LOCALIZATION OF EQUIVARIANT COHOMOLOGY FOR COMPACT AND NON-COMPACT GROUP ACTIONS

A.A. BYTSENKO
DEPARTAMENTO DE FÍSICA, UNIVERSIDADE ESTADUAL DE LONDRINA, CAIXA POSTAL 6001, LONDRINA-PARANÁ, BRAZIL
E-MAIL: ABYTS@UEL.BR
M. LIBINE
DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, LEDERLE RESEARCH TOWER, 710 NORTH PLEASANT STREET, AMHERST, MA 01003
E-MAIL: MATVEI@MATH.UMASS.EDU
F.L. WILLIAMS
DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, LEDERLE RESEARCH TOWER, 710 NORTH PLEASANT STREET, AMHERST, MA 01003
E-MAIL: WILLIAMS@MATH.UMASS.EDU

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ABSTRACT. We give a brief introduction to the Berline-Vergne localization formula for the finite-dimensional setting and indicate how the Duistermaat-Heckman formula is derived from it. We consider applications of the localization formula when it is specialized to a maximal dimensional co-adjoint orbit. In particular, the case when the co-adjoint orbit is a quotient G/T of a connected Lie group G modulo a maximal torus T is analyzed in detail. We describe also a generalization of the localization formula to non-compact group actions.

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1. INTRODUCTION

In 1982 J.J. Duistermaat and G. Heckman [13] found a formula which expressed certain oscillatory integrals over a compact symplectic manifold as a sum over critical points of a corresponding phase function. In this sense these integrals are localized, and their stationary-phase approximation is exact with no error terms occurring. The ideas and techniques of localization extended to infinite-dimensional settings have proved to be quite useful and indeed central for many investigations in theoretical physics - investigations ranging from supersymmetric quantum mechanics, topological and supersymmetric field theories, to integrable models and low-dimensional gauge theories, including two-dimensional Yang-Mills theory [40].

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Path integral localization appears in the work of M. Semenov-Tjan-Schanskii [38], which actually pre-dates [13].

E. Witten was the first to propose an extension of the Duistermaat-Heckman (D-H) formula to an infinite-dimensional manifold - namely to the loop space LM of smooth maps from the circle S^1 to a compact orientable manifold M. In this case a purely formal application of the D-H formula to the partition function of $N = 1/2$ supersymmetric quantum mechanics yields a correct formula for the index of a Dirac operator [1]. Further arguments in this direction were presented with mathematical rigor by J.-M. Bismut in [7, 8].

The various generalizations of D-H generally require formulations in terms of equivariant cohomology. One has, for example, the Berline-Vergne (B-V) localization formula [3, 4, 5, 6] which expresses the integral of an equivariant cohomology class as a sum over zeros of a vector field to which that class is related; also see [44, 9, 27, 40] for example, a broader formulation of the localization formula.

This paper is organized as follows. In Sections 2-5 our remarks are designed to provide the reader with a brief introduction to the B-V localization formula, and to indicate how the D-H formula is derived from it (see also [11]). We limit our discussion, in particular, to the finite-dimensional setting as our idea is to convey the basic flavor of these formulas. This introduction should prepare readers for quite more ambitions discussions found in [6, 27, 40], for example. In Sections 6 we consider applications of the D-H localization formula when it is specialized to a maximal dimensional co-adjoint orbit. We pay attention to the case when the co-adjoint orbit is a quotient G/T of a connected Lie group G (in particular the unitary group $G = U(n)$) modulo a maximal torus T. Finally in Section 7 we describe a generalization of the localization formula to non-compact group actions.

The role of equivariant cohomology in physical theories will continue to grow as it has grown in past years. In particular it will be an indispensable tool for topological theories of gauge, strings, and gravity.

2. The equivariant cohomology space $H(M, X, s)$

For an integer $j \geq 0$ let $\Lambda^j M$ denote the space of smooth complex differential forms of degree j on a smooth manifold M. $d : \Lambda^j M \to \Lambda^{j+1} M$ will denote exterior differentiation, and for a smooth vector field X on M,

$$\theta(X) : \Lambda^j M \to \Lambda^j M, \quad \iota(X) : \Lambda^j M \to \Lambda^{j-1} M \tag{1}$$

will denote Lie and interior differentiation by X, respectively:

$$\begin{aligned} (\theta(X)\omega)(X_1, ..., X_j) &= X\omega(X_1, ..., X_j) \\ &- \sum_{\ell=1}^{j} \omega(X_1, ..., X_{\ell-1}, [X, X_\ell], X_{\ell+1}, ..., X_j)\,, \end{aligned} \tag{2}$$

$$(\iota(X)\omega)(X_1, ..., X_{j-1}) = \omega(X, X_1, ...X_{j-1}) \tag{3}$$

for $\omega \in \Lambda^j M$ and for $X_1, ..., X_j \in VM =$ the space of smooth vector fields on M. One has the familiar rules

$$\theta(X) = d\iota(X) + \iota(X)d\,, \tag{4}$$

$$d\theta(X) = \theta(X)d, \ \theta(X)\iota(X) = \iota(X)\theta(X)\,, \tag{5}$$

$$\iota(X) \circ \iota(X) = 0\,, \quad d \circ d = 0. \tag{6}$$

For a complex number s let

$$d_{X,s} \stackrel{def}{=} d + s^{-1}\iota(X) \text{ on } \Lambda M = \bigoplus_{j\geq 0} \Lambda^j M. \tag{7}$$

Then by (4)–(6), $d_{X,s}\theta(X) = \theta(X)d_{X,s}$ and $d_{X,s}^2 = s^{-1}\theta(X)$. Hence the subspace

$$\Lambda_X M = \{\omega \in \Lambda M | \theta(X)\omega = 0\}, \tag{8}$$

of ΛM is $d_{X,s}-$ invariant and $d_{X,s}^2 = 0$ on $\Lambda_X M$. It follows that we can define the cohomology space

$$H(M, X, s) = Z(M, X, s)/B(M, X, s) \tag{9}$$

for $Z(M, X, s)$ = kernel of $d_{X,s}$ on $\Lambda_X M$, $B(M, X, s) = d_{X,s}\Lambda_X M$. The space $H(M, X, s)$ appears to depend on the parameter s. However it is not difficult to show that for $s \neq 0$ there is an isomorphism of $H(M, X, s)$ onto $H(M, X, 1)$. For $X = 0$, $H(M, 0, s)$ is the ordinary de Rham cohomology of M.

We shall be interested in the case when M has a smooth Riemannian structure $<,>$, and when M is oriented and even-dimensional. Thus let $\omega \in \Lambda^{2n}M - \{0\}$, $\dim M = 2n$, define the orientation of M. In this case we assume moreover that X is a Killing vector field:

$$X < X_1, X_2 > = < [X, X_1], X_2 > + < X_1, [X, X_2] > \tag{10}$$

for $X_1, X_2 \in VM$. If $p \in M$ is a zero of X (i.e. $X_p = 0$) then there is an induced linear map $\mathfrak{L}_p(X)$ of the tangent space $T_p(M)$ of M at p such that

$$\mathfrak{L}_p(X)(Y_p) = [X, Y]_p \quad \text{for } Y \in VM. \tag{11}$$

Because of (10) one has that $\mathfrak{L}_p(X)$ is skew-symmetric; i.e.

$$< \mathfrak{L}_p(X)V_1, V_2 >_p = - < V_1, \mathfrak{L}_p(X)V_2 >_p \quad \text{for} \quad V_1, V_2 \in T_p(M). \tag{12}$$

Let

$$f_p(X) : T_p(M) \oplus T_p(M) \to \mathbb{R} \tag{13}$$

be the corresponding skew-symmetric bilinear form on T_pM:

$$f_p(X)(V_1, V_2) = < V_1, \mathfrak{L}_p(X)V_2 >_p \text{ for } V_1, V_2 \in T_pM. \tag{14}$$

In order to apply some standard linear algebra to the real inner product space $(T_p(M), <,>_p)$, we suppose $\mathfrak{L}_p(X)$ is a non-singular linear operator on $T_p(M)$: $\det\mathfrak{L}_p(X) \neq 0$; equivalently, this means that the bilinear form $f_p(X)$ is non-degenerate. Then one can find an ordered orthonormal basis $e = e^{(p)} = \{e_i = e_j^{(p)}\}_{j=1}^{2n}$ of $T_p(M)$ such that

$$\begin{aligned} \mathfrak{L}_p(X)e_{2j-1} &= \lambda_j e_{2j}, \\ \mathfrak{L}_p(X)e_{2j} &= -\lambda_j e_{2j-1}, \quad \text{for } 1 \leq j \leq n, \end{aligned} \tag{15}$$

where each $\lambda_j \in \mathbb{R} - \{0\}$. In other words, relative to e the matrix of $\mathfrak{L}_p(X)$ has the form

$$\mathfrak{L}_p(X) = \begin{bmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_n \\ & & & \lambda_n & 0 \end{bmatrix} \tag{16}$$

Moreover, interchanging e_1, e_2 if necessary, we can assume that e is positively oriented: $\omega_p(e_1, ..., e_{2n}) > 0$. Finally, consider the Pfaffian $\mathrm{Pf}_e(\mathfrak{L}_p(X))$ of $\mathfrak{L}_p(X)$ relative to e:

$$\mathrm{Pf}_e(\mathfrak{L}_p(X)) = \frac{1}{n!}\left[f_p(X) \wedge ... \wedge f_p(X)\right](e_1, ..., e_{2n}). \tag{17}$$

$\mathrm{Pf}_e(\mathfrak{L}_p(X))$ satisfies

$$\mathrm{Pf}_e(\mathfrak{L}_p(X))^2 = \det \mathfrak{L}_p(X), \tag{18}$$

$$\mathrm{Pf}_e(\mathfrak{L}_p(X)) = (-1)^n \lambda_1 \cdots \lambda_n. \tag{19}$$

If $e' = \{e'_j\}_{j=1}^{2n}$ is another ordered, positively oriented orthogonal basis of $T_p(M)$ then

$$\mathrm{Pf}_{e'}(\mathfrak{L}_p(X)) = \mathrm{Pf}_e(\mathfrak{L}_p(X)). \tag{20}$$

Equation (20) means that we can define a square-root of $\mathfrak{L}_p(X)$ by setting

$$[\det \mathfrak{L}_p(X)]^{1/2} = (-1)^n \mathrm{Pf}_e(\mathfrak{L}_p(X)). \tag{21}$$

That is, the square-root is independent of the choice e of an ordered, positively oriented orthogonal basis of $T_p(M)$. By (19) we have

$$[\det \mathfrak{L}_p(X)]^{1/2} = \lambda_1 \cdots \lambda_n. \tag{22}$$

The reader is reminded that the hypotheses $X_p = 0$ and $\det(\mathfrak{L}_p(X)) \neq 0$ were imposed, with X a Killing vector field.

