

# RECURRENT POINTS OF FUZZY DYNAMICAL SYSTEMS

M. BURGIN

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF CALIFORNIA, LOS ANGELES  
LOS ANGELES, CA 90095-1555  
E-MAIL: MBURGIN@MATH.UCLA.EDU

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**ABSTRACT.** The main goal of the present paper is to reconsider classical results from the theory of dynamical systems, making them appropriate for treating systems about which we have imprecise, vague, uncertain or incomplete information. Here we study qualitative properties of dynamical systems related to recurrent points. According to the basic Poincaré recurrence theorem, the set of recurrent points has full measure in any subset of the classical dynamical system space. In general, this property is invalid for fuzzy dynamical systems. In this paper, we consider fuzzy dynamical systems of two types. It is demonstrated that for fuzzy dynamical systems of the first type, the Poincaré property of recurrent points remains true if the fuzzy measure in the system space satisfies some, rather weak, additional conditions. It is also proved that infinitely recurrent points have the same property. At the same time, for fuzzy dynamical systems of the second type, the Poincaré property of recurrent points is invalid, and it is possible to find only sufficiently big subsets of recurrent points. In contrast to fuzzy dynamical systems of the first type, infinitely recurrent points of fuzzy dynamical systems of the second type do not have even this property.

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## 1. INTRODUCTION

The Poincaré recurrence theorem is one of the basic results of ergodic theory (cf. [1, 19, 28]). It has many applications in mathematical physics. This theorem has been extensively used in physics and caused many controversies. It states that if a mechanical system governed by Newton's laws has a fixed total energy that restricts its dynamics to bounded subsets of its phase space, the system will eventually return as closely as you like to any given initial set of states. In the kinetic theory of gases, the system will eventually return to any neighborhood of a given initial set of molecular positions and velocities.

There are other applications. For instance, the Poincaré recurrence theorem explains why, according to the contemporary cosmological model, the universe is pulsating under the indicated initial conditions [17]. In [16], relations between the Poincaré recurrence theorem and number theory are considered.

However, the Poincaré recurrence theorem is inherently classical and hence many branches of modern physics, such as quantum mechanics or the theory of gauge fields, do not provide necessary conditions for application of this theorem and many other results of ergodic theory.

Researchers suggest (cf., for example, [4]) that "quantum" in these physical theories may be interpreted as "fuzzy", i.e., that the most characteristic feature of quantum observable is their "fuzziness". Thus, to model quantum systems, we need fuzzy dynamical systems.

Another cause of imprecision in physics is measurement. Any real measurement is never absolutely precise but gives only approximate results. So, the data that we process are not absolutely precise. This inaccuracy leads to the inaccuracy in the result of data processing. The problem is to estimate the resulting inaccuracy.

Two more aspects that demand fuzzy models are incompleteness of information in control processes and finite precision of computations. For instance, computing points of a trajectory of a system, we can find these points only with a finite precision.

As we know imprecision and fuzziness is much larger in social and behavioral sciences than in natural sciences. This prevents valid application of the Poincaré recurrence theorem in social and behavioral sciences. All this shows that constructions and methods developed in the classical theory of dynamical systems are only approximations to what exists in reality. In many situations such approximations have been giving a sufficiently adequate representation of studied phenomena. However, scientists and, especially, engineers have discovered many cases in which such methods did not work because classical approach is too rough, using abstractions that are too far from reality. Fuzzy dynamical systems give an appropriate framework for capturing system uncertainty and imprecision.

We study here two cases of fuzziness: fuzzy dynamical systems of the first type are defined in spaces with a fuzzy measure, while fuzzy dynamical systems of the second type employ transformations that can change the measure of a set, but not essentially. Indeed, when we make measurement, there are no means to get absolutely precise results. Consequently, it is impossible to test all conditions of classical measures, as well as to assure that some transformation completely preserves a given measure.

In the second part of the paper, going after introduction, elements of the theory of fuzzy measures are presented to make the exposition more complete. In the third part, fuzzy dynamical systems are introduced. The fourth part contains a study of recurrent points in fuzzy dynamical systems.

