

COMPLETE SEMI-DYNAMICAL SYSTEMS

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ABSTRACT. In this paper generalized vector fields are studied. The notion of left invariant vector fields is extended by the use of top spaces. As a result a new kind of dynamics is extrapolated. By the topological properties of topological generalized groups, topological complete semi-dynamical systems are studied. A method for constructing a complete semi-dynamical system on a generalized coset space is deduced. An application in the genetic space and the whether system are considered.

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1. INTRODUCTION

In this paper we would like to use generalized groups [12] in order to introduce a new kind of dynamics on top spaces [14]. As a result we will deduce a generalization of dynamical systems. We recall that a generalized group is a non-empty set T admitting an operation called multiplication, with the following properties:

- (i) $(xy)z = x(yz)$ for all x, y, z in T ;
- (ii) for each x in G there exists a unique z in G such that $xz = zx = x$ (we denote z by $e(x)$);
- (iii) for each x in T there exists y in T such that $xy = yx = e(x)$.

Recently Araújo and Konieczny [1] have proven that a generalized group is equivalent to a "completely simple semigroup [8]". This structure is not equivalent to completely regular semigroups or Vagner generalized groups [17]. The second condition of generalized groups also make them different from semi-groups admitting relative inverse [2, 3].

We employ Araújo and Konieczny's theorem [1] and choose the above definition for

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a completely simple semigroup. We also recall that [14] if X and Y are two non-empty sets, G is a group, and $s : Y \times X \rightarrow G$ is a mapping, then $P = X \times G \times Y$ with the multiplication $(x, g, y)(x', g', y') = (x, gs(y, x')g', y')$ is a completely simple semigroup which is called Rees matrix semigroup. In fact Rees' theorem [5, 7, 8] implies that each completely simple semigroup is isomorphic to a Rees matrix semigroup.

A generalized group T is called a topological generalized group [13] if it satisfies the following conditions:

1. T is a Hausdorff topological space;
2. Two mappings m_1 from T to T and m_2 from $T \times T$ to T are defined by $m_1(x) = x^{-1}$ and $m_2(x, y) = xy$, are continuous mappings.

A topological generalized group T is called a top space [10] if:

- i) The topological space T is a smooth manifold of dimension t ;
- ii) The mappings $m_1 : T \rightarrow T$ defined by $m_1(u) = u^{-1}$ and the mapping $m_2 : T \times T \rightarrow T$ defined by $m_2(u_1, u_2) = u_1 u_2$ are smooth maps.

Example 1.1 The n -dimensional Euclidean space $\mathbf{R}^n$ with the product

$$((x^1, \dots, x^n), (y^1, \dots, y^n)) \mapsto \left(\frac{nx^1 + \sum_{i=1}^n y^i}{n}, \dots, \frac{nx^n + \sum_{i=1}^n y^i}{n} \right)$$

is a top space which is not a Lie group with this product. The subset $G = \{(a, \dots, a) : a \in \mathbf{R}\}$ is a Lie group with this product. If F is a foliation $\{H_b : b \in \mathbf{R} \text{ and } H_b = \{(x^1, \dots, x^n) : x^1 + \dots + x^n = b\}\}$ then each leaves of this foliation attract a point in G and the product of top space makes a dynamics on G . We have illustrated this situation (in the case of $\mathbf{R}^2$) in figure 1.

Example 1.2 If $(H, <, >)$ is a finite dimensional real Hilbert space, then H with the product $xy = x + \langle u, y \rangle u$ is a top space, where u is a chosen unit vector.

In example 1.2 if x, y are from the hyperplane L orthogonal to u , then $xy = x$, which implies that this hyperplane is the left zero subsemigroup of idempotents. If x and y are in the span Ru of u , say $x = ru$ and $y = su$, then $xy = ru + \langle u, su \rangle u = ru + su = x + y$ and so here is one of subgroups. So H has a decomposition to a left zero semigroup and a group. In fact if $x = l_1 + g_1, y = l_2 + g_2$, where $g_i \in Ru$ and $l_i \in L$, then $xy = l_1 l_2 + g_1 g_2$.

We recall that [5], if X and Y are Hausdorff topological spaces, G is a topological group, and $s : Y \times X \rightarrow G$ is a continuous mapping, then the Rees matrix $P = X \times G \times Y$ is a topological generalized group which is called paragroup.

Theorem 1.1 If X and Y are smooth manifolds, G is a Lie-group, and $s : Y \times X \rightarrow G$ is a smooth mapping, then the paragroup $P = X \times G \times Y$ is a top space [14].

A vector field X on a top space T is called a left invariant vector field [14] if $(l_g)_*p(X(p)) = (X \circ l_g)(p)$ where $g, p \in T$, $l_g : T \rightarrow T$ is defined by $l_g(x) = gx$, and $(l_g)_*p$ is the derivative of l_g at p .