3. The localization formula

As before we are given an oriented, $2n-$dimensional Riemannian manifold $(M, \omega, <\ ,>)$. Now assume that G is a compact Lie group which acts smoothly on M, say on the left, and that the metric $<,>$ is $G-$invariant. Let $\mathfrak{g}$ denote the Lie algebra of G. Given $X \in \mathfrak{g}$, there is an induced vector field $X^* \in VM$ on M: for $\phi \in C^\infty(M)$, $p \in M$

$$(X^*\varphi)(p) = \frac{d}{dt}\varphi(\exp(tX) \cdot p)|_{t=0}. \tag{23}$$

Since $<,>$ is $G-$invariant, one knows that X^* is a Killing vector field. X^* is said to be non-degenerate if, for every zero $p \in M$ of X^*, the induced linear map $\mathfrak{L}_p(X^*) : T_p(M) \to T_p(M)$ is non-singular. Since X^* is a Killing vector field, $\mathfrak{L}_p(X^*)$ is skew-symmetric with respect to the inner product structure $<,>_p$ on $T_p(M)$, as we have noted, and the non-singularity of $\mathfrak{L}_p(X^*)$ means that we can construct the square-root

$$[\det \mathfrak{L}_p(X^*)]^{1/2} = (-1)^n \mathrm{Pf}_e(\mathfrak{L}_p(X^*)) = \lambda_1 \cdots \lambda_n, \tag{24}$$

as in (21) and (22).

For a form $\tau \in \Lambda M = \bigoplus \Lambda^j M$ we write $\tau_j \in \Lambda^j M$ for its homogeneous $j-$th component,

$$\tau = (\tau_0, ..., \tau_{2n}) = \sum_{j=0}^{2n} \tau_j, \tag{25}$$

and we write $[\tau]$ for the cohomology class of τ in case $\tau \in Z(M, X, s)$ for $X \in VM$, $s \in \mathbb{C}$; i.e. $d_{X,s}\tau = 0$ for $d_{X,s}$ in (7). When M is compact, in particular, one can integrate any $2n-$form (as M is orientable). Thus we can define

$$\int_M \tau = \int_M \tau_{2n}, \tag{26}$$

and in fact we can define

$$\int_M [\tau] = \int_M \tau = \int_M \tau_{2n}. \tag{27}$$

The integral $\int_M [\tau]$ really does depend only on the class $[\tau]$ of τ. Therefore the following result holds:

Proposition 1. *If $\tau' \in B(M, X, s)$ then by a quick computation using Stokes' theorem one sees that*

$$\int_M \tau' = 0. \tag{28}$$

Similarly if $p \in M$ with $X_p = 0$ then $\tau_0'(p) = 0$ for $\tau' \in B(M, X, s)$.

Proof. Indeed, if we write $\tau' = d_{X,s}\beta$ for $\beta \in \Lambda_X M$ then one has

$$\begin{aligned} \tau' &= (s^{-1}\iota(X)\beta_1,\ d\beta_0 + s^{-1}\iota(X)\beta_2,\ d\beta_1 + s^{-1}\iota(X)\beta_3,\ d\beta_2 + s^{-1}\iota(X)\beta_4, \\ &\quad ...,\ d\beta_{2n-2} + s^{-1}\iota(X)\beta_{2n},\ d\beta_{2n-1}) \\ &= d\beta_0 + s^{-1}\iota(X)\beta_0 + d\beta_1 + s^{-1}\iota(X)\beta_1 + d\beta_2 + s^{-1}\iota(X)\beta_2 + \\ &\quad ... + d\beta_{2n} + s^{-1}\iota(X)\beta_{2n}. \end{aligned} \tag{29}$$

Thus $\tau_0'(p) = [s^{-1}\iota(X)\beta_1]_{X=X_p} = s^{-1}\beta_{1p}(X_p) = 0$, and

$$\int_M \tau' = \int_M d\beta_{2n-1} = 0, \tag{30}$$

which proves (28). □

It follows that the map $p^* : H(M, X, s) \to \mathbb{R}$ given by

$$p^*[\tau] = [\tau_0 \equiv s^{-1}\iota(X)\beta_1]_{X=X_p} = \tau_0(p) \text{ for } X_p = 0 \tag{31}$$

is well-defined. In [3, 4, 5], N. Berline and M. Vergne, following some ideas of R. Bott in [10], established the following localization theorem, where the choice $s^{-1} = -2\pi\sqrt{-1}$ is made.

Theorem 1. *Assume as above that M and G are compact and that the Riemannian metric $<,>$ on M is $G-$invariant; i.e. each $a \in G$ acts as an isometry of M. For $X \in \mathfrak{g}$, the Lie algebra of G, assume that the induced vector field X^* on M (see*

(23)) is non-degenerate; thus the square-root in (24) is well-defined (and is non-zero) for $p \in M$ a zero of X^* (i.e. $X^*_p = 0$). Then for any cohomology class $[\tau] \in H(M, X^*, s)$ one has

$$\int_M [\tau] = (-1)^{n/2} \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} \frac{p^*[\tau]}{[\det \mathfrak{L}_p(X^*)]^{1/2}}; \tag{32}$$

see (27), (31).

For concrete applications of Theorem 1 we shall need to construct concrete cohomology classes in $H(M, X^*, s)$. The construction of such classes requires that a bit more be assumed about M and G. Suppose for example that M has a symplectic structure $\sigma : \sigma \in \Lambda^2 M$ is a closed two-form (i.e. $d\sigma = 0$) such that for every $p \in M$ the corresponding skew-symmetric form

$$\sigma_p : T_p(M) \oplus T_p(M) \to \mathbb{R} \tag{33}$$

is non-degenerate. In particular M is oriented by the Liouville form

$$\omega_\sigma = \frac{1}{n!} \sigma \wedge \cdots \wedge \sigma \in \Lambda^{2n} M - \{0\}. \tag{34}$$

Suppose also that there is a map $J : \mathfrak{g} \to C^\infty(M)$ which satisfies

$$\iota(X^*)\sigma + dJ(X) = 0, \quad \forall X \in \mathfrak{g}, \tag{35}$$

an equality of one-forms. The existence of such a map J amounts to the assumption that the action of G on M is Hamiltonian, a point which we shall return to later.

Proposition 2. *For a given J let us define for each $X \in \mathfrak{g}$ the form $\tau^X \in \Lambda M$ by*

$$\tau^X \stackrel{def}{=} (J(X), 0, s\sigma, 0, ..., 0); \tag{36}$$

see (25). Then, we have that $\tau^X \in Z(M, X^*, s)$.

Proof. Since $J(X)$ is a function, then $\iota(X^*)J(X) = 0$. Therefore by (4)–(6) and (35),

$$\theta(X^*)J(X) = \iota(X^*)dJ(X) = -\iota(X^*)^2\sigma = 0 \tag{37}$$

and

$$\theta(X^*)\sigma = d\iota(X^*)\sigma + \iota(X^*)d\sigma = d\iota(X^*)\sigma \quad (\text{as } d\sigma = 0) = -d^2 J(X) = 0. \tag{38}$$

¿From the definition (36) it follows that

$$\theta(X^*)\tau^X = (\theta(X^*)J(X), 0, \theta(X^*)s\sigma, 0, ..., 0) = 0, \tag{39}$$

which by (8) means that $\tau^X \in \Lambda_{X^*} M$. Also from the definitions (7) and (35), we have

$$\begin{aligned} d_{X^*,s}\tau^X &= (d + s^{-1}\iota(X^*))\tau^X \\ &= dJ(X) + s^{-1}\iota(X^*)J(X) + ds\sigma + s^{-1}\iota(X^*)s\sigma \\ &= -\iota(X^*)\sigma + \iota(X^*)\sigma = 0, \end{aligned} \tag{40}$$

which verifies the claim, where again we have used that $\iota(X^*)J(X) = 0$, $d\sigma = 0$. □

Thus, for a given J, we have for each $X \in \mathfrak{g}$ a cohomology class $[\tau^X] \in H(M, X^*, s)$.

4. The class $[e^{c\tau^X}]$

In the next Section the Duistermaat-Heckman formula will be derived by a direct application of Theorem 1. The main point is the construction of an appropriate cohomology class. Namely for the cocycle $\tau^X \in Z(M, X^*, s)$ in (36) we wish to construct for $c \in \mathbb{C}$ a well-defined form $e^{c\tau^X}$ which also is an element of $Z(M, X^*, s)$.

Thus again suppose that J which satisfies (35) is given. For $X \in \mathfrak{g}$ let

$$\tau_0 = J(X),\ \tau_1 = 0,\ \tau_2 = s\sigma,\ \tau_j = 0 \quad \text{for} \quad 3 \le j \le 2n, \tag{41}$$

and let $\tau = \tau^X$. That is, by (36),

$$\tau = (\tau_0, \tau_1, \tau_2, ..., \tau_{2n}) = (\tau_0, 0, \tau_2, 0, 0, ..., 0). \tag{42}$$

If ω_1, ω_2 are forms of degree p, q respectively, then ω_1 and ω_2 commute if either p or q is even, since

$$\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1. \tag{43}$$

In particular τ_0 and τ_2 commute. Now if A and B are commuting matrices one has $e^{A+B} = e^A \cdot e^B$. Since τ_0 and τ_2 commute we should have, formally for any complex number c,

$$c\tau = c\tau_0 + c\tau_2 \Rightarrow e^{c\tau} = e^{c\tau_0} \cdot e^{c\tau_2} = e^{c\tau_0}(1 + c\tau_2 + c^2\tau_2^2/2! + c^3\tau_2^3/3! + ...), \tag{44}$$

with

$$\tau_2^j = \tau_2 \wedge \cdots \wedge \tau_2 \quad (j \quad \text{times}) \ \in \Lambda^{2j}M. \tag{45}$$

Since $\Lambda^{2j}M = 0$ for $j > n$ we can take $\sum_{j=0}^{\infty} c^j\tau_2^j/j!$ to mean $\sum_{j=0}^{n} c^j\tau_2^j/j!$. That is, thinking of $c\tau_2^j/j!$ as $(0, 0, ..., c\tau_2^j/j!, 0, ..., 0)$ and 1 as $(1, 0, 0, ..., 0)$ for $1 \in C^\infty(M)$, we are therefore lead to define $e^{c\tau}$ by

$$e^{c\tau} = \left(e^{c\tau_0}, 0, e^{c\tau_0}c\tau_2, 0, e^{c\tau_0}\frac{1}{2!}c^2\tau_2^2, 0, e^{c\tau_0}\frac{1}{3!}c^3\tau_2^3, 0, ..., 0, e^{c\tau_0}\frac{1}{n!}c^n\tau_2^n\right) \in \Lambda M; \tag{46}$$

which we can compare to expression (25). Now $\iota(X^*)e^{c\tau_0} = 0$ (as $e^{c\tau_0}$ is a function), and $de^{c\tau_0} = ce^{c\tau_0}d\tau_0$. That is, by (4)–(6),

$$\begin{aligned} \theta(X^*)e^{c\tau_0}{}^{\cdot} &= c[\iota(X^*)e^{c\tau_0}d\tau_0 + e^{c\tau_0}\iota(X^*)d\tau_0] \\ &= c\iota(X^*)e^{c\tau_0}d\tau_0 = ce^{c\tau_0}\iota(X^*)d\tau_0, \end{aligned} \tag{47}$$

since

$$\begin{aligned} \tau_0 &= J(X) \Rightarrow \text{(by (4) – (6), (35))} \quad \iota(X^*)d\tau_0 = -\iota(X^*)^2\sigma = 0 \\ &\Rightarrow \theta(X^*)e^{c\tau_0} = 0. \end{aligned} \tag{48}$$

More generally,

$$\begin{aligned} \theta(X^*)e^{c\tau_0}(c^j\tau_2^j)/j! &= (\theta(X^*)e^{c\tau_0})(c^j\tau_2^j)/j! + e^{c\tau_0}(c^j/j!)\theta(X^*)\tau_2^j \\ &= e^{c\tau_0}(c^j/j!)\theta(X^*)\tau_2^j \ \text{(by (48))} = 0, \end{aligned} \tag{49}$$

again by the fact that $\theta(X^*)$ is a derivation and the fact that $\theta(X^*)\tau_2 = s\theta(X^*)\sigma$ with $\theta(X^*)\sigma = 0$ (as observed earlier).