According to the basic Poincaré recurrence theorem, the set of recurrent points has full measure in any subset of the classical dynamical system space. In general, this property is invalid for fuzzy dynamical systems. Some examples are given in this paper. However, it is demonstrated (Theorem 4.1) that for fuzzy dynamical systems of the first type, the Poincaré property of recurrent points remains true if the fuzzy measure in the system space satisfies some, rather weak, additional conditions. Examples show that these conditions are essential for validity of the

result. It is also proved (Corollary 4.3) that infinitely recurrent points have the same property. At the same time, for fuzzy dynamical systems of the second type, the Poincaré property of recurrent points is invalid, and it is possible (Theorem 4.2) to find only sufficiently big subsets of recurrent points. In contrast to fuzzy dynamical systems of the first type, infinitely recurrent points of fuzzy dynamical systems of the second type do not have even this property, i.e., their measures can be equal to zero.

Other researcher studied other types of fuzzy dynamical systems and their properties (cf., for example, [10, 11, 14, 15, 21, 25]).

## 2. FUZZY MEASURES

Sugeno [29] introduced the concept of fuzzy measure.

Let  $\mathbf{P}(X)$  be the set of all subsets of a set  $X$  and  $\mathbf{A}$  be a Borel field of sets from  $\mathbf{P}(X)$ , specifically,  $\mathbf{A} \subseteq \mathbf{P}(X)$ .

**Definition 2.1** A fuzzy measure on  $\mathbf{A}$  in  $X$  is a function  $g : \mathbf{A} \rightarrow [0, 1]$  that assigns to each set from  $\mathbf{A}$  a number in the unit interval  $[0, 1]$  so that the following conditions are valid:

(FM1)  $\emptyset, X \in \mathbf{A}$ ,  $g(\emptyset) = 0$ , and  $g(X) = 1$ ;

(FM2) For any  $A$  and  $B$  from  $\mathbf{A}$ , the inclusion  $A \subseteq B$  implies  $g(A) \leq g(B)$ ;

(FM3) For any non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $\mathbf{A}$ , the following equality is valid:

$$(1) \quad g\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} g(A_n);$$

(FM4) For any non-increasing sequence  $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq A_{n+1} \supseteq \dots$  of sets from  $\mathbf{A}$ , the following equality is valid:

$$(2) \quad g\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} g(A_n).$$

Then this definition was improved further through elimination of the fourth condition [30, 33].

**Definition 2.2** A fuzzy measure on  $\mathbf{A}$  in  $X$  is a function  $g : \mathbf{A} \rightarrow [0, 1]$  that assigns to each set from  $\mathbf{A}$ , a number in the unit interval  $[0, 1]$  so that the following conditions are valid:

(1)  $g(\emptyset) = 0$  and  $g(X) = 1$ ;

(2) For any  $A$  and  $B$  from  $\mathbf{A}$ , the inclusion  $A \subseteq B$  implies  $g(A) \leq g(B)$ ;

(3) For any non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $\mathbf{A}$ , the following equality is valid:

$$(3) \quad g\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} g(A_n).$$

**Remark 2.1** The condition  $g(X) = 1$  defines a normed fuzzy measure. However, because fuzzy measure is always bounded, it is equivalent to a normed fuzzy measure.

Bannon [3] demonstrated that many measures with infinite universe studied by different researchers, such as probability measures, belief functions, plausibility measures, and so on, are fuzzy measures in the sense of Sugeno.

We take even more general definition of a fuzzy measure from [20].

Let  $P(X)$  be the set of all subsets of a set  $X$  and  $\mathbf{B}$  be an algebra of sets from  $P(X)$ , specifically,  $\mathbf{B} \subseteq P(X)$ .

**Definition 2.3** A fuzzy measure on  $\mathbf{B}$  in  $X$  is a function  $g: \mathbf{B} \rightarrow [0, 1]$  that assigns to each set from  $\mathbf{B}$  a number in the unit interval  $[0, 1]$ , is normed (1) and monotone (2):

(1)  $\emptyset, X \in \mathbf{B}$ ,  $g(\emptyset) = 0$  and  $g(X) = 1$ ;

(2) For any  $A$  and  $B$  from  $\mathbf{B}$ , the inclusion  $A \subseteq B$  implies  $g(A) \leq g(B)$ .

Then  $\mathbf{B}$  is called the algebra of fuzzy measurable sets with respect to the fuzzy measure  $g$ .

**Remark 2.2** For simplicity, we will write about fuzzy measures on the space  $X$ , while all sets from  $\mathbf{B}$  are called fuzzy measurable.

Popular examples of fuzzy measures are possibility, belief and plausibility measures.