Example 1.3 The subset $T = \{(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) : 0 < \theta < 2\pi, 0 < \phi < \pi\}$ of S^2 with the product :

$$((\sin \phi_1 \cos \theta_1, \sin \phi_1 \sin \theta_1, \cos \phi_1), (\sin \phi_2 \cos \theta_2, \sin \phi_2 \sin \theta_2, \cos \phi_2)) \mapsto (\sin \phi_1 \cos \theta_2, \sin \phi_1 \sin \theta_2, \cos \phi_1)$$

is a top space [14] (see figure 2). In fact for all $p \in S^2$, $e(p) = p^{-1} = p$. The vector field $X : T \rightarrow \mathbf{R}^3$ is defined by $X(x, y, z) = (-y, x, 0)$ is a left invariant vector field

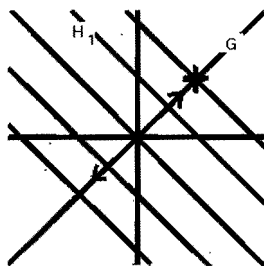


FIGURE 1

on T .

Theorem 1.2 X is a left invariant vector field on T if and only if $(l_g)_*e(p)(X(e(p))) = X(ge(p))$ for all $p, g \in T$ [14].

Theorem 1.3 Let T be a top space and let $e(T)$ be a finite set, then the set of left invariant vector fields on T is a Lie-algebra under the Lie bracket operation [14].

2. GENERALIZED VECTOR FIELDS

Generalized vector fields are means for creation of parameters for new dynamics. In a special case there is a physical history behind this notion. In fact if one considers a manifold in a vector space with a physical vector field then the restriction of physical vector field to that manifold will be a generalized vector field on that manifold. So a generalized vector field on a manifold may not be tangent to it. For the definition of a generalized vector field we assume that T and M are two C^{r+1} manifolds, and $T(T)$ is the tangent bundle of T . We are assuming that $V : T \rightarrow M$ and $\psi : T(T) \rightarrow M$ are C^r maps.

Definition 2.1 (V, M, ψ) is called a C^r generalized vector field on T if for all $p \in T$, there exists a C^1 map $\alpha : (-\epsilon, \epsilon) \rightarrow T$ (for some $\epsilon > 0$) so that $\alpha(0) = p$ and $V(\alpha(t)) = \psi(\alpha'(t))$. α is called a generalized integral curve passing through p . α may not be unique. For example if $M = \mathbb{R}^n$, $\psi = 0$ and $V = 0$ then there exists a lot of generalized integral curves.

One can imagine a generalized vector field as a source which is a manifold as a

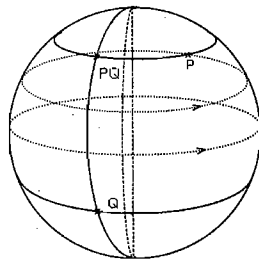


FIGURE 2

subset of another manifold.

Example 2.1 Let $T = S^1$, $M = \mathbf{R}^3$; $\psi(x, y) = (x, -y, x + y)$ and $V(x, y) = (-y, -x, -y + x)$, then $(V, \mathbf{R}^3, \psi)$ is a C^∞ generalized vector field on S^1 (see figure 3). One can imagine this generalized vector field as the velocity vectors of liquid which inject to the space by using of the mean which is similar to S^1 .

If (V, M, ψ) is a C^r generalized vector field then there exists a vector field $\tilde{V}$ (may not be unique) such that $\psi \circ \tilde{V} = V$.

Definition 2.2 A generalized vector field (V, M, ψ) is called a left invariant generalized vector field on T if there exists a left invariant vector field $\tilde{V}$ on T such that $\psi \circ \tilde{V} = V$.

Example 2.2 Let T be the top space which is introduced in example 1.3. Moreover let $V : T \rightarrow \mathbf{R}^4$ and $\psi : \mathbf{R}^3 \rightarrow \mathbf{R}^4$ be defined by $V(x, y, z) = (-y, x, -y, x - y)$ and $\psi(x, y, z) = (x, y + z, x + z, x + y)$ respectively. Then $(V, \mathbf{R}^4, \psi)$ is a left invariant generalized vector field.

Let $\tilde{V}$ be a left invariant vector field corresponding to a left invariant generalized vector field (V, M, ψ) on a top space T . Moreover let ψ be a one-to-one mapping. Then for all $g \in T$ we can define $\eta^g : M \rightarrow M$ in the form:

$$\eta^g(m) = \begin{cases} V(gt) & \text{if } m = V(t) \\ m & \text{if } m \notin \text{Im}V \end{cases} \quad (1)$$

Lemma 2.1 For all $g \in T$, the mapping η^g is well-defined.

Proof. Let $g \in T$ be given. Then we have two cases:

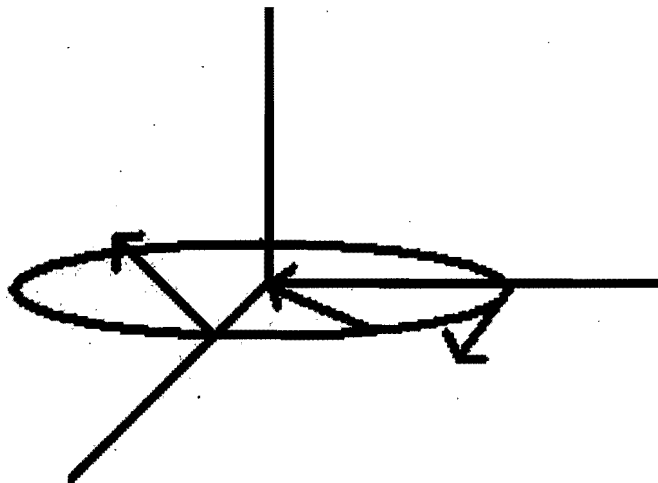


FIGURE 3

- i) If $m_1 = m_2 \notin \text{Im} V$, then $\eta^g(m_1) = m_1 = m_2 = \eta^g(m_2)$.
 ii) If $m_1 = m_2 \in \text{Im} V$, then there exist $t_1, t_2 \in T$ such that $m_1 = V(t_1)$ and $m_2 = V(t_2)$. So $\psi \circ \tilde{V}(t_1) = \psi \circ \tilde{V}(t_2)$.