Proposition 3. *By (46) we see therefore that*

$$\theta(X^*)e^{c\tau} = 0 \Rightarrow e^{c\tau} \in \Lambda_{X^*}M, \tag{50}$$

by (8). Therefore, we obtain $d_{X^*,s}e^{c\tau} = 0$.

By (29) and (46)

$$\begin{aligned} d_{X^*,s}e^{c\tau} &= (0,\, d\beta_0 + s^{-1}\iota(X^*)\beta_2,\, 0,\, d\beta_2 + s^{-1}\iota(X^*)\beta_4,\, 0, \\ &\ldots\ ,\, d\beta_{2n-2} + s^{-1}\iota(X^*)\beta_{2n},\, 0) \end{aligned} \tag{51}$$

for $\beta_{2j} = e^{c\tau_0}c^j\tau_2^j/j!$. Using that

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg\omega_1}\omega_1 \wedge d\omega_2 \tag{52}$$

for forms ω_1, ω_2 of homogeneous degree and that $e^{c\tau_0}, \tau_2$ are of even degree, we get

$$de^{c\tau_0}\tau_2^j = de^{c\tau_0} \wedge \tau_2^j + e^{c\tau_0} \wedge d\tau_2^j, \tag{53}$$

where

$$\begin{aligned} d\tau_2^j &= 0 \text{ (by (52)) since } d\tau_2 = sd\sigma = 0 \\ &\Rightarrow\ d\beta_{2j} = (c^j/j!)e^{c\tau_0}dc\tau_0 \wedge \tau_2^j \\ &= -(c^{j+1}/j!)e^{c\tau_0}(\iota(X^*)\sigma) \wedge \tau_2^j, \end{aligned} \tag{54}$$

by (35). Similarly

$$\iota(X^*)e^{c\tau_0}\tau_2^j = (\iota(X^*)e^{c\tau_0})\tau_2^j + e^{c\tau_0}\iota(X^*)\tau_2^j = e^{c\tau_0}\iota(X^*)\tau_2^j, \tag{55}$$

where

$$\iota(X^*)\tau_2^j = j\tau_2^{j-1} \wedge \iota(X^*)\tau_2 \tag{56}$$

since $\iota(X^*)$ also satisfies the derivative property (52), and since $i(X^*)\tau_2$ and τ_2 commute as $\deg\tau_2 = 2$. It follows

$$\begin{aligned} j\tau_2^{j-1} \wedge \iota(X^*)\tau_2 &\Rightarrow \iota(X^*)\beta_{2j}e^{c\tau_0}(c^j/(j-1)!)\tau_2^{j-1} \wedge \iota(X^*)\tau_2 \\ &= e^{c\tau_0}(c^j/(j-1)!)\tau_2^{j-1} \wedge \iota(X^*)s\sigma \\ &\Rightarrow\ s^{-1}\iota(X^*)\beta_{2j+2} = e^{c\tau_0}(c^{j+1}/j!)\tau_2^j \wedge \iota(X^*)\sigma. \end{aligned} \tag{57}$$

That is, by (54) and (57),

$$d\beta_{2j} + s^{-1}\iota(X^*)\beta_{2j+2} = 0 \text{ (again as } \iota(X^*)\tau_2 \text{ and } \tau_2 \text{ commute)}, \tag{58}$$

which by (51) establishes our claim.

Hence the following result is holds:

Theorem 2. *Suppose $J : \mathfrak{g} \to C^\infty(M)$ which satisfies (35) is given, where σ is a symplectic structure on M. Recall that for $X \in \mathfrak{g}$, equation (36) defines a cocycle $\tau^X \in Z(M, X^*, s)$. Similarly for $c \in \mathbb{C}$, define $e^{c\tau^X}$ by (46):*

$$e^{c\tau^X} = \left(e^{cJ(X)}, 0, e^{cJ(X)}cs\sigma, 0, e^{cJ(X)}\frac{c^2}{2!}(s\sigma)^2, 0, \ldots, 0, e^{cJ(X)}\frac{c^n}{n!}(s\sigma)^n\right) \in \Lambda M, \tag{59}$$

for $\dim M = 2n$. Then also $e^{c\tau^X} \in Z(M, X^*, s)$, and thus we have the cohomology class $\left[e^{c\tau^X}\right] \in H(M, X^*, s)$; see (7), (9), (23).

5. The Duistermaat-Heckman formula

Theorem 2 contains the basic assumption that a function $J : \mathfrak{g} \to C^\infty(M)$ exists which satisfies condition (35). As pointed out earlier this assumption amounts to the assumption that the action of G on M is Hamiltonian – a point which we will now explain.

Given the symplectic structure σ on M there is a duality $Y \leftrightarrow \beta_Y$ between smooth vector fields $Y \in VM$ and smooth one-forms $\beta_Y \in \Lambda^1 M$ on M:

$$\beta_Y(X) = \sigma(Y, X) \quad \text{for every } X \in VM. \tag{60}$$

$Y \in VM$ is called a Hamiltonian vector field if β_Y is exact: $\beta_Y = d\zeta$ for some $\zeta \in C^\infty(M)$. Let HVM denote the space of Hamiltonian vector fields on M. Actually HVM is a Lie algebra. For example, given any $\zeta \in C^\infty(M)$, the smooth one-form $d\zeta$ corresponds (by the aforementioned duality) to a smooth vector field Y_ζ on M. Thus $Y_\zeta \in HVM$ and by (3) and (60) we have for every $X \in VM$,

$$(\iota(Y_\zeta)\sigma)(X) = \sigma(Y_\zeta, X) = d\zeta(X) \Rightarrow d\zeta = \iota(Y_\zeta)\sigma. \tag{61}$$

The equation

$$[\zeta_1, \zeta_2] = Y_{\zeta_1}\zeta_2 \quad \text{for } \zeta_1, \zeta_2 \in C^\infty(M) \tag{62}$$

defines the Poisson bracket $[\,,\,]$ on $C^\infty(M)$ which converts $C^\infty(M)$ into a Lie algebra such that the map $\wp : \zeta \to Y_\zeta : C^\infty(M) \to HVM$ is a Lie algebra homomorphism; i.e.

$$[Y_{\zeta_1}, Y_{\zeta_2}] = Y_{[\zeta_1, \zeta_2]}. \tag{63}$$

The (left) action of G on M is called symplectic if $X^* \in HVM$, $\forall X \in \mathfrak{g}$; see (23). Now the map $X \to X^* : \mathfrak{g} \to VM$ is not a Lie algebra homomorphism since

$$[X_1, X_2]^* = -[X_1^*, X_2^*] \quad \text{for } X_1, X_2 \in \mathfrak{g}. \tag{64}$$

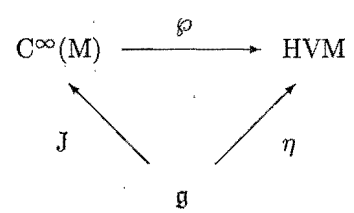
If we define

$$\eta : \mathfrak{g} \to VM \quad \text{by } \eta(X) = (-X^*) = (-X)^*, \tag{65}$$

then we do obtain a homomorphism:

$$\eta([X_1, X_2]) = -[X_1, X_2]^* = [X_1^*, X_2^*] = [-\eta(X_1), -\eta(X_2)] = [\eta(X_1), \eta(X_2)]. \tag{66}$$

In other words if the action of G is symplectic then $\eta : \mathfrak{g} \to HVM$ is a Lie algebra homomorphism. The (left) action of G on M is called Hamiltonian if it is symplectic and if the Lie algebra homomorphism $\eta : \mathfrak{g} \to HVM$ has a lift to $C^\infty(M)$ – i.e. if there exists a Lie algebra homomorphism $J : \mathfrak{g} \to C^\infty(M)$ such that the diagram



is commutative: $\eta = \wp \circ J$, or

$$-X^* = Y_{J(X)} \quad \text{for every } X \in \mathfrak{g}. \tag{67}$$

We note that such a J will indeed satisfy condition (35). Namely, by (61) and (67),

$$dJ(X) = \iota(Y_{J(X)})\sigma = -\iota(X^*)\sigma \quad \text{for } X \in \mathfrak{g}. \tag{68}$$

The triple (M, σ, J), for J subject to (5.4), is called a Hamiltonian $G-$space [22, 44]. The basic example of a Hamiltonian $G-$space is that of an orbit $\mathcal{O}$ in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ under the co-adjoint action of G on $\mathfrak{g}^*$, which is induced by the adjoint action of G on $\mathfrak{g}$, and where σ is chosen as the Kirillov symplectic form on $M = \mathcal{O}$, and J is given by a canonical construction. Namely for a linear functional f on $\mathfrak{g}$, $f \in \mathfrak{g}^*$,

$$(a \cdot f)(X) = f(Ad(a^{-1})X) \quad \text{for } a \in G,\ X \in \mathfrak{g}. \tag{69}$$

We shall recall how the (well-known) symplectic structure σ on $\mathcal{O}$ is obtained (due to A.A. Kirillov) and how the lifting J is canonically constructed. Thus we exhibit $(\mathcal{O}, \sigma = \sigma_{\mathcal{O}}, J = J_{\mathcal{O}})$ as a key example of a Hamiltonian $G-$space. For this purpose it is convenient to regard the orbit of f as a homogeneous space: $\mathcal{O} \simeq G/G_f$ where G_f is the stabilizer of f:

$$G_f = \{a \in G | a \cdot f = f\}. \tag{70}$$

G_f is a closed subgroup of G with Lie algebra $\mathfrak{g}_f$ given by

$$\mathfrak{g}_f = \{X \in \mathfrak{g} \mid f([X, Y]) = 0 \quad \forall Y \in \mathfrak{g}\}. \tag{71}$$

Let τ^f be the corresponding Maurer-Cartan one-form on G. That is, $\tau^f \in \Lambda^1 G$ is the unique left-invariant one-form on G subject to the condition

$$\tau^f(X)(1) = f(X) \qquad \forall X \in \mathfrak{g}. \tag{72}$$

Let $\pi : G \to G/G_f$ denote the quotient map.

Theorem 3. *G/G_f has a symplectic structure σ which is uniquely given by $\pi^*\sigma = d\tau^f$.*

Here $\pi^*\omega_1$ denotes the pull-back of a form ω_1. The form σ is also left-invariant; i.e. $\ell_a^*\sigma = \sigma$ where $\ell_a : G/G_f \to G/G_f$ denotes left translation by $a \in G$. Given $X \in \mathfrak{g}$ define $\psi_X : G/G_f \to \mathbb{R}$ by

$$\psi_X(aG_f) = f(Ad(a^{-1})X) = (a \cdot f)(X) \tag{73}$$

for $a \in G$; ψ_X is well-defined by (70). One can show by computation that

$$d\psi_X = -\iota(X^*)\sigma. \tag{74}$$

That is, by (60), $\beta_{-X^*} = d\psi_X \Rightarrow -X^*$ (or X^*) is Hamiltonian for each $X \in \mathfrak{g}$; i.e. the action of G on G/G_f is symplectic.

