

A CLASS OF ONE-DIMENSIONAL MODELS WITH UNIQUE GROUND STATES THAT ADMITS PHASE TRANSITIONS

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(Received 2 October 2004)

ABSTRACT. We will construct a class of one-dimensional models with unique ground states and two spin variables that have a countable number of extreme limits Gibbs states. The work is a modification and extension of some previous work. We will show that it is an improvement over such work.

AMS Classification: 82-08

Keywords: Hamiltonian, Gibbs State, Ground State, Markov Chain, Spin Variable

1. INTRODUCTION

Recently, the uniqueness of limit Gibbs states in one-dimensional models became an interesting object of research of many authors. Several issues related to such models were investigated. Sufficient conditions for the uniqueness of limit Gibbs states in one-dimensional models were investigated in [1], [2], [3], [7], [9], [12], [17], [18].

Although in usual cases there can be no phase transitions, there are several models with different properties that admit phase transitions, [4], [5], [8], [13], [16], [20], [21].

Of particular interest is the work of Kerimov [10, 11] in which the models were defined in different ways but the results were obtained depending on the convergence of the infinite product to a number $>1/2$. Also Kerimov and Mallak [13, 15] used the same approach for constructing the models but for specific values and again the results obtained depend on the convergence of the infinite product to a number $>1/2$.

In this paper we extend the results of the models in [10], [11], [13], [15] to a general class of models. In Section 2, we define the class of models from classical statistical mechanics. Such models will be interpreted as an inhomogeneous Markov chains with two states in Section 3. In Section 4, we will present the main result together

with supporting arguments. We will conclude with some comments in the last section.

2. THE CLASS OF MODELS

We will start by defining the Hamiltonian, and to do so, let us choose two real numbers p and r such that $\frac{1}{2} < p < 1$ and $\sum_{i=1}^{\infty} r^i = q = 1 - p$. Then define two sequences of real numbers a_k, b_k with $a_k = p + \sum_{i=1}^{k-1} r^i$ & $b_k = p + \sum_{i=1}^k r^i$, a sequence of intervals I_k by $I_k = [a_k, b_k)$ and a sequence of real numbers s_k by $s_k = \frac{a_k + b_k}{2}$. Finally, consider a sequence of intervals O_{-l} defined by $O_{-l} = [-c_l, -c_{l-1})$ where $c_1 = 0$, $c_l = \sum_{i=1}^l n^{3i+2}$, and n is a natural number satisfying $\frac{1}{r} < n$.

Now we are ready to define the class of models. Consider a model of the classical statistical mechanics on the one-dimensional lattice Z^1 with the Hamiltonian

$$H(w(y)) = \sum_{x \in Z^1, x < 0} U_1(\xi(x), \eta(O_{-l(x)-1})) - \sum_{x \in Z^1, x \geq 0} U_2(\xi(x)), \quad (2.1)$$

where the spin variable takes two values $\cong \alpha, \gamma$, $y = (x)_{x \in Z^1}$, and $\eta(O_{-l(x)})$ is the restriction of the configuration $w(y)$ to the set $O_{-l(x)}$ where $l(x)$ in (2.1) is defined by the condition $x \in O_{-l(x)}$.

The interaction potential $U_1(\xi(x), \eta(O_{-l(x)-1}))$ in (2.1) takes place between the points x and the left neighbor interval $O_{-l(x)-1}$. While $U_1(\xi(x), \eta(O_{-l(x)-1}))$ specifies the interaction between the spin variable $\xi(x)$ and the restriction of the configuration $w(y)$ to the interval $O_{-l(x)-1}$. Here $U_1(\xi(x), \eta(O_{-l(x)-1}))$ are defined as follows:

$$\begin{cases} U_1(\xi(x) = \alpha, \\ \eta(O_{-l(x)-1})) = 0 & \text{if } \frac{\#\alpha}{c_l - c_{l-1}} = 1 \\ \\ U_1(\xi(x) = \gamma, \\ \eta(O_{-l(x)-1})) = \infty & \text{if } \frac{\#\gamma}{c_l - c_{l-1}} = 1 \\ \\ U_1(\xi(x) = \alpha, \\ \eta(O_{-l(x)-1})) = -\ln(s_k) & \text{if } \frac{\#\alpha}{c_l - c_{l-1}} \in I_k \\ \\ U_1(\xi(x) = \gamma, \\ \eta(O_{-l(x)-1})) = -\ln(1 - s_k) & \text{if } \frac{\#\gamma}{c_l - c_{l-1}} \in I_k \\ \\ U_1(\xi(x) = \alpha, \\ \eta(O_{-l(x)-1})) = -\ln(p) & \text{if } \frac{\#\alpha}{c_l - c_{l-1}} \notin I_k \text{ for any } k. \end{cases}$$

and

$$\begin{cases} U_1(\xi(x) = \gamma, \\ \eta(O_{-l(x)-1})) = -\ln(q) & \text{if } \frac{\#\gamma}{c_l - c_{l-1}} \notin I_k \text{ for any } k. \end{cases}$$

Here $U_2(\xi(x))$ plays the role of the external field defined by:

$$U_2(\xi(x)) = \begin{cases} 1 & \text{if } \xi(x) = \alpha \\ 0 & \text{if } \xi(x) = \gamma. \end{cases}$$

Notice that by $\frac{\#(\cdot)}{c_l - c_{l-1}}$ we mean the density of spins in the set O_{-l} which is
$$\sum_{x \in Z^1; x \in O_{-l}} \frac{\xi(x)}{c_l - c_{l-1}}.$$

Now, let $I_N = [-N, N]$ and $[-N, -1] = \cup_{i=1}^f O_{-i}$. Suppose that the boundary conditions $\xi^k(x)$, $x \in Z^1 - I_N$ are fixed. The Hamiltonian in the subset I_N is given by

$$H_N(w(y)|\xi^k(x)) = \sum_{x=-N}^{-1} U_1(\xi(x), \eta(O_{-l(x)-1})) - \sum_{x=0}^N U_2(\xi(x)).$$

The restriction of the configuration $w(y)$ to the interval I_N will be denoted by $w_N(y)$ and the set of all configurations $w_N(y)$ will be denoted by $W(N)$.

The finite-volume Gibbs state in $W(N)$ at inverse temperature $\beta = T^{-1}$ and boundary conditions $\xi^k(x)$ are defined by

$$P_N^k(w_N(y)|\xi^k(x)) = E_N^{-1} \exp(-\beta H_N(w_N(y)|\xi^k(x)))$$

where $E_N = \sum_{w_N(y) \in W(N)} \exp(-\beta H_N(w_N(y)|\xi^k(x)))$ is the partition function.

An extreme limit Gibbs state is the weak limit of the finite-volume Gibbs states. It is well known that the set of all limits Gibbs states coincides with the closed convex hull of the set of weak limits of finite-volume Gibbs states [6].

A configuration $w^{gr}(y)$ is said to be a ground state, if for any perturbation $w'(y)$ of the configuration $w^{gr}(y)$ the expression $H(w'(y)) - H(w^{gr}(y))$ is non-negative.

3. THE MODELS INTERPRETED AS INHOMOGENEOUS MARKOV CHAINS

From the construction of the Hamiltonian of the models (2.1), they can be interpreted as inhomogeneous Markov chains with two states α, γ , (see [6], [14], [19]), starting at minus infinity, whose transition probabilities are defined by the following rule:

At $\beta = 1$, if the density of particles (α) in $B_{-l(x)-1}$ is 1, then the probability that $\xi(x) = \alpha$ is 1, if the density of particles in $B_{-l(x)-1}$ belongs to the interval I_k , then the probability that $\xi(x) = \alpha$ is s_k , and if the density of particles in $O_{-l(x)-1}$ does not belong to any interval I_k , then the probability that $\xi(x) = \alpha$ is p . If the point belongs to the interval $[0, \infty)$ then the probability that $\xi(x) = \alpha$ is $\frac{e}{e+1}$.