On the other hand: $((l_g)_{*t_1}(\tilde{V}(t_1))) = (\tilde{V} \circ l_g)(t_1)$ and $((l_g)_{*t_2}(\tilde{V}(t_2))) = (\tilde{V} \circ l_g)(t_2)$. Hence $(\tilde{V} \circ l_g)(t_1) = (\tilde{V} \circ l_g)(t_2)$. Therefore $\tilde{V}(gt_1) = \tilde{V}(gt_2)$. So $V(gt_1) = V(gt_2)$. $\square$

Example 2.3 The set of real numbers $\mathbf{R}$ with addition is a Lie group. If $V : \mathbf{R} \rightarrow \mathbf{R}^2$ and $\psi : T(\mathbf{R}) \rightarrow \mathbf{R}^2$ defined by $V(x) = (0, x^2)$ and $\psi(x) = (0, x^2)$, then $(V, \mathbf{R}, \psi)$ is a generalized vector field which is not left invariant. In fact in this case $\tilde{V}(x) = x$ which is not left invariant. Moreover since η^1 is not well defined (pay attention to $V(1)$ and $V(-1)$) then V is not a left invariant generalized vector field.

Theorem 2.1 Let $\tilde{V}$ be a left invariant vector field on a top space T corresponding to a C^r generalized vector field (V, M, ψ) , and let η^g be the mapping which is defined by (1). Then

- i) $\eta^g \circ V = V \circ l_g$, for all $g \in T$;
 ii) $\psi((l_g)_{*t}(\tilde{V}(t))) = (\eta^g \circ V)(t)$, for all $g, t \in T$.

Proof. Let $t \in T$, $g \in T$ and $v \in \tilde{V}(T)$ be given. Then:

$$\begin{aligned}
 i) \quad (\eta^g \circ V)(t) &= \eta^g(V(t)) = V(gt) = (V \circ l_g)(t). \\
 ii) \quad (\psi \circ (l_g)_{*t} \circ \tilde{V})(t) &= (\psi \circ \tilde{V} \circ l_g)(t) \\
 &= (V \circ l_g)(t) \\
 &= (\eta^g \circ V)(t) \quad \text{by (i).} \quad \square
 \end{aligned}$$

Now we state the main theorem of this section.

Theorem 2.2 With the assumptions of theorem 2.1, the set $(M, \{\eta^g : g \in T\})$ has the following properties:

- i) For all $m \in M$ there exists η^g such that m is a fixed point of η^g ;
- ii) $\eta^{g_1} \circ \eta^{g_2} = \eta^{g_1 g_2}$ for all $g_1, g_2 \in T$.

Proof. If $m \in M$ we have two cases:

Case 1. If $m \notin V(T)$, then $\eta^g(m) = m$ for all $g \in T$.

Case 2. If $m \in V(T)$, then there exists $t \in T$ such that $m = V(t)$, and we have:

$$\eta^{e(t)}(m) = V(e(t)t) = V(t) = m.$$

Now let $g_1, g_2 \in T$ be given. Then:

$$\begin{aligned} \eta^{g_1} \circ \eta^{g_2} \circ V &= \eta^{g_1} \circ (V \circ l_{g_2}) = (\eta^{g_1} \circ V) \circ l_{g_2} \\ &= (V \circ l_{g_1}) \circ l_{g_2} = V \circ (l_{g_1} \circ l_{g_2}) \\ &= V \circ l_{g_1 g_2} = \eta^{g_1 g_2} \circ V. \end{aligned}$$

So by definition of η we have

$$\eta^{g_1} \circ \eta^{g_2} = \eta^{g_1 g_2}. \square$$

Corollary 2.2.1 In theorem 2.2 if T is a Lie group, then $(M, \{\eta^g : g \in T\})$ is a dynamical system on M .

Proof. It is enough to show that η^e is the identity mapping, where e is the identity of the Lie group T . We have two cases:

Case 1. If $m \notin V(T)$, then $\eta^e(m) = m$.

Case 2. If $m = V(t)$ for some $t \in T$, then $\eta^e(m) = \eta^{e(t)}(m) = V(e(t)t) = V(t) = m$.