Theorem 4. *The action G on G/G_f is Hamiltonian.*

Proof. To see that this action is Hamiltonian we must construct a lift $J : \mathfrak{g} \to C^\infty(G/G_f)$ of $\eta : X \to -X^*$. Namely define J by

$$J(X) = \psi_X \quad \text{for } \psi_X \text{ in (73).} \tag{75}$$

Recall that $\wp : C^\infty(M) \to HVM$ is given by $\wp(\psi) = Y_\psi$. That is, by (61) and (74), $\wp(\psi_X) = -X^* = \eta(X)$, which shows that J does satisfy the commutative diagram

(see above). The final step is to show that J is a homomorphism. Let $X_1, X_2 \in \mathfrak{g}$, $a \in G$. The Poisson bracket is given by (62):

$$\begin{aligned}
[J(X_1), J(X_2)](\pi(a)) &= \left(Y_{J(X_1)} J(X_2)\right)(\pi(a)) \\
&= \left(\wp(J(X_1)) J(X_2)\right)(\pi(a)) \\
&= \left(\eta(X_1) J(X_2)\right)(\pi(a)) \quad \text{(again by (5.4))} \\
&= \left((-X_1^*)\psi_{X_2}\right)(\pi(a)) \quad \text{(by (75))} \\
&= \frac{d}{dt}\psi_{X_2}\left((\exp(-tX_1)) \cdot \pi(a)\right)|_{t=0} \quad \text{(by (23))} \\
&= \frac{d}{dt}\psi_{X_2}\left(\pi((\exp(-tX_1)) \cdot a)\right)|_{t=0} \\
&= \frac{d}{dt} f\left(Ad(a^{-1}\exp(X_1))X_2\right)|_{t=0} \quad \text{(by (73))} \\
&= \frac{d}{dt} f\left(Ad(a^{-1})Ad(\exp(X_1))X_2\right)|_{t=0} \\
&= \frac{d}{dt}(a \cdot f)\left(Ad(\exp(X_1))X_2\right)|_{t=0} \quad \text{(by (73))} \\
&= (a \cdot f)([X_1, X_2]) = f\left(Ad(a^{-1})[X_1, X_2]\right). \qquad (76)
\end{aligned}$$

On the other hand

$$\begin{aligned}
J([X_1, X_2])(\pi(a)) &= \psi_{[X_1,X_2]}(\pi(a)) \quad \text{(by (75))} \\
&= f\left(Ad(a^{-1})[X_1, X_2]\right) \quad \text{(by (73))} \qquad (77)
\end{aligned}$$

which proves that $[J(X_1), J(X_2)] = J([X_1, X_2])$. □

We are now in position to state the Duistermaat-Heckman formula – in a form directly derivable from Theorem 1.

Theorem 5. *Suppose as above that (M, σ, J) is a Hamiltonian $G-$space where G and M are compact. Orient M by the Liouville form ω_σ in (34). Then for $c \in \mathbb{C}$ and for $X \in \mathfrak{g}$ with X^* non-degenerate, we have*

$$\int_M e^{cJ(X)}\omega_\sigma = \left(\frac{2\pi}{c}\right)^n \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}. \qquad (78)$$

Here, as in Theorem 1, some $G-$invariant Riemannian metric $<,>$ on M has been selected, and the square-root in (78) is that in (24).

Proof. The proof of (78) is quite simple, given Theorem 1. Namely, given the lifting J (where we have noted that (5.4) implies (35)) let $c_J(X) = \left[e^{c\tau^X}\right]$ be the cohomology class constructed in Theorem 2, for $c \in \mathbb{C}$, $X \in \mathfrak{g}$. By (31) and (59)

$$p^* c_J(X) = e^{cJ(X)(p)} \quad \text{for } X_p^* = 0, \qquad (79)$$

and by (27) and (59)

$$\int_M c_J(X) = (sc)^n \int_M e^{cJ(X)} \frac{\sigma^n}{n!} = (sc)^n \int_M e^{cJ(X)}\omega_\sigma. \qquad (80)$$

On the other hand given that X^* is non-degenerate, the localization formula (32) gives

$$\int_M c_J(X) = (-1)^{\frac{n}{2}} \sum_{\substack{p\in M,\\ p=\text{a zero of } X^*}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}, \tag{81}$$

by (79). That is, by (80) and (81) we obtain exactly formula (78), as desired. □

Note that for $X \in \mathfrak{g}$, $Y \in VM$, and $p \in M$,

$$\begin{aligned} dJ(X)_p(Y_p) &= [dJ(X)(Y)](p) = [(-\iota(X^*)\sigma)(Y)](p) \text{ (as } J \text{ satisfies (35))} \\ &= -\sigma(X^*, Y)(p) \text{ (by (3))} = -\sigma_p(X^*_p, Y_p). \end{aligned} \tag{82}$$

Hence $dJ(X)_p = 0$ if $X^*_p = 0$, and conversely $dJ(X)_p = 0 \Rightarrow X^*_p = 0$ since σ_p is non-degenerate. (78) can therefore be expressed as

$$\int_M e^{cJ(X)}\omega_\sigma = \left(\frac{2\pi}{c}\right)^n \sum_{\substack{p\in M,\\ p=\text{a critical point of } J(X)}} \frac{e^{cJ(X)(p)}}{[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}}}, \tag{83}$$

where the critical points of $J(X)$ are those where $dJ(X)$ vanishes.

Recall that the asymptotic behaviour of an oscillatory integral $\mathcal{I}(f,t) = \int_M e^{\sqrt{-1}tf(x)}dx$, M = some space, for large t is given by the stationary-phase approximation - the dominant terms of this approximation being governed by the critical points of the phase $f(x)$. If we choose $c = \sqrt{-1}t$, for $t \in \mathbb{R}$, in (83), in particular, we see that the D-H formula can be viewed as an exactness result in a stationary-phase approximation of the integrals $\int_M e^{\sqrt{-1}tJ(X)}\omega_\sigma$, as our remarks of Section 1 indicated. For extended and much broader discussions of material introduced here, the two references [6, 40] are especially recommended. The reference [40] in particular serves as a vast source of information for the needs of physicists. Further reading of interest is found in the references [22, 2, 41, 39, 32, 28, 42, 29, 43, 14, 30, 16, 12, 31, 37].

6. Harish-Chandra, Itzykson-Zuber integral formulas

Some very practical and beautiful applications of the general D-H localization formula (Theorem 5) result when it is specialized, for example, to maximal dimensional co-adjoint orbits $\mathcal{O}$. To be specific, of special interest is the case where $\mathcal{O}$ is a quotient G/T of a compact, connected Lie group G modulo a maximal torus T. For physically important reasons one often concentrates on the case of the unitary group $G = U(n)$. Integration over matrix groups G, which amounts to integration over G/T for $T-$invariant functions, has well-known importance for diverse areas as quantum gravity [17], integrable systems [33], quantum chromodynamics [15], etc. The Itzykson-Zuber (I-Z) integration formula (integration over $U(n)$) [20], for example, occurs crucially in matrix models (the Ising model on a random surface) where one considers the coupling of conformal matter with two-dimensional quantum gravity. This formula also appears in work on higher-dimensional lattice gauge theories. Harish-Chandra-Itzykson-Zuber integration over the symplectic group G is involved in the computation of the mean of products of characteristic polynomials of random matrices in certain ensembles [15]. Specialization of the D-H formula also

leads to the Kirillov integral formula for irreducible representations of G, which has relevance for geometric quantization theory.

We establish in this section therefore the useful reductions of Theorem 5 in the case of G/T, and we carry out calculations in the special, but important, case $G = U(n)$. As a new localization formula will be presented in the next section (due to the second named author) for a *non-compact* group, it is helpful here to gain further understanding of localization in the compact case.

G will denote a compact, connected Lie group with Lie algebra $\mathfrak{g}$ on which G acts via the adjoint representation Ad. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ (i.e. $\mathfrak{t}$ is a maximal abelian subalgebra) and an $Ad(G)$-invariant innner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. For $\mathfrak{z}$ the center of $\mathfrak{g}$, one has decompositions

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 = \mathfrak{t} \oplus \mathfrak{m}, \qquad \mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_1 \tag{84}$$

for $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ the commutator of $\mathfrak{g}$, $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{g}_1$, $\mathfrak{m} = [\mathfrak{t}_1, \mathfrak{g}_1]$. If $T = \exp \mathfrak{t}$, which is a maximal torus in G with Lie algebra $\mathfrak{t}$, then $\mathfrak{m}$ is $Ad(T)-$invariant which means that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}$ induces a natural $G-$invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on $M \stackrel{\text{def}}{=} G/T$, where one can regard $\mathfrak{m}$ as the tangent space to M at its origin 0. An element $x \in \mathfrak{g}$ is called a *regular* element if its centralizer $\mathfrak{g}_x \stackrel{\text{def}}{=} \{y \in \mathfrak{g} | [y, x] = 0\}$ is of minimal dimension among all other centralizers: $\dim \mathfrak{g}_x \leq \dim \mathfrak{g}_y$ for all $y \in \mathfrak{g}$. It is known that one can choose a regular element $x_0 \in \mathfrak{t}_1$ such that $\mathfrak{t} = \mathfrak{g}_{x_0}$, $T = G_{f_{x_0}}$ for $G_{f_{x_0}}$ given in (70) where $f_{x_0} \in \mathfrak{g}^*$ (the dual space of $\mathfrak{g}$) is given by $f_{x_0}(x) = \langle x, x_0 \rangle$ for $x \in \mathfrak{g}$. Thus M is a co-adjoint orbit $\mathcal{O}$ of maximal dimension, since $\mathfrak{g}_{x_0}$ is of minimal dimension.