4. MAIN RESULT

Before we give our main result, let us give the following lemma:

Lemma: Any model with Hamiltonian satisfying (2.1) has a unique ground state.

Proof: It can be readily verified that the configuration $w^{gr}(y) = (\alpha)_{\forall x \in Z^1}$ is the only ground state of the model (2.1). It follows directly if we notice that, in each case the probability that $\xi(x) = \alpha$ is $> \frac{1}{2}$.

Now let us proof this lemma formally:

Let us show that the configuration $w^{gr}(y) = (\alpha)_{\forall x \in Z^1}$ is a ground state of the model (2.1). Let w' be a finite perturbation of the configuration $w^{gr}(y)$. Then the expression $H(w'(y)) - H(w^{gr}(y))$ is non-negative, indeed

$$\begin{aligned} H(w'(y)) - H(w^{gr}(y)) &= \sum_{x \in Z^1, x < 0} (U_1(\xi'(x), \eta'(O_{-l(x)-1}) - U_1(\xi^{gr}(x), \eta^{gr}(O_{-l(x)-1}))) \\ &+ \sum_{x \in Z^1, x \geq 0} (U_2(\xi^{gr}(x) - U_2(\xi'(x))) = \sum_1 + \sum_2. \end{aligned}$$

Now let $U_1(\xi'(x), \eta'(O_{-l(x)-1}) - U_1(\xi^{gr}(x), \eta^{gr}(O_{-l(x)-1}))$ be a non-zero term of $\sum_1$.

Now if $\xi' = \alpha$ then due to the definitions this term is equal to $-\ln s_k - 0$, for some

k and hence is positive. If $\xi' = \gamma$ then again due to the definitions this term is, equal to $\infty - \ln(1 - s_k)$, for some k and is positive. On the other hand, all non-zero terms of $\sum_2$ are 1. Thus, the configuration $w^{gr}(y) = (\alpha)_{\forall x \in Z^1}$ is a ground state of the model (2.1).

Now let us prove that $w^{gr}(y) = (\alpha)_{\forall x \in Z^1}$ is a unique ground state. Assume that the configuration $w^1(y)$ is a ground state of the model (2.1) and let the set $Z(\xi)$ of all points $x' \in Z'$ such that $\xi^1(x') = \gamma$ be non-empty.

If $Z(\xi) \cap [0, \infty)$ is not empty and contains a point x' , we define a configuration $w^{1,1}(y)$ by $\xi^{1,1}(x') = \alpha$ and $\xi^{1,1}(x) = \xi^1(x)$, $\forall x \neq x'$.

Now $U_2(\xi^{1,1}(x)) - U_2(\xi^1(x)) = -1$ and have a contradiction with the fact that $w^1(y)$ is a ground state. If $Z(\xi) \cap (-\infty, -1]$ is not empty, we consider the point $x' = \max_{x \in Z(\xi) \cap (-\infty, -1]} x$ and define a configuration $w^{1,1}(y)$ by the following rule: $\xi^{1,1}(x') = \alpha$ and $\xi^{1,1}(x) = \xi^1(x)$, $\forall x \neq x'$. Now $H(w^{1,1}(y)) - H(w^1(y))$ is either $-\ln s_k + \ln(1 - s_k)$, for some k or $-\infty$ and since $s_k > \frac{1}{2}$, the expression $-\ln s_k + \ln(1 - s_k) < 0$ and again we have a contradiction with the fact that $w^1(y)$ is a ground state. The proof of the lemma is complete.

Now we are ready to give our main result.

Theorem: The model (2.1) has a countable number of extreme limits Gibbs states, although any model with Hamiltonian satisfying (2.1) has a unique ground state at $\beta = 1$.