So η^e is the identity mapping on M . $\square$

3. A GENERALIZATION OF DYNAMICAL SYSTEMS

If T is a Lie group, then corollary 2.2.1 implies that $\{\eta^g : g \in G\}$ is a dynamical system on M . So property (i) of theorem 2.2 can be replaced with the existence of identity mapping in the definition of dynamical systems. Hence properties (i) and (ii) can be a base for the definition of systems which time values of their evolution operators are members of a completely simple semigroups. These systems make a special category which is a sub-category of the category of semi-dynamical systems. More precisely let G be a completely simple semigroup and $D = \{\eta^g : g \in G\}$ be a family of maps from a set M to M , then we have the following definition.

Definition 3.1 (M, D, G) is called a complete semi-dynamical system if:

- i) $\eta^{g_1} \circ \eta^{g_2} = \eta^{g_1 g_2}$ for all $g_1, g_2 \in G$;
- ii) For all $m \in M$ there exists $g \in G$ such that m is a fixed point of η^g .

We use the abbreviation CSS instead of "complete semi-dynamical system".

Example 3.1 The set $G = \mathbb{R} \times (\mathbb{R} - \{0\})$ with multiplication $(g_1, g_2)(h_1, h_2) := (g_2 h_1, g_2 h_2)$ is a generalized group.

If

$$D = \left\{ \begin{array}{ll} \varphi^{(g_1, g_2)} : & \varphi^{(g_1, g_2)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad \text{and} \quad (g_1, g_2) \in G \\ & (x_1, x_2) \mapsto (g_2 x_1, g_2 x_2) \end{array} \right\},$$

then $(\mathbb{R}^2, D, G)$ is a CSS.

Example 3.2 Let G be a completely simple semigroup and

$$D := \left\{ \begin{array}{ccc} \varphi^g : \varphi^g : G & \rightarrow & G \\ x & \mapsto & gx \end{array} \text{ and } g \in G \right\}.$$

Then (G, D, G) is a CSS.

We know that the set $G_a := \{g \in G : e(g) = e(a)\}$ where $a \in G$, with the multiplication of G is a group. The following theorem shows that CSSs are the generalization of dynamical systems.

Theorem 3.1 Let (M, D, G) be a CSS and suppose that there exists $a \in G$ such that: for all $m \in M$ there exists $g \in G_a$ so that $\varphi^g(m) = m$. Then (M, D, G_a) is a dynamical system.

Proof. It is enough to show that $\varphi^{e(a)} = I_M$, where I_M is the identity mapping on M .

For $m \in M$, there exists $g \in G_a$ such that $\varphi^g(m) = m$. Hence $\varphi^{e(a)}(m) = \varphi^{e(a)}(\varphi^g(m)) = \varphi^g(m) = m$. Thus (M, D, G_a) is a dynamical system. $\square$

Corollary 3.1 Let (M, D, G) be a CSS and G be a group, then (M, D, G) is a dynamical system.

Proof. If $a \in G$, then $G = G_a$ and the corollary follows from the theorem 3.1. $\square$

Theorem 3.2 Let (M, D, G) be a CSS and $m \in M$, moreover let $S_m = \{g \in G : \varphi^g(m) = m\}$. Then S_m is a generalized subgroup [7] of G .

Proof. If $g \in S_m$, then: $\varphi^{e(g)}(m) = \varphi^{e(g)}(\varphi^g(m)) = \varphi^g(m) = m$ and $\varphi^{g^{-1}}(m) = \varphi^{g^{-1}}(\varphi^g(m)) = \varphi^{e(g)}(m)$. So $e(g) \in S_m$ and $g^{-1} \in S_m$.

If $g_1, g_2 \in S_m$, then $\varphi^{g_1 g_2}(m) = \varphi^{g_1}(\varphi^{g_2}(m)) = m$. Thus S_m is a generalized subgroup of G . $\square$

We now define the relation $\sim$ on M in the form:

$$m_1 \sim m_2 \Leftrightarrow (\varphi^{g_1}(m_1) = m_2 \text{ and } \varphi^{g_2}(m_2) = m_1 \text{ for some } g_1, g_2 \in G.)$$

It is easy to show that $\sim$ is an equivalence relation.

Definition 3.2 If $p \in M$, then $O^G(p) = \{m \in M : m \sim p\}$ is called the G-orbit of p .

Definition 3.3 The function $g \mapsto \varphi^g x$ is called the G-motion of the point x and the set $T^G(x) := \{\varphi^g x : g \in G\}$ is called the G-trajectory of a point x .

Proposition 3.1 If $x \in M$, then $T^G(x)$ is a disjoint union of G-orbits.

Proof. M is disjoint union of G-orbits. We show that if $O^G(a) \cap T^G(x) \neq \emptyset$ for some $a \in M$, then $O^G(a) \subseteq T^G(x)$.

If $y \in O^G(a) \cap T^G(x)$, then $y \sim a$ and $y = \varphi^g(x)$ for some $g \in G$. If $b \in O^G(a)$, then $b \sim y$ and $b = \varphi^h(y)$ for some $h \in G$. Hence $b = \varphi^{hg}(x) \in T^G(x)$. Thus $O^G(a) \subseteq T^G(x)$, and proposition follows. $\square$

Theorem 3.3 Let G be a normal generalized group and (M, D, G) be a CSS, moreover let every G-motions be one-to-one. Then for all $x \in M$:

- a) $T^G(x)$ with multiplication $\varphi^g(x)\varphi^{g'}(x) = \varphi^{gg'}(x)$ is a completely simple semigroup;
- b) The set S_x is a generalized normal subgroup [11] of G ;
- c) There exists a bijection $\psi : G/S_x \rightarrow T^G(x)$.