The key ingredient in the D-H formula (78) is the determinant there and its square root. In order to describe these in the present setting a bit more notation is necessary. Choose a system of positive roots P contained in the roots $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, the complexifications of $(\mathfrak{g}, \mathfrak{t})$. $\mathfrak{g}_1^{\mathbb{C}}$ is semisimple, $\mathfrak{t}_1^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_1^{\mathbb{C}}$, and P is related to a system of positive roots Δ^+ in the root system $\Delta = \Delta(\mathfrak{g}_1^{\mathbb{C}}, \mathfrak{t}_1^{\mathbb{C}})$ of $(\mathfrak{g}_1^{\mathbb{C}}, \mathfrak{t}_1^{\mathbb{C}})$ by a bijection $\alpha \leftrightarrow \tilde{\alpha} : \Delta^+ \leftrightarrow P$ where $\tilde{\alpha}(Z + x) \stackrel{\text{def}}{=} \alpha(x)$ for $Z + x \in \mathfrak{z}^{\mathbb{C}} \oplus \mathfrak{t}_1^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}}$. The main point here is that one can choose an orthonormal basis $\mathcal{B} = \{e_\alpha, f_\alpha\}_{\alpha \in \Delta^+}$ of $\mathfrak{m}$ such that

$$[H, e_\alpha] = -\sqrt{-1}\alpha(H) f_\alpha, \qquad [H, f_\alpha] = \sqrt{-1}\alpha(H) e_\alpha \qquad \text{for } H \in \mathfrak{t}_1 \tag{85}$$

(by compact Lie group structure theory), so that if a particular ordering $\{\alpha\}_{j=1}^{n=\frac{1}{2}\dim M}$ of Δ^+ is picked then the matrix of $ad_H : \mathfrak{m} \to \mathfrak{m}$ relative to $\mathcal{B}$ assumes the form

$$ad_H = \begin{pmatrix} 0 & \sqrt{-1}\alpha_1(H) & & & \\ -\sqrt{-1}\alpha_1(H) & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sqrt{-1}\alpha_n(H) \\ & & & -\sqrt{-1}\alpha_n(H) & 0 \end{pmatrix}, \tag{86}$$

as in equation (11), for $H \in \mathfrak{t}_1$. Moreover an element $X \in \mathfrak{t}$ is regular $\iff$ $\alpha(X) \neq 0$ for all $\alpha \in \Delta$; and for a regular element $X \in \mathfrak{t}_1$, and $p \in M$ with $X_p^* = 0$ (see (23)), say $p = \pi(a)$ for $a \in G$, where $\pi : G \to M = G/T$ is the natural map, $\mathfrak{L}_p(X^*)$ in (8) considered as a map from $\mathfrak{m}$ to $\mathfrak{m}$ is calculated to be given by $\mathfrak{L}_p(X^*) = -ad_{Ad(a^{-1})X}$ where $X_p^* = 0 \Longrightarrow Ad(a^{-1})X \in \mathfrak{t}_1$. By (86) therefore, and

the discussion in Sections 2, 3 one sees that X^* is nondegenerate and one can make the following choice of square root of $\det \mathfrak{L}_p(X^*)$, again for $p = \pi(a)$, $a \in G$:

$$[\det \mathfrak{L}_p(X^*)]^{\frac{1}{2}} = \prod_{\alpha \in \Delta^+} \alpha(\sqrt{-1} Ad(a^{-1})X), \tag{87}$$

where we assume that $\mathcal{B}$ is positively oriented with respect to the form $(\omega_\sigma)_0$ where ω_σ is the Liouville form

$$\omega_\sigma = \frac{\overbrace{\sigma \wedge \ldots \wedge \sigma}^{\text{n-times}}}{n!} \tag{88}$$

in (34), σ being the symplectic form on the orbit $\mathcal{O} = M = G/T = G/G_{f_{x_0}}$ and (again) 0 being the origin T of M. The latter assumption is satisfied if x_0 satisfies

$$\sqrt{-1}\alpha(x_0) > 0, \qquad \forall \alpha \in \Delta^+. \tag{89}$$

The final point to make here is that (by computation) $X^*_{p=\pi(a)} = 0$ (for any regular element $X \in \mathfrak{t}$) if and only if

$$a \in N_G(\mathfrak{t}) \stackrel{\text{def}}{=} \{a \in G |\, Ad(a)\mathfrak{t} = \mathfrak{t}\}, \tag{90}$$

the normalizer of $\mathfrak{t}$ in G. Here $N_G(\mathfrak{t})$ is also the normalizer

$$N_G(T) \stackrel{\text{def}}{=} \{a \in G \,|\, aTa^{-1} = T\} \tag{91}$$

of T in G. On the other hand, the Weyl group of (G,T) is $W \stackrel{\text{def}}{=} N_G(T)/T$ and one sees therefore that the map $W \to M$ given by $w \mapsto p = \pi(a)$ for a coset $w = aT \in W$, $a \in N_G(T)$, is a well-defined bijection of W onto the set $F^X \stackrel{\text{def}}{=} \{p \in M \,|\, X^*_p = 0\}$, which is the set that one sums over in (78). Recalling that $T = G_{f_{x_0}}$, one has that the Hamiltonian lift $J : g \to \mathcal{C}^\infty(M)$, given in general by (73), (75), is given in the present situation by

$$J(Y)(aT) \stackrel{\text{def}}{=} f_{x_0}(Ad(a^{-1})Y) \stackrel{\text{def}}{=} \langle Ad(a^{-1})Y, x_0\rangle \quad \text{for } a \in G. \tag{92}$$

Putting the pieces together we see that the D-H localization formula (78) reduces to the following concrete formula for $M = G/T$, $c \in \mathbb{C}$, $\dim M = 2n$:

$$\begin{aligned} &\int_{G/T} e^{c\langle Ad(a^{-1})X, x_0\rangle} \cdot \frac{\overbrace{\sigma \wedge \ldots \wedge \sigma}^{\text{n-times}}}{(2\pi)^n n!}(aT) \\ &= c^{-n} \sum_{w \in W,\, w = aT,\, a \in N_G(T)} \frac{e^{c\langle Ad(a^{-1})X, x_0\rangle}}{\prod_{\alpha \in \Delta^+} \alpha(\sqrt{-1} Ad(a^{-1})X)} \\ &= c^{-n} \sum_{w \in W} \frac{e^{c(w \cdot \lambda_{x_0})(X)}}{\prod_{\alpha \in \Delta^+} (w \cdot \alpha)(\sqrt{-1}X)}, \end{aligned} \tag{93}$$

where $\lambda_{x_0} \in \mathfrak{t}_1^*$ is given by $\lambda_{x_0}(H) \stackrel{\text{def}}{=} \langle H, x_0\rangle$ for $H \in \mathfrak{t}_1$ (i.e. $\lambda_{x_0} \doteq f_{x_0}|_{\mathfrak{t}_1}$) and

$$(w \cdot \lambda)(H) \stackrel{\text{def}}{=} \lambda(Ad(a^{-1})H) \qquad \text{for } w = aT, \quad a \in N_G(T). \tag{94}$$

Here we also assume that (as in (89)) x_0 satisfies the positivity condition $\sqrt{-1}\alpha(x_0) > 0$ for all $\alpha \in \Delta^+$.

We have noted that $y \in \mathfrak{t}_1$ is regular $\iff$ $\alpha(y) \neq 0$ for all $\alpha \in \Delta^+$, and that $\alpha \leftrightarrow \widetilde{\alpha}$ is a bijection of Δ^+ and P. Since for $Y = Z + y \in \mathfrak{z} \oplus \mathfrak{t}_1 = \mathfrak{t}$ one has that $\mathfrak{g}_Y = \mathfrak{g}_y$, it follows that Y is regular $\iff$ y is regular $\iff$ $\beta(Y) \neq 0$ for all $\beta \in P$, and the localization formula in (93) *extends* directly from regular elements X, x_0 in $\mathfrak{t}_1$ to regular elements X, x_0 in $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_1$: for $\omega \stackrel{\text{def}}{=} [(2\pi)^n n!]^{-1} \overbrace{\sigma \wedge \ldots \wedge \sigma}^{\text{n-times}}$,

$$\int_{G/T} e^{c\langle Ad(a^{-1})X, x_0\rangle} \omega(aT) = c^{-n} \sum_{w \in W} \frac{e^{c(w \cdot \lambda_{x_0})(X)}}{\prod_{\beta \in P}(w \cdot \beta)(\sqrt{-1}X)} \tag{95}$$

for x_0 satisfying $\sqrt{-1}\beta(x_0) > 0$ for all $\beta \in P$, where $\lambda_{x_0} \in \mathfrak{t}^*$ is given by $\lambda_{x_0}(H) \stackrel{\text{def}}{=} \langle H, x_0\rangle$ for $H \in \mathfrak{t}$ and $(w \cdot \lambda)(Y) \stackrel{\text{def}}{=} \lambda(Ad(a^{-1})Y)$ for $Y \in \mathfrak{t}^{\mathbb{C}}$, $w = aT \in W$, and $a \in N_G(T)$. The symplectic structure σ on G/T is given by Theorem 3. Thus in (95) we have arrived at Harish-Chandra's integral formula [19, 5, 6], which in essence computes the Fourier transform of the measure ω.

For the unitary group $G = U(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{u}(n) =$ the space of skew Hermitian matrices of degree n, one has the following data: $\langle X, Y\rangle \stackrel{\text{def}}{=} -\operatorname{Tr} XY$ for $X, Y \in \mathfrak{g}$, $\mathfrak{t} =$ the space of diagonal matrices with entries $\sqrt{-1}\theta_1, \ldots, \sqrt{-1}\theta_n$ for the $\theta_j \in \mathbb{R}$, $T =$ the group of diagonal matrices with entries $e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_n}$, $\mathfrak{g}_1 = \mathfrak{su}(n)$, $\mathfrak{t}_1 =$ matrices in $\mathfrak{t}$ with zero trace, $\mathfrak{z} =$ matrices in $\mathfrak{t}$ with all entries equal, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_1^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \{\widetilde{\alpha}_{rs}\}_{r \neq s}$, where $\widetilde{\alpha}_{rs}(H) = H_r - H_s$ for $\operatorname{diag}(H_1, \ldots, H_n)$ the space of complex diagonal matrices, $\Delta(\mathfrak{g}_1^{\mathbb{C}}, \mathfrak{t}_1^{\mathbb{C}}) =$ the set of restrictions of elements of $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ to the trace zero matrices $\mathfrak{t}_1^{\mathbb{C}}$ in $\mathfrak{t}^{\mathbb{C}}$.

$$P = \{\widetilde{\alpha}_{rs} |\, 1 \leq r < s \leq n\}, \qquad \Delta^+ = \{\widetilde{\alpha}|_{\mathfrak{t}_1^{\mathbb{C}}} \mid \widetilde{\alpha} \in P\}. \tag{96}$$

For $1 \leq j \leq n$, let $H_j \in \mathfrak{t}$ be the element with zero diagonal entries except the j^{th} entry which is $\sqrt{-1}$. If $a \in N_G(T)$ then $aH_ja^{-1} = H_{\sigma(j)}$ for some permutation σ of the set $\{1, 2, \ldots, n\}$, since $Ad(a)H_j = aH_ja^{-1} \in \mathfrak{t}$ has the same eigenvalues of H_j. One can show that the map $a \mapsto \sigma$ defines an isomorphism $aT \mapsto \sigma$ of the Weyl group $W = N_G(T)/T$ of $U(n)$ onto the symmetric group S_n on n letters such that for $(\lambda, H) \in (\mathfrak{t}^{\mathbb{C}})^* \times \mathfrak{t}^{\mathbb{C}}$ the action of $w \in W$,

$$(w \cdot \lambda)(H) \stackrel{\text{def}}{=} \lambda(Ad(a^{-1})H) = \lambda(a^{-1}Ha), \qquad w = aT, \tag{97}$$

goes over to the action of S_n on $(\mathfrak{t}^{\mathbb{C}})^*$ given by

$$(\sigma \cdot \lambda)(H = \operatorname{diag}(H_1, \ldots, H_n)) = \lambda\left(\operatorname{diag}(H_{\sigma(1)}, \ldots, H_{\sigma(n)})\right). \tag{98}$$

Sometimes it is convenient to change signs and work with $\sigma^- \stackrel{\text{def}}{=} -\sigma$ in place of σ. Then J, ω_σ are replaced by $J^- = -J$, $\omega_{\sigma^-} = (-1)^n\omega_\sigma$. Formula (95) for the choice $c = -\sqrt{-1}$ then assumes the form

$$\int_{G/T} e^{\sqrt{-1}\langle Ad(a^{-1})x, x_0\rangle} \cdot \frac{\omega_{\sigma^-}(aT)}{(2\pi)^n} = \sum_{w \in W} \frac{e^{\sqrt{-1}(w \cdot \lambda_{x_0})(x)}}{\prod_{\beta \in P}(w \cdot \beta)(x)} \tag{99}$$

for regular elements $x, x_0 \in \mathfrak{t}$, but where we now assume that $i\beta(x_0) \stackrel{\eta}{<} 0$ for all $\beta \in P$. That is, for

$$x_0 = \operatorname{diag}\left(\sqrt{-1}t_1, \ldots, \sqrt{-1}t_n\right), \quad x = \operatorname{diag}\left(\sqrt{-1}\theta_1, \ldots, \sqrt{-1}\theta_n\right) \in \mathfrak{t} \tag{100}$$

the regularity condition is that the diagonal entries are all distinct, and condition η is that $t_1 > t_2 > \ldots > t_n$.