Proof: First of all notice that for each k , there exists a configuration $w^k(y)$, such that the value of the density of the particles in each block O_{-l} for all sufficiently large $l = l(k)$ belongs to the interval

$$I_k : \sum_{x \in Z^1; x \in O_{-l}} \frac{\xi(x)}{c_l - c_{l-1}} \in I_k.$$

Let P^k be a limit Gibbs state corresponding to the boundary conditions $\xi^k(x)$. Let us define a random variable, $\chi_{-l} = \sum_{x \in Z^1; x \in O_{-l}} \frac{\xi(x)}{c_l - c_{l-1}}$, which is actually $\frac{\#\alpha}{c_l - c_{l-1}}$

(the density of particles in the set O_{-l}).

We prove the theorem by showing that, $\forall k$ and l , with $l > k$, and at sufficiently large N :

$$P_N^k(\chi_{-l} \in I_k) \geq \delta > 0 \quad (4.1)$$

where P_N^k is the Gibbs distribution corresponding to the boundary conditions $\xi^k(x)$, $x \in Z^1 - [-N, N]$.

Actually if we prove (4.1) then the theorem is proven. From (4.1) it follows that $P_N^k(\chi_{-l} \in I_k) \geq \delta > 0$, $\forall k$. Now if P^k were unique (does not depend on k , say P), then $\sum_k P(\chi_{-l} \in I_k) = 1$, but this is impossible because of (4.1).

Suppose that $[-N, -1] = \cup_{i=1}^f O_{-i}$. It readily follows from the definition of the potential that all spin variables $\xi(x)$, $x \in [0, \infty)$ are independent. They take α, β with respective probabilities $\frac{e}{e+1}$, $\frac{1}{1+e}$. Hence the restriction of the Gibbs distribution P_N^k to the set $\xi(x)$, $x \in [-N, -1]$ can be treated as a one-sided inhomogeneous Markov chain with two states starting at minus infinity [6], [14], [19]. Thus,

$$P_N^k(\chi_{-l} \in I_k) \geq P_N^k\left(\cap_{i=l}^f \chi_{-i} \in I_k\right) = P_N^k(\chi_{-f} \in I_k) \prod_{i=f-1}^l P_N^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k).$$

Now we estimate the expression inside the product. By the definition of the potential

$$P_N^k(\xi(x) = \alpha | \chi_{-i-1} \in I_k) = s_k.$$

Let us define the sequence of positive numbers $\epsilon_k = r^{k+1}$ (from the construction $r < \frac{1}{2}$). By the law of large numbers,

$$\begin{aligned} & P_N^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) \\ & \geq P_N^k(|\chi_{-i} - P_k| < \epsilon_k | \chi_{-i-1} \in I_k) \\ & \geq 1 - \frac{1}{|O_{-i}| \epsilon_k^2} \\ & = 1 - \frac{r^{2k+2}}{n^{3l+2}} \\ & > 1 - \frac{1}{n^{3l-2k}} \end{aligned}$$

is valid since $\frac{1}{r} < n$. Now, since $l > k$, $P_N^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) > 1 - n^{-l}$.

Hence, $P_N^k(\chi_{-f} \in I_k) \prod_{i=f-1}^l P_N^k(\chi_{-i} \in I_k | \chi_{-i-1} \in I_k) > \prod_{i=f-1}^l (1 - n^{-i}) > \prod_{i=1}^{\infty} (1 - n^{-i}) \geq \delta > 0$. (since $\sum_{i=1}^{\infty} (\frac{1}{n})^i$ is convergent). Hence, the result is obtained.

5. COMMENTS

In Kerimov [10,11], the models were defined in two different ways but the results were obtained depending on the convergence of the infinite product to a number $> 1/2$. While in Kerimov, Mallak [13] and Mallak [15] the models were constructed in a similar way for specific values of n and r and again the results were obtained depending on the convergence of the infinite product to a number $> 1/2$. Consequently, our result presented in this paper is considered as an improvement of these models mentioned above.

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