Proof. a) Let $\varphi^g(x) \in T^G(x)$ be given, moreover let $\varphi^g(x)\varphi^h(x) = \varphi^h(x)\varphi^g(x) = \varphi^g(x)$, for some $\varphi^h(x) \in T^G(x)$. Then $gh = hg = g$. So $h = e(g)$. We also have: $\varphi^g(x)\varphi^{e(g)}(x) = \varphi^{e(g)}(x)\varphi^g(x) = \varphi^g(x)$. Hence $e(\varphi^g(x)) = \varphi^{e(g)}(x)$. Similarly

$$(\varphi^g(x))^{-1} = \varphi^{g^{-1}}(x).$$

Let $\varphi^g(x)$, $\varphi^h(x)$ and $\varphi^k(x)$ in $T^G(x)$ be given. Then:

$$\begin{aligned} \varphi^g(x)(\varphi^h(x)\varphi^k(x)) &= \varphi^g(x)(\varphi^{hk}(x)) = \varphi^{g(hk)}(x) = \varphi^{(gh)k}(x) \\ &= \varphi^{gh}(x)\varphi^k(x) = (\varphi^g(x)\varphi^h(x))\varphi^k(x). \end{aligned}$$

Thus $T^G(x)$ is a completely simple semigroup.

b) By theorem 3.2, S_x is a generalized subgroup of G . Obviously " $\psi : G \rightarrow T^G(x), g \mapsto \varphi^g(x)$ " is a completely simple semigroup homomorphism and for $a \in G$ we have:

$$S_x \cap G_a = \begin{cases} \phi & \text{if } \varphi^{e(a)}(x) \neq x \\ \text{Kernel } \psi_a & \text{if } \varphi^{e(a)}(x) = x \end{cases}$$

where $\psi_a = \psi|_{G_a}$. So S_x is a generalized normal subgroup of G .

c) The relation $\psi : G/S_x \rightarrow T^G(x)$ defined by $g(S_x \cap G_g) \mapsto \varphi^g(x)$ is a mapping, because if $g_1(S_x \cap G_{g_1}) = g_2(S_x \cap G_{g_2})$, for some $g_1, g_2 \in G$, then there exists $g \in G$ such that $g_1, g_2 \in G_g$ and $G_{g_1} = G_{g_2} = G_g$. Hence $g_2^{-1}g_1(S_x \cap G_g) = S_x \cap G_g$. So $g_2^{-1}g_1 \in S_x$. Thus $\varphi^{g_1}(x) = \varphi^{g_2}(x)$.

If $\varphi^{g_1}(x) = \varphi^{g_2}(x)$, then $g_1 = g_2$. Consequently we have

$$g_1(S_x \cap G_{g_1}) = g_2(S_x \cap G_{g_2}).$$

Therefore ψ is a bijection. $\square$

Theorem 3.4 Let (M, D, G) be a CSS and for all $g \in G$, φ^g be an onto map, then D with combination of maps as multiplication is a group.

Proof. Let $\varphi^g \in D$ and $y \in M$ be given. Then there exists $x \in M$ such that $\varphi^g(x) = y$ and we have:

$$\varphi^{e(g)}(y) = \varphi^{e(g)}(\varphi^g(x)) = \varphi^g(x) = y.$$

So $\varphi^{e(g)} = I_M$. Thus D is a group. $\square$

The following example shows that, in theorem 3.4, G may not be a group.

Example 3.3 Let G be a completely simple semigroup and M be a non-empty set. Then $(M, \{I_M\}, G)$ is a CSS.

4. TOPOLOGICAL COMPLETE SEMI-DYNAMICAL SYSTEMS

In this section we assume that G is a topological generalized group [9], M is a topological space and (M, D, G) is a CSS.

Lemma 4.1 The mapping $e : G \rightarrow G$ defined by $g \mapsto e(g)$, is a continuous mapping.

Proof. Let $e(g) \in G$ and a neighborhood V of $e(g)$ be given. Then there exist neighborhoods U_1 and U_2 of g and g^{-1} respectively such that $m_2(U_1 \times U_2) \subseteq V$, where $m_2 : G \times G \rightarrow G$ is defined by $(g_1, g_2) \mapsto g_1 g_2$. Moreover there exists a neighborhood U_3 of g such that $m_1(U_3) \subseteq U_2$, where $m_1 : G \rightarrow G$ is defined by $h \mapsto h^{-1}$. If $W = U_1 \cap U_3$, then $e(W) \subseteq V$. Thus e is a continuous map. $\square$

If $a \in G$, then lemma 4.1 shows that G_a is a closed generalized normal subgroup of G . So G/G_a is a topological generalized group [9].

Definition 4.1 A CSS (M, D, G) is called a Topological Complete Semi-dynamical System (TCSS) if the mapping

$$\begin{aligned} \varphi : G \times M &\rightarrow M \\ (g, x) &\mapsto \varphi^g(x) \end{aligned}$$

is a continuous mapping.

