-1}(\tilde{x})$, $n(z) \leq N$. So for any $z \in p^{-1}(\tilde{x})$, if y is any point of X with the property $(x, y) \in B_N$ then, $y \in V \subseteq p^{-1}(U)$. But if $y \in X$ and $\delta(x, y) < \alpha 2^{-(N+1)}$ then, $(x, y) \in B_N$ and therefore $y \in V \subseteq p^{-1}(U)$. So if $\tilde{y}$ is any point in $\tilde{X}$ and $\tilde{\delta}(\tilde{x}, \tilde{y}) < \alpha 2^{-(N+1)}$ then, $y \in U$. $\square$

Corollary 2.10: *The following statements hold:*

(1) *There exists $\eta > 0$ and $\mathcal{U}_2$ of f such that, if $g \in \mathcal{U}_\epsilon \cap \mathcal{U}_1 \cap \mathcal{U}_2$ and $d(x, y) < \eta$ then*

$$\tilde{\delta}(p(x), p(y)) < \alpha/4.$$

(2) *For all $g \in \mathcal{U}_1 \cap \mathcal{U}_2$, if $\tilde{\delta}(\tilde{x}, \tilde{y}) < \alpha/2$, then $d(x, y) < \alpha$ for any $x \in p^{-1}(\tilde{x})$ and*

$$y \in p^{-1}(\tilde{y}).$$

Proof: For the proof of (1), suppose that, y is sufficiently close to x then, $(x, y) \in U_2$ and the corollary 2.8 implies that $\tilde{\delta}(p(x), p(y)) < \alpha/4$. For the proof of (2), let $\tilde{\delta}(\tilde{x}, \tilde{y}) < \alpha/2$, $x \in p^{-1}(\tilde{x})$ and $y \in p^{-1}(\tilde{y})$. Then $\tilde{\delta}(x, y) < \alpha/4$. Thus by corollary 2.8, $(x, y) \in U_0$. So $d(x, y) < \alpha$.

Finally we introduce the mapping $\tilde{g}$ on $\tilde{X}$ by:

$$\tilde{g}(p(x)) = p(g(x))$$

Now we are ready to prove the main theorem of this article. Before it, we recall that:

Remark: If C is a continuum in $\tilde{X}$ then, $\overline{p^{-1}(C)}$ is a continuum in X .

Theorem 2.11: The mapping $\tilde{g} : \tilde{X} \rightarrow \tilde{X}$ is a cw-expansive homeomorphism with expansivity constant $\alpha/2$.

Proof: Clearly $\tilde{g}$ is a homeomorphism on $\tilde{X}$. To see that it is expansive. Consider a nontrivial continuum C of $\tilde{X}$ with $\sup_{n \in \mathbb{Z}} \text{diam } \tilde{g}^n(C) < \alpha/2$. From corollary 2.10,

$$\sup_{n \in \mathbb{Z}} \text{diam } g^n(\overline{p^{-1}(C)}) < \alpha/2.$$

So lemma 2.1 implies that $\sup_{n \in \mathbb{Z}} \text{diam } g^n(\overline{p^{-1}(C)}) < \epsilon$. Thus any two point in $\overline{p^{-1}(C)}$ has the same class and this means that C is trivial. $\square$

REFERENCES

- [1] Hertz J.R., There are no stable points for continuum-wise expansive homeomorphisms, preprint.
- [2] Kato H., Continuum-wise expansive homeomorphisms, *Canad. Math.* 45, no. 3, (1993) 576-598.
- [3] Kelly J.L., General Topology, *New York: Van Nostrand*, 1995.
- [4] Nadler Jr. Sam B., Continuum theory, *Marcel Dekker Inc., New York*, 1992.