If we write $\sigma(x) \stackrel{\text{def}}{=} \text{diag}\,(\sqrt{-1}\theta_{\sigma(1)}, ..., \sqrt{-1}\theta_{\sigma(n)})$ for $\sigma \in S_n$, then for

$$P(n) \stackrel{\text{def}}{=} \frac{1}{2}n(n-1) = \frac{1}{2}\text{dim}U(n)/T, \tag{101}$$

we obtain from the above remarks and data for $U(n)$ its localization formula:

$$\begin{aligned} & \int_{U(n)/T} e^{-\sqrt{-1}\text{Tr}(xax_0a^{-1})} \cdot \frac{\overbrace{\sigma^- \wedge \ldots \wedge \sigma^-}^{\text{P(n)-times}}}{P(n)!(2\pi)^{P(n)}}(aT) \\ = & \frac{1}{(-1)^{P(n)/2}} \frac{1}{\prod_{r<s}(\theta_r - \theta_s)} \sum_{\sigma \in S_n} (\text{sgn}\sigma) e^{-\sqrt{-1}\text{Tr}(\sigma(x)x_0)} \\ = & \frac{1}{(-1)^{P(n)/2}} \frac{1}{\prod_{r<s}(\theta_r - \theta_s)} \det\left[e^{\sqrt{-1}\theta_i t_j}\right] \end{aligned} \tag{102}$$

by (99), where we have used that

$$\prod_{r<s}(\theta_\sigma(r) - \theta_\sigma(s)) = (\text{sgn}\,\sigma) \prod_{r<s}(\theta_r - \theta_s) \tag{103}$$

and the expansion $\det A = \sum_{\sigma \in S_n} (\text{sgn}\,\sigma) A_{1\sigma(1)} \ldots A_{n\sigma(n)}$ of a determinant.

Formula (102) leads to the Itzykson-Zuber formula as we now indicate. Define a measure μ_0 on G by

$$\int_G f(a)\, d\mu_0(a) = \int_{G/T} \left[\int_T f(at)\, dt\right] \frac{\omega_\sigma^-(aT)}{(2\pi)^{\frac{1}{2}\text{dim}\, G/T}} \tag{104}$$

where dt denotes normalized Haar measure on T: $\int_T 1\, dt = 1$; f is any continuous function on G. ¿From the G-invariance of σ (see the remarks following Theorem 3) it follows that ω_{σ^-} is also G-invariant and that μ_0 is therefore a Haar measure on G. The choice $f = 1$ gives $\int_G 1\, d\mu_0 = v(G/T)$ for

$$v(G/T) \stackrel{\text{def}}{=} \int_{G/T} \frac{\omega_{\sigma^-}}{(2\pi)^{\frac{1}{2}\text{dim}G/T}}, \tag{105}$$

which means that $\mu \stackrel{\text{def}}{=} \mu_0[v(G/T)]^{-1}$ is normalized Haar measure on G. To compute $v(G/T)$ for $G = U(n)$ choose $\theta_j = \varepsilon(n-j)$ in (100) for x. Then the determinant in (102) is Vandermonde's determinant $= \prod_{r<s}(e^{\sqrt{-1}\varepsilon t_r} - e^{\sqrt{-1}\varepsilon t_s})$, and $\theta_r - \theta_s = \varepsilon(s-r)$. The right hand side of formula (102) becomes

$$\frac{1}{(-1)^{P(n)/2}} \frac{1}{\prod_{r<s}(s-r)} \prod_{r<s} \frac{(e^{\sqrt{-1}\varepsilon t_r} - e^{\sqrt{-1}\varepsilon t_s})}{\varepsilon} \tag{106}$$

whose limit as $\varepsilon \to 0$ is (by L'Hospital's rule) $\prod_{r<s} \frac{t_r - t_s}{s-r}$. On the other hand, the limit as $\varepsilon \to 0$ of the left hand side of formula (102) is $v(U(n)/T)$, since $x \to 0$ as $\varepsilon \to 0$. Thus we see that (for $P(n) \stackrel{\text{def}}{=} \frac{1}{2}n(n-1) = \frac{1}{2}\text{dim}U(n)/T$)

$$v(U(n)/T) \stackrel{\text{def}}{=} \int_{U(n)/T} \frac{\omega_{\sigma^-}}{(2\pi)^{P(n)}} = \prod_{r<s} \frac{t_r - t_s}{s-r} = \frac{\prod_{r<s}(t_r - t_s)}{\prod_{k=0}^{n-1} k!} \tag{107}$$

is the symplectic volume of the co-adjoint orbit defined by x_0 in (100). Finally, in (102) choose $f(a) \stackrel{\text{def}}{=} e^{-\sqrt{-1}\text{Tr}(xax_0a^{-1})}$ for x, x_0 in (100). Then for $b \in T$, $f(ab) =$

$f(a)$ as b commutes with x_0. Keeping in mind that $\mu \stackrel{\text{def}}{=} \mu_0[v(U(n)/T)]^{-1}$ is normalized Haar measure on $U(n)$, we obtain from equations (102), (104), (107)

$$\begin{aligned}
&\int_{U(n)} e^{-\sqrt{-1}\mathrm{Tr}(xax_0a^{-1})}\, d\mu(a) \\
&= \frac{1}{v(U(n)/T)} \int_{U(n)/T} e^{-\sqrt{-1}\mathrm{Tr}(xax_0a^{-1})} \cdot \frac{\omega_{\sigma^-}(aT)}{(2\pi)^{P(n)}} \\
&= \frac{(\prod_{k=0}^{n-1} k!) i^{-P(n)}}{\prod_{r<s}(t_r - t_s)} \frac{1}{\prod_{r<s}(\theta_r - \theta_s)} \det\left[e^{e^{\sqrt{-1}}\theta_i t_j}\right], \qquad (108)
\end{aligned}$$

which is the Itzykson-Zuber formula [20], obtained here under the very general umbrella of localized equivariant cohomology – and under the above assumptions that x, x_0 in (100) satisfy $t_1 > t_2 > \ldots > t_n$ with $\theta_r \neq \theta_s$ for $r \neq s$.

Formula (108) can be formulated in the general context of an arbitrary compact, connected Lie group G that we have been considering. For this we need a formula for $v(G/T)$ which replaces that in (107) for $U(n)$. It is given as follows. Given $\beta \in P$ there is a unique element $H_\beta \in \sqrt{-1}\mathfrak{t}_1$ such that for every $H \in t^{\mathbb{C}}$, $\beta(H) = \langle H, H_\beta \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{g}^{\mathbb{C}}$ that naturally extends $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. If $2\delta_P \stackrel{\text{def}}{=} \sum_{\beta \in P} \beta$, then

$$v(G/T) \stackrel{\text{def}}{=} \int_{G/T} \frac{\omega_{\sigma^-}}{(2\pi)^{\frac{1}{2}\dim G/T}} = \prod_{\beta \in P} \frac{\beta(-\sqrt{-1}x_0)}{\delta_P(H_\beta)}, \qquad (109)$$

again for $x_0 \in \mathfrak{t}$ regular with $\sqrt{-1}\beta(x_0) < 0$ for all $\beta \in P$. For $G = U(n)$, for example, with x_0 in (100), we have seen that $P = \{\widetilde{\alpha_{rs}} \mid 1 \leq r < s \leq n\}$ where $\widetilde{\alpha_{rs}}(H) = H_r - H_s$ for $H = \mathrm{diag}\,(H_1, ..., H_n) \in \mathfrak{t}^{\mathbb{C}}$. Therefore $\prod_{\beta \in P} \beta(-\sqrt{-1}x_0) = \prod_{r<s}(t_r - t_s)$. Also $\langle Z, W \rangle = \mathrm{Tr}(Z\overline{W}^t)$ for $Z, W \in \mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$, $H_\beta = \mathrm{diag}(0, ..., 1, ..., -1, ..., 0)$ for $\beta = \widetilde{\alpha_{rs}}$, where $1, -1$ appear in the r^{th} and s^{th} row, and $\delta_P(H_\beta) = s - r$. Thus in this case formula (109) reduces to formula (107). Given (109) one can now repeat the argument that followed (107), where now one takes $f(a) = e^{\sqrt{-1}\langle Ad(a^{-1})x, x_0\rangle}$ in (104) for $x \in \mathfrak{t}$ also regular. For $t \in T$ we still have $f(at) = f(a)$ since

$$\langle Ad((at)^{-1})x, x_0\rangle = \langle Ad(t^{-1})Ad(a^{-1})x, x_0\rangle = \langle Ad((a^{-1})x, Ad(t)x_0\rangle \qquad (110)$$

(by the $Ad(G)$–invariance of $\langle \cdot, \cdot \rangle$) $= \langle Ad(a^{-1})x, x_0\rangle$, as

$$T = \{b \in G \mid Ad(b)x = y\ \forall y \in \mathfrak{t}\}. \qquad (111)$$

Following the argument exactly as given for $U(n)$ one obtains (again for $\sqrt{-1}\beta(x_0) < 0$ for all $\beta \in P$) by (99), (104), (109)

$$\int_G e^{\sqrt{-1}\langle Ad(a^{-1})x, x_0\rangle}\, d\mu(a) = \prod_{\beta \in P} \frac{\delta_P(H_\beta)}{\beta(-\sqrt{-1}x_0)} \sum_{w \in W} \frac{e^{\sqrt{-1}(w \cdot \lambda_{x_0})(x)}}{\prod_{\beta \in P}(w \cdot \beta)(x)}, \qquad (112)$$

which is a formulation of the Itzykson-Zuber formula for an arbitrary compact, connected Lie group G.

7. Localization formula for non-compact group actions

In this section we describe a generalization of the B-V localization formula to non-compact group actions. In the classical localization formula (32) it was assumed that both the manifold M and the group G are compact. The compactness of

M ensures convergence of the integral $\int_M [\tau]$, and the compactness of G implies existence of a $G-$invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on M which was used in the proof of (32). When G is not compact such a Riemannian metric may not exist. Now, pick an element $X \in \mathfrak{g}$. Then M being compact and the space $Z(M, X, s) =$ kernel of $d_{X,s}$ on $\Lambda_X M$ being non-zero together imply that the vector field X^* comes from the action of some *compact* group G' and we could apply the B-V localization formula (32) to G' instead of G. Thus, in order to have a truly new result where the action of G does not factor through action of some compact group we must allow non-compact manifolds.