Example 4.1 The set $G = \mathbb{R}^2$ with multiplication

$$\begin{aligned} \cdot : G \times G &\rightarrow G \\ ((a_1, b_1), (a_2, b_2)) &\mapsto \left(\frac{2a_1 + a_2 + b_2}{2}, \frac{2b_1 + a_2 + b_2}{2} \right) \end{aligned}$$

is a completely simple semigroup. If

$$D = \left\{ \begin{array}{l} \varphi^{(g_1, g_2)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ (x_1, x_2) \mapsto \left(\frac{2g_1 + x_1 + x_2}{2}, \frac{2g_2 + x_1 + x_2}{2} \right) \end{array} \right\}$$

then $(\mathbb{R}^2, D, G)$ is a TCSS.

Let N be a generalized normal subgroup of G , $\Gamma_N := \{g \in G \mid N \cap G_g \neq \emptyset\}$ and $\pi : \Gamma_N \rightarrow G/N$ in the form $g \mapsto gNg$ be the map corresponding to N [13], then the topologies of Γ_N and G/N are defined in the following forms respectively.

$$\tau_{\Gamma_N} := \{V \subseteq \Gamma_N : V \text{ is open in } G_g \text{ for all } g \in \Gamma_N\} \cup \Gamma_N$$

and

$$\tau_{G/N} := \{U \subseteq G/N : \pi^{-1}(U) \text{ is open in } \Gamma_N\}.$$

Theorem 4.1 Let f from G onto E be a continuous homomorphism of topological generalized groups and let kernel of f be a closed subset of G . Moreover let (M, D, E) be a TCSS, then $(M, D^f, G/\ker f)$ is a TCSS, where $D^f := \{\varphi^{gNg} : \varphi^{gNg}(x) = \varphi^{f(g)}(x) \text{ for all } x \in M\}$.

Proof. If $g_1Ng_1 = g_2Ng_2$, then $g_1 = g_2n$ and $f(n) = f(e(g_1)) = f(e(g_2))$. So $f(g_1) = f(g_2n) = f(g_2)f(n) = f(g_2)f(e(g_2)) = f(g_2e(g_2)) = f(g_2)$. Hence $\varphi^{g_1Ng_1}(x) = \varphi^{g_2Ng_2}(x)$, for all $x \in M$. Thus D^f is a set of functions from M to M . Let $g_1Ng_1, g_2Ng_2 \in G/N$ and $x \in M$ be given, then

$$\begin{aligned} (\varphi^{g_1Ng_1} \circ \varphi^{g_2Ng_2})(x) &= \varphi^{g_1Ng_1}(\varphi^{f(g_2)}(x)) = \varphi^{f(g_1)}(\varphi^{f(g_2)}(x)) \\ &= \varphi^{f(g_1)f(g_2)}(x) = \varphi^{f(g_1g_2)} = \varphi^{g_1g_2Ng_1g_2}(x). \end{aligned}$$

So $\varphi^{g_1Ng_1} \circ \varphi^{g_2Ng_2} = \varphi^{g_1g_2Ng_1g_2}$.

If $x \in M$, then there exists $h \in E$ such that $\varphi^h(x) = x$. By assumption there exists $g \in G$ so that $f(g) = h$. Hence $\varphi^{gNg}(x) = x$.

Thus $(M, D^f, G/N)$ is a CSS.

We now show that $\psi : G/N \times M \rightarrow M$
 $(gNg, x) \mapsto \varphi^{gNg}(x)$ is a continuous map.

We know that $\varphi : E \times M \rightarrow M$
 $(h, x) \mapsto \varphi^h(x)$ is a continuous map. Let a neighborhood

V of $\varphi^{gNg}(x)$ be given, where $x \in M$ and let $gNg \in G/N$. By the continuity of φ there exists neighborhoods U and W of x and $f(g)$ respectively, such that $\varphi(W \times U) \subseteq V$. The continuity of f implies that there exists a neighborhood T of g such that $f(T) \subseteq W$. We know that the mapping $\pi : G \rightarrow G/N$
 $g \mapsto gNg$ is an open

map. So $R = \pi(T \cap \Gamma_N)$ is a neighborhood of gNg and $\psi(R \times U) \subseteq V$. Thus ψ is a continuous map. $\square$

We now present a method for constructing new TCSSs.

Theorem 4.2 Let $(M_1, \{\varphi_1^g\}, G_1)$, $(M_2, \{\varphi_2^g\}, G_2)$ be TCSSs, and let

$$D = \left\{ \begin{array}{ccc} \varphi^{(g_1, g_2)} : M_1 \times M_2 & \rightarrow & M_1 \times M_2 : (g_1, g_2) \in G \times G \\ (x, y) & \mapsto & (\varphi_1^{g_1}(x), \varphi_2^{g_2}(y)) \end{array} \right\},$$

then $(M_1 \times M_2, D, G_1 \times G_2)$ is a TCSS.

Proof. The diagram

$$\begin{array}{ccc} G_1 \times G_2 \times M_1 \times M_2 & & \\ & \searrow \varphi & \\ & \psi \downarrow & \\ G_1 \times M_1 \times G_2 \times M_2 & \xrightarrow{\varphi_1 \times \varphi_2} & M_1 \times M_2 \end{array}$$

is a commutative diagram, where $\psi(g_1, g_2, m_1, m_2) = (g_1, m_1, g_2, m_2)$. So φ is continuous. Other properties could also be deduced. $\square$

5. GENETIC SPACE

Each genetic code consists of an alphabet of four letters. In DNA the four letters are adenine, guanine, cytosine, and thymine. In RNA the four letters are adenine, guanine, cytosine, and uracil. To modelize each genetic code we consider each genetic code of n letters as a mapping ω from $\{1, 2, \dots, n\}$ to $\{0, 1, 2, 3\}$, where 0, 1, 2 are identification numbers for adenine, guanine, and cytosine respectively, 3 is the identification number for either thymine or uracil. If we denote $\omega(i)$ by ω_i , then a genetic code with n letters can be denoted by the n -tuple $(\omega_1, \omega_2, \dots, \omega_n)$ where $\omega_i \in \{0, 1, 2, 3\}$.

Let $G = \{(\omega_1, \omega_2, \dots, \omega_n) : n \in \mathbb{N}, \text{ and } \omega_i \in \{0, 1, 2, 3\}\}$ be the set of genetic codes. Then G with the following operation is a generalized group.

If $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, and $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ are two members of G , then their product is denoted by $\omega * \eta$. To define the product $*$ we are going to have the following two cases:

Case 1. If $n \leq m$, then for $1 \leq i \leq m$ we define

$$(\omega * \eta)_i = \begin{cases} \omega_i & \text{if } i \text{ is an odd number and } 1 \leq i \leq n \\ \eta_i & \text{if } i \text{ is an even number or } n < i \leq m \end{cases}$$

Case 2. If $m < n$, then for $1 \leq i \leq n$ we define

$$(\omega * \eta)_i = \begin{cases} \omega_i & \text{if } i \text{ is an odd number} \\ \eta_i & \text{if } i \text{ is an even number or } m < i \leq n \end{cases}$$

The above definition shows a method for combining ω and η . Moreover $\omega * \eta$ is a genetic code of the length $\max\{m, n\}$.

If $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, and $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ are two members of G and $m \geq n$,

then we define $d(\omega, \eta) = \sum_{k=1}^m \frac{\rho_{\omega_k \eta_k}}{2^k}$, where

$$\rho_{\omega_k \eta_k} = \begin{cases} 1 & \text{if } \omega_k \neq \eta_k \text{ or } k > n \\ 0 & \text{if } \omega_k = \eta_k \text{ and } k \leq n \end{cases}$$

Moreover we define $d(\eta, \omega) = d(\omega, \eta)$.

The straightforward calculation shows that (G, d) is a metric space. Moreover G is a topological generalized group. Now let D be the set of $\eta^g : G \rightarrow G$ defined by $\eta^g(x) = g * x$, where $g, x \in G$. Then (G, D, G) is a topological complete semi-dynamical system.

If we put $x = (1, 1)$, then $T^G(x) = \{(g_1, 1), (g_1, 1, g_3, \dots, g_n) : n \in \{3, 4, 5, \dots\} \text{ } g_i \in \{0, 1, 2, 3\}\}$.

6. THE WEATHER SYSTEM AS A GENERALIZED VECTOR FIELD

The Lorenz system [9]

$$\begin{cases} \dot{x}_1 = \sigma x_5 - \sigma x_1 \\ \dot{x}_2 = -\beta x_2 + x_1 x_5 \\ \dot{x}_5 = \rho x_1 - x_5 - x_1 x_2 \end{cases} \quad \sigma, \rho, \beta > 0 \quad (1)$$

has considered from different points of views [4, 6, 10, 18]. This system has created new problems in mathematics, and the proof of the existence of attractors, such as attractors of its geometrical model [4, 15], was an important result on this structure [16].

In this section we use of this system and Hamiltonian systems to construct a generalized vector field on a submanifold of $\mathbf{R}^6$. For this goal we will try to find $f_3(x_1, x_2, x_3, x_4, x_5, x_6)$, $f_4(x_1, x_2, x_3, x_4, x_5, x_6)$, and $f_6(x_1, x_2, x_3, x_4, x_5, x_6)$ such that the system:

$$\begin{cases} \dot{x}_1 = \sigma x_5 - \sigma x_1 \\ \dot{x}_2 = -\beta x_2 + x_1 x_5 \\ \dot{x}_3 = f_3(x_1, x_2, x_3, x_4, x_5, x_6) \\ \dot{x}_4 = f_4(x_1, x_2, x_3, x_4, x_5, x_6) \\ \dot{x}_5 = \rho x_1 - x_5 - x_1 x_2 \\ \dot{x}_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6) \end{cases} \quad \sigma, \rho, \beta > 0$$

be a C^∞ Hamiltonian system on $\mathbf{R}^6$. Moreover we will find a submanifold of $\mathbf{R}^6$ such that $f_3 = f_4 = f_6 = 0$ on it, then the restriction of the Hamiltonian vector field on that submanifold will be a generalized vector field. These mean that we are looking for $f_3(x_1, x_2, x_3, x_4, x_5, x_6)$, $f_4(x_1, x_2, x_3, x_4, x_5, x_6)$, and $f_6(x_1, x_2, x_3, x_4, x_5, x_6)$ such that:

$\frac{\partial f_i}{\partial x_1}(\sigma x_5 - \sigma x_1) + \frac{\partial f_i}{\partial x_2}(-\beta x_2 + x_1 x_5) + \frac{\partial f_i}{\partial x_5}(\rho x_1 - x_5 - x_1 x_2) = 0$, for $i = 3, 4, 6$. So we are looking for a C^∞ function $H(x_1, x_2, x_3, x_4, x_5, x_6)$ such that:

$$\begin{cases} \dot{x}_1 = \sigma x_5 - \sigma x_1 = \frac{\partial H}{\partial x_4} \\ \dot{x}_2 = -\beta x_2 + x_1 x_5 = \frac{\partial H}{\partial x_5} \\ \dot{x}_3 = f_3(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{\partial H}{\partial x_3} \\ \dot{x}_4 = f_4(x_1, x_2, x_3, x_4, x_5, x_6) = -\frac{\partial H}{\partial x_1} \\ \dot{x}_5 = \rho x_1 - x_5 - x_1 x_2 = -\frac{\partial H}{\partial x_2} \\ \dot{x}_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6) = -\frac{\partial H}{\partial x_6} \end{cases} \quad \sigma, \rho, \beta > 0$$

and

$$\begin{aligned} \frac{\partial^2 H}{\partial x_1 \partial x_6}(\sigma x_5 - \sigma x_1) + \frac{\partial^2 H}{\partial x_2 \partial x_6}(-\beta x_2 + x_1 x_5) + \frac{\partial^2 H}{\partial x_5 \partial x_6}(\rho x_1 - x_5 - x_1 x_2) &= 0 \\ -\frac{\partial^2 H}{\partial x_1^2}(\sigma x_5 - \sigma x_1) - \frac{\partial^2 H}{\partial x_2 \partial x_1}(-\beta x_2 + x_1 x_5) - \frac{\partial^2 H}{\partial x_5 \partial x_1}(\rho x_1 - x_5 - x_1 x_2) &= 0 \\ -\frac{\partial^2 H}{\partial x_1 \partial x_3}(\sigma x_5 - \sigma x_1) - \frac{\partial^2 H}{\partial x_2 \partial x_3}(-\beta x_2 + x_1 x_5) - \frac{\partial^2 H}{\partial x_5 \partial x_3}(\rho x_1 - x_5 - x_1 x_2) &= 0 \end{aligned}$$

After straightforward calculation one can deduce:

i) $H = -\sigma x_1 x_4 + \frac{1}{2} x_1 x_5^2 - \rho x_1 x_2 + x_5 x_2 + \frac{1}{2} x_1 x_2^2 + \int v(x_3, x_6) dx_6 + w(x_3)$
where $v(x_3, x_6)$ and $w(x_3)$ are arbitrary C^∞ functions;

ii) $\beta x_2^2 - \beta \rho x_2 + x_5^2 = 0$ (*).

Conditions $f_3 = f_6 = 0$ imply that $v(x_3, x_6) = w(x_3) = 0$. Moreover $f_4 = 0$ implies that $-\sigma x_4 + \frac{1}{2}x_5^2 - \rho x_2 + \frac{1}{2}x_2^2 = 0$ (**). (*) and (**) determine a submanifold of $\mathbf{R}^6$. Moreover the restriction of the Hamiltonian system

$$\begin{cases} \dot{x}_1 = \sigma x_5 - \sigma x_1 \\ \dot{x}_2 = -\beta x_2 + x_1 x_5 \\ \dot{x}_3 = 0 \\ \dot{x}_4 = \sigma x_4 - \frac{1}{2}x_5^2 + \rho x_2 - \frac{1}{2}x_2^2 \\ \dot{x}_5 = \rho x_1 - x_5 - x_1 x_2 \\ \dot{x}_6 = 0 \end{cases} \quad \sigma, \rho, \beta > 0$$

on this submanifold is

$$\begin{cases} \dot{x}_1 = \sigma x_5 - \sigma x_1 \\ \dot{x}_2 = -\beta x_2 + x_1 x_5 \\ \dot{x}_3 = 0 \\ \dot{x}_4 = 0 \\ \dot{x}_5 = \rho x_1 - x_5 - x_1 x_2 \\ \dot{x}_6 = 0 \end{cases} \quad \sigma, \rho, \beta > 0,$$

which is a generalized vector field look a like the Lorenz system. So by this method we can inject a fluid to $\mathbf{R}^6$ which is similar to the Lorenz fluid.

7. CONCLUSION

Generalized vector fields can be a mathematical base for describing some physical phenomena. They prepared a reasonable base for complete semi-dynamical systems, and as a result a mathematical description for genetic space and an evolution by generalized group product deduced. Although we paid attention to genetic code but these results can apply for coding.

In section 6 we present a Hamiltonian description of the Lorenz system. So it seems by adding dimension to the space it will be reasonable to associate each complicated vector field with a simple generalized vector field which will be the subject of new research. Moreover the genetic space implies that: consideration of dynamical behaviour of complete semi-dynamical systems is also a reasonable research topic.

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