At a first glance it appears that the formula fails when G is not compact. For example, let us consider $G = SL(2, \mathbb{R})$ and let us take an element f in the dual of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ defined by

$$f : \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto b - c. \tag{113}$$

Let $\mathcal{O} \subset \mathfrak{sl}(2, \mathbb{R})^*$ denote the co-adjoint orbit of f. Like all co-adjoint orbits, $\mathcal{O}$ possesses a canonical symplectic structure σ which is the top degree part of the equivariantly closed form $s^{-1}\tau^X = s^{-1}(J(X), 0, s\sigma)$. Although $\mathcal{O}$ is not compact, the symplectic volume $\int_{\mathcal{O}} \sigma$ still exists as a distribution on $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathfrak{sl}(2, \mathbb{R})'_{split} \subset \mathfrak{sl}(2, \mathbb{R})$ be the open subset consisting of $X \in \mathfrak{sl}(2, \mathbb{R})$ with distinct real eigenvalues. In other words, $\mathfrak{sl}(2, \mathbb{R})'_{split}$ consists of all elements in $\mathfrak{sl}(2, \mathbb{R})$ conjugate to $\mathrm{diag}(\lambda, -\lambda)$, for some $\lambda \in \mathbb{R} - \{0\}$. Now, if we take any element $X \in \mathfrak{sl}(2, \mathbb{R})'_{split}$, then one can see that the vector field X^* on $\mathcal{O}$ generated by X has no zeroes. Thus, if there were a fixed point integral localization formula like in the case of compact groups, this formula would suggest that the distribution determined by $\int_{\mathcal{O}} \sigma$ vanishes on the open set $\mathfrak{sl}(2, \mathbb{R})'_{split}$. But it is known that the restriction of $\int_{\mathcal{O}} \sigma$ to $\mathfrak{sl}(2, \mathbb{R})'_{split}$ is *not* zero.

On the other hand, recent results from representation theory, namely the two character formulas for representations of reductive Lie groups due to M. Kashiwara, W. Rossmann, W. Schmid and K. Vilonen described in [34], [35] strongly suggest that the B-V localization formula should extend to actions of non-compact groups. Heuristically, the failure of the localization formula in the above example can be attributed to the lack of zeroes of X^*, as if they "ran away to infinity."

This discussion demonstrates two immediate challenges to having a localization formula when the acting group G is not compact. First of all we must allow non-compact manifolds M or homology cycles with infinite support. But then we need to worry about convergence of the integral $\int_M \tau$. We will resolve this problem by restricting the class of forms that we will integrate and by introducing a new (weaker) notion of convergence of integrals in the sense of distributions on $\mathfrak{g}$. Secondly, for arbitrary non-compact manifolds or cycles with infinite support, the zeroes of X^* tend to "run away to infinity." Since we cannot have a localization formula for all manifolds and cycles, we will specify a class of cycles for which all zeroes of X^* are accounted for and the localization formula holds.

The statement of the new localization formula uses the language of algebraic geometry. We consider pairs of Lie groups: a real group G sitting inside a complex

one $G_{\mathbb{C}}$. For example:

(114)

$$\begin{array}{cccccc} GL(n,\mathbb{R}) & \subset & GL(n,\mathbb{C}) & SL(n,\mathbb{R}) & \subset & SL(n,\mathbb{C}) \\ GL^{+}(n,\mathbb{R}) & \subset & GL(n,\mathbb{C}) & SO(n) & \subset & SL(n,\mathbb{C}) \\ U(n) & \subset & GL(n,\mathbb{C}) & SU(n) & \subset & SL(n,\mathbb{C}) \end{array} \qquad Sp(n,\mathbb{R}) \subset Sp(n,\mathbb{C})$$

More precisely, we fix a connected complex algebraic linear reductive Lie group $G_{\mathbb{C}}$ which is defined over $\mathbb{R}$. We will be primarily interested in a real Lie subgroup $G \subset G_{\mathbb{C}}$ lying between the group of real points $G_{\mathbb{C}}(\mathbb{R})$ and the identity component $G_{\mathbb{C}}(\mathbb{R})^0$.

Our ambient space will be the holomorphic cotangent space T^*M of a smooth complex projective variety M on which $G_{\mathbb{C}}$ acts algebraically. We will also assume that the maximal complex torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ (i.e. the maximal abelian subgroup of $G_{\mathbb{C}}$ isomorphic to the product of several copies of $\mathbb{C}-\{0\}$) acts on M with isolated fixed points. Then there are only finitely many fixed points because M is compact. Let σ denote the canonical complex algebraic holomorphic symplectic form on T^*M.

For a real closed submanifold $N \subset M$, we define the real conormal space

$$T^*_N M = \{\xi \in T^*M |\ \mathrm{Re}\,\xi|_{T_N} = 0\}. \tag{115}$$

The homology cycles over which we will integrate equivariant forms will include the real conormal spaces $T^*_N M$ associated to real closed $G-$invariant submanifolds $N \subset M$ and equipped with some orientation. An interesting example is $G = GL(n,\mathbb{R}) \subset GL(n,\mathbb{C}) = G_{\mathbb{C}}$ acting naturally on a complex Grassmanian $Gr_{\mathbb{C}}(k,n)$. Let N be the real Grassmanian $Gr_{\mathbb{R}}(k,n)$ sitting inside $Gr_{\mathbb{C}}(k,n)$ and the homology cycle $C = T^*_{Gr_{\mathbb{R}}(k,n)} Gr_{\mathbb{C}}(k,n)$.

The ordinary homology cycle is defined as a *finite* sum of simplices which has no boundary. Here we will consider chains C which are possibly *infinite* sums of simplices. In order to be able to compute the boundary of C we require that every point in the ambient space has an open neighborhood which intersects only finitely many simplices. Then the boundary ∂C makes sense and we say that a chain is a (Borel-Moore homology) cycle if $\partial C = 0$. We denote by $|C|$ the support of C. The Borel-Moore cycles $C \subset T^*M$ over which we will integrate will be subject to the following three properties:

- C is $G-$invariant;
- C is real Lagrangian, i.e. $\mathrm{Re}\,\sigma|_C \equiv 0$ and $\dim_{\mathbb{R}} C = \dim_{\mathbb{R}} M$;
- C is *conic*, i.e. invariant under the scaling action of positive reals $\mathbb{R}^{>0}$ on T^*M (but not necessarily under the actions of $\mathbb{C}-\{0\}$ or $\mathbb{R}-\{0\}$).

Intuitively, these cycles C consist of portions of real conormal spaces which piece together so that there is no boundary left.

Example 1. The first non-trivial example comes from the action of $G_{\mathbb{C}} = SL(2,\mathbb{C})$ on the projective space $M = \mathbb{C}P^1$ by projective transformations. (Recall that $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere; also $n = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{C}P^1 = 1$.) The group $G = SL(2,\mathbb{R})$ acts on $\mathbb{C}P^1$ with exactly three different orbits: two open hemispheres and one circle which is their common boundary. We can position $\mathbb{C}P^1$ in space so that the Eastern and Western hemispheres are stable under the $SL(2,\mathbb{R})-$action. Let H denote one of these two open hemispheres, say, the Western one; and let $S^1 \subset \mathbb{C}P^1$ denote the circle containing the Greenwich meridian; $S^1 = \partial H$.

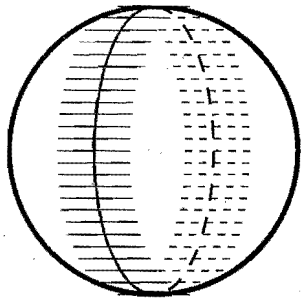


FIGURE 1. Cycle $T^*_{S^1}\mathbb{C}P^1$.

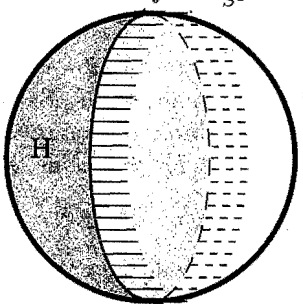


FIGURE 2. Cycle C_H.

One possible choice of the cycle C is the real conormal space $T^*_{S^1}\mathbb{C}P^1$ equipped with some orientation (Figure 1). Another interesting choice of C is the cycle

$$C_H = H \bigcup \{\xi \in T^*_{S^1}\mathbb{C}P^1 |\ \mathrm{Re}\langle \xi, v\rangle \geq 0\,, \quad \text{for all tangent vectors } v \text{ pointing outside of } H\} \tag{116}$$

(Figure 2). Its orientation is determined by the orientation on $\mathbb{C}P^1$ which induces the orientation on H and which in turn determines the orientation on all of C_H. □

Let U be a maximal compact subgroup of $G_{\mathbb{C}}$. For instance, if $G_{\mathbb{C}}$ is $GL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$ one can take the group U to be $U(n)$ or $SU(n)$ respectively. Then, letting $\mathfrak{u}$ and $\mathfrak{g}_{\mathbb{C}}$ denote the Lie algebras of U and $G_{\mathbb{C}}$ respectively, we have an isomorphism $\mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}_{\mathbb{C}}$. We denote by $\Lambda^{(p,q)}M$ the space of complex-valued differential forms of type (p,q) on M, that is the space of forms which are $p-$holomorphic and $q-$antiholomorphic. Recall that $n = \frac{1}{2}\dim_{\mathbb{R}} M$. We consider forms $\alpha^X = (\alpha^X_0, \alpha^X_1, \ldots, \alpha^X_{2n}) \in \Lambda M$ depending on $X \in \mathfrak{g}_{\mathbb{C}}$ and which satisfy the following three conditions:

- The assignment $X \mapsto \alpha^X \in \Lambda M$ depends holomorphically on $X \in \mathfrak{g}_{\mathbb{C}}$;
- For each $k \in \mathbb{N}$ and each $X \in \mathfrak{g}_{\mathbb{C}}$,

$$\alpha^X_{2k} \in \bigoplus_{\substack{p+q=2k \\ p \geq q}} \Lambda^{(p,q)}M;$$

- For each $X \in \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$, we have $\alpha^X \in Z(M, X, s)$, i.e.

$$d_{X,s}\alpha^X = 0 \qquad \text{and} \qquad \theta(X)\alpha^X = 0.$$

Example 2. A $U-$equivariant characteristic form $\alpha^X \in \Lambda M$ depending on $X \in \mathfrak{u}$ associated to a U-equivariant vector bundle over M (see Section 7.1 of [6]) satisfies the third condition. Since it depends on $X \in \mathfrak{u}$ polynomially, α^X extends uniquely from $\mathfrak{u}$ to $\mathfrak{g}_{\mathbb{C}}$ so that the first condition is satisfied. Finally, for each $X \in \mathfrak{g}_{\mathbb{C}}$,

$$\alpha^X \in \bigoplus_k \Lambda^{(k,k)} M, \tag{117}$$

so that the second condition is satisfied too. This is the most important class of forms satisfying these conditions. □

Let $J(X) : T^*M \to \mathbb{C}$ be the ordinary moment map:

$$J(X) : \xi \mapsto \langle \xi, X^* \rangle, \qquad \xi \in T^*M,\ X \in \mathfrak{g}_{\mathbb{C}}. \tag{118}$$

The integrals will be defined as distributions on $\mathfrak{g}$, so let $\varphi \in \mathcal{C}_c^\infty(\mathfrak{g})$ be a test function, and let dX denote the Lebesgue measure on $\mathfrak{g}$. The new localization formula will apply to integrals of the following kind:

$$\int_C \Big(\int_{\mathfrak{g}} e^{J(X)(\xi)+\sigma} \wedge \varphi(X)\alpha^X \, dX \Big)_{2n}, \qquad X \in \mathfrak{g},\ \xi \in |C| \subset T^*M. \tag{119}$$

The inside integral $\int_{\mathfrak{g}} e^{J(X)(\xi)+\sigma} \wedge \varphi(X)\alpha^X \, dX$ is essentially the Fourier transform of $\varphi(X)\alpha^X$ which decays rapidly in the imaginary directions of $\mathfrak{g}_{\mathbb{C}}^* \simeq \mathfrak{g}^* \oplus i\mathfrak{g}^*$. We denote by $\operatorname{supp}(\sigma|_C)$ the closure in T^*M of the set of smooth points of the support $|C|$ where $\sigma|_{|C|} \neq 0$. Then integral (119) converges provided that the moment map J, regarded as a map

$$J : T^*M \ni \xi \mapsto J(\cdot)(\xi) \in \mathfrak{g}_{\mathbb{C}}^*, \tag{120}$$

is *proper* on $\operatorname{supp}(\sigma|_C)$ (meaning that the J-preimage of every compact set in $\mathfrak{g}_{\mathbb{C}}^*$ is compact in $\operatorname{supp}(\sigma|_C)$). In particular, (119) is well-defined when J is proper on $|C|$.

Now the main result of [23] says that if the support of φ lies in $\mathfrak{g}'$ ($\mathfrak{g}$ without a finite number of certain hypersurfaces) then the integral (119) can be rewritten as

$$\int_C \Big(\int_{\mathfrak{g}} e^{J(X)(\xi)+\sigma} \wedge \varphi(X)\alpha^X \, dX \Big)_{2n} = \int_{\mathfrak{g}} F_\alpha(X)\varphi(X)\, dX, \tag{121}$$

where F_α is a function on $\mathfrak{g}'$ given by the formula

$$F_\alpha(X) = (-2\pi s)^n \sum_{\substack{p \in M, \\ p = \text{a zero of } X^*}} m_p(X) \frac{\alpha_0^X(p)}{[\det \mathfrak{L}_p(X^*)]^{1/2}}, \tag{122}$$

and each $m_p(X)$ is a certain integer multiplicity. The function F_α is invariant under the action of $G \cap U$ obtained by restricting the adjoint action of G on $\mathfrak{g}$.

Remark 1. *Perhaps the most striking new feature of this localization formula is the presence of integer multiplicities $m_p(X)$'s. Each multiplicity $m_p(X)$ equals the local contribution of p to the Lefschetz fixed point formula, as generalized to sheaf cohomology by M. Goresky and R. MacPherson* [18]. *Sheaves are a generalization of the notion of vector bundles over a manifold. There is a recent construction due to M. Kashiwara which associates to each sheaf $\mathcal{F}$ on M a cycle in T^*M called the* characteristic cycle *of $\mathcal{F}$. For example, the characteristic cycle of a vector*

bundle over M of rank k is the manifold M itself regarded as a cycle in T^*M and taken with multiplicity k. Any cycle C satisfying the three conditions above can be realized as a characteristic cycle $Ch(\mathcal{F})$ of some G-equivariant sheaf $\mathcal{F}$ ([21], [36]). The multiplicities are determined in [23] *in terms of local cohomology of $\mathcal{F}$, where $\mathcal{F}$ is any sheaf with characteristic cycle $Ch(\mathcal{F}) = C$.*

Remark 2. *The reason why the localization formula is stated in terms of distributions is that when the support of C is not compact the integral*

$$\int_C (e^{J(X)(\xi)+\sigma} \wedge \alpha^X)_{2n} \tag{123}$$

practically never converges.

The set $\mathfrak{g}'$ is essentially the set of regular semisimple elements of $\mathfrak{g}$ on which the denominators $[\det \mathfrak{L}_p(X^*)]^{1/2}$ do not vanish.

In the special case when $C = M$ as oriented cycles, C is U-invariant, each multiplicity $m_p(X)$ equals 1 and this theorem can be easily deduced from the classical B-V localization formula (32).

Notice that the cycle C is invariant with respect to the action of the group G which need not be compact, while the form $\alpha^X \in \Lambda M$, $X \in \mathfrak{g}_{\mathbb{C}}$, is required to be equivariant with respect to a different group U only, and U may not preserve the cycle C.

This localization formula has many interesting applications. The most important of them is a geometric proof of the integral character formula for representations of real reductive Lie groups [24]. Article [25] gives a very accessible introduction to [24] and explains the key ideas used there by way of examples and illustrations.

Example 3. In the setting of Example 1 we consider the group $G_{\mathbb{C}} = SL(2,\mathbb{C})$ acting on the projective space $M = \mathbb{C}P^1$, and take $G = SL(2,\mathbb{R})$. In this situation the set $\mathfrak{g}'$ is the set of regular semisimple elements

$$\mathfrak{sl}(2,\mathbb{R})^{rs} = \{X \in \mathfrak{sl}(2,\mathbb{R}) |\ X \text{ has two distinct (real or complex) eigenvalues}\}. \tag{124}$$

The vector field generated by each $X \in \mathfrak{sl}(2,\mathbb{R})^{rs}$ has exactly two zeroes on $\mathbb{C}P^1$ located diameterally opposite to each other. The elements of $\mathfrak{sl}(2,\mathbb{R})^{rs}$ come in two flavors. We call an element $X \in \mathfrak{sl}(2,\mathbb{R})^{rs}$ *elliptic* if it has purely imaginary eigenvalues or, equivalently, if it is conjugate to $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ for some $\lambda \in \mathbb{R} - \{0\}$. We also call an element $X \in \mathfrak{sl}(2,\mathbb{R})^{rs}$ *split* if it has real eigenvalues or, equivalently, if it is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R} - \{0\}$.

Consider the cycle

$$C_H = H \bigcup \{\xi \in T^*_{S^1}\mathbb{C}P^1 |\ \mathrm{Re}\langle \xi, v\rangle \geq 0 \text{ for all tangent vectors } v \text{ pointing outside of } H\} \tag{125}$$

introduced in Example 1. If $X \in \mathfrak{sl}(2,\mathbb{R})^{rs}$ is elliptic, then the vector field X^* has one zero in the open hemisphere H and the other zero in the open hemisphere opposite to H. The multiplicity

$$m_p(X) = \begin{cases} 1 & \text{if } p \in H \\ 0 & \text{if } p \notin H \end{cases} \qquad X \text{ is elliptic, } p \text{ is a zero of } X^* \tag{126}$$

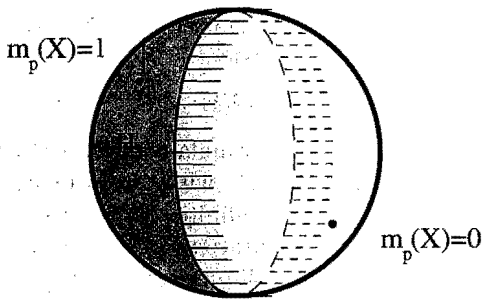


FIGURE 3. Zeroes of X^*, X elliptic.

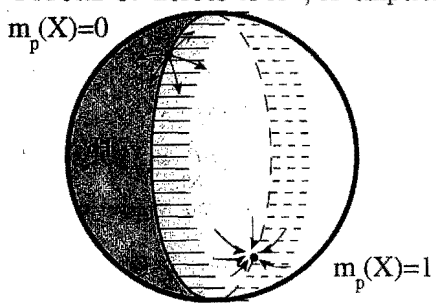


FIGURE 4. Zeroes of X^*, X split.

(Figure 3). This is hardly surprising since only those zeroes of X^* are expected to make any contribution to the integral which lie in the support of the cycle.

If $X \in \mathfrak{sl}(2,\mathbb{R})^{rs}$ is split, then both zeroes of the vector field X^* lie on the boundary $S^1 = \partial H$. While the zeroes appear to be symmetric at first, one of them counts and the other one does not. The symmetry is broken by the fact that one of these zeroes is stable (the vector field X^* points towards it) and the other zero is unstable (the vector field X^* points away from it). The multiplicity

$$m_p(X) = \begin{cases} 1 & \text{if } p \text{ is stable} \\ 0 & \text{if } p \text{ is unstable} \end{cases} \qquad X \text{ is split, } p \text{ is a zero of } X^* \tag{127}$$

(Figure 4). This phenomenon is new and does not have analogues in compact group actions. □

Another interesting application of the localization formula (122) is a generalization of the Riemann-Roch-Hirzebruch integral formula to $\mathcal{D}$-modules. Its statement can be found in [26] and it uses the language of $\mathcal{D}$-modules (sheaves of modules over the sheaf of linear differential operators), but its flavor can be illustrated by the following example.

As before, $G_{\mathbb{C}}$ is a connected complex algebraic linear reductive Lie group defined over $\mathbb{R}$ and acting algebraically on a smooth complex projective variety M, and $G \subset G_{\mathbb{C}}$ is a real Lie subgroup lying between the group of real points $G_{\mathbb{C}}(\mathbb{R})$ and the identity component $G_{\mathbb{C}}(\mathbb{R})^0$. Take the sheaf of sections $\mathcal{O}(\mathbf{E})$ of a $G_{\mathbb{C}}-$equivariant algebraic line bundle $(\mathbf{E}, \nabla_{\mathbf{E}})$ over a $G_{\mathbb{C}}-$invariant open algebraic subset $O \subset M$ with a $G_{\mathbb{C}}$-invariant algebraic flat connection $\nabla_{\mathbf{E}}$.

Let $O_{\mathbb{R}} \subset M$ be an open G-invariant subset (which may or may not be $G_{\mathbb{C}}$-invariant) and consider the cohomology spaces

$$H^p(O_{\mathbb{R}}, \mathcal{O}(\mathbf{E})). \tag{128}$$

The classical Riemann-Roch-Hirzebruch formula computes the index of $\mathbf{E}$, i.e. the alternating sum $\sum_p (-1)^p \dim H^p(O_{\mathbb{R}}, \mathcal{O}(\mathbf{E}))$ with $O_{\mathbb{R}} = O = M$. For general $O_{\mathbb{R}}$ and O, however, these dimensions can be infinite. To work around this problem we regard the vector spaces (128) as representations of G, and, as a substitute for the index, we ask for the character of the virtual representation $\sum_p (-1)^p H^p(O_{\mathbb{R}}, \mathcal{O}(\mathbf{E}))$. (Recall that for finite-dimensional representations the value of the character at the identity element $e \in G$ equals the dimension of the representation.) This character is given by the integral formula (119) with concrete choices of the cycle C and the form α^X.

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