**Example 2.1** Let us consider some set  $X$  and its power set  $P(X)$ .

**Definition 2.4** A possibility measure in  $X$  is a partial function  $Pos: P(X) \rightarrow [0, 1]$  that is defined on a subset  $\mathbf{A}$  of  $P(X)$  and satisfies the following axioms (cf. [18, 32, 33]):

(Po1)  $\emptyset, X \in \mathbf{A}$ ,  $Pos(\emptyset) = 0$ , and  $Pos(X) = 1$ ;

(Po2) For any  $A$  and  $B$  from  $\mathbf{A}$ , the inclusion  $A \subseteq B$  implies  $Pos(A) \leq Pos(B)$ ;

(Po3) For any system  $\{A_i; i \in I\}$  of sets from  $\mathbf{A}$ ,

$$(4) \quad Pos\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} Pos(A_i).$$

A possibility measure is a fuzzy measure in the sense of Definition 2.3, but it is not always a fuzzy measure in the sense of Definitions 2.1 and Definition 2.2 [24].

**Example 2.2** Possibility is also described by a more general class of measures (cf. [8, 23, 32]).

**Definition 2.5** A quantitative possibility measure in  $X$  is a function  $P: \mathbf{P}(X) \rightarrow [0, 1]$  that satisfies the following axioms:

(Po1)  $\emptyset, X \in \mathbf{A}$ ,  $P(\emptyset) = 0$ , and  $P(X) = 1$ ;

(Po2) For any  $A$  and  $B$  from  $P(X)$ ,  $P(A \cup B) = \max\{P(A), P(B)\}$ .

A quantitative possibility measure is a fuzzy measure in the sense of Definition 2.3.

**Example 2.3** Dual to a quantitative possibility measure is a necessity measure [8, 23].

**Definition 2.6** A quantitative necessity measure in  $X$  is a function  $N: P(X) \rightarrow [0, 1]$  that satisfies the following axioms:

(Po1)  $N(\emptyset) = 0$ , and  $N(X) = 1$ ;

(Po2) For any  $A$  and  $B$  from  $P(X)$ ,  $N(A \cap B) = \min\{N(A), N(B)\}$ .

A quantitative necessity measure is a fuzzy measure in the sense of Definition 2.3.

Possibility and necessity measures are important in support logic programming [2].

It uses fuzzy measures for reasoning under uncertainty and approximate reasoning in expert systems, based on the logic programming style.

**Definition 2.7** A fuzzy measure  $g$  in  $X$  is called:

a) additive if for any  $A$  and  $B$  from  $\mathbf{B}$ , the equality  $A \cap B = \emptyset$  implies

$$(5) \quad g(A \cup B) = g(A) + g(B).$$

b) super-additive if for any  $A$  and  $B$  from  $\mathbf{B}$ , the equality  $A \cap B = \emptyset$  implies

$$(6) \quad g(A \cup B) \geq g(A) + g(B).$$

c) subadditive if for any  $A$  and  $B$  from  $\mathbf{B}$ , the equality  $A \cap B = \emptyset$  implies

$$(7) \quad g(A \cup B) \leq g(A) + g(B).$$

Let  $g$  be an additive fuzzy measure.

**Lemma 2.1** If  $g(A) = 0$ , then  $g(A \cup B) = g(B)$ .

Let  $k$  be a non-negative real number.

**Definition 2.8** A fuzzy measure  $g$  in  $X$  is called  $k$ -additive if for any  $A$  and  $B$  from  $\mathbf{B}$ , the equality  $A \cap B = \emptyset$  implies

$$(8) \quad g(A \cup B) \geq \frac{1}{2}(1+k)(g(A) + g(B)).$$

**Lemma 2.2** Any super-additive fuzzy measure  $g$  is  $k$ -additive for any  $k < 2$ .

If we take a non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $X$ , then the sequence of numbers  $g(A_1) \subseteq g(A_2) \subseteq \dots \subseteq g(A_n) \subseteq g(A_{n+1}) \subseteq \dots$  is non-decreasing, bounded and thus, has the limit  $\lim g(A_n)$ .

**Definition 2.9** A fuzzy measure  $g$  in  $X$  is called continuous if for any non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $\mathbf{B}$ , the union  $\bigcup_{n=1}^{\infty} A_n$  belongs to  $\mathbf{B}$  and the following equality is valid

$$(9) \quad g\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} g(A_n).$$

Let  $r$  be a non-negative real number.

**Definition 2.10** A fuzzy measure  $g$  in  $X$  is called  $r$ -continuous if for any non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $\mathbf{B}$ , the union  $\bigcup_{n=1}^{\infty} A_n$  belongs to  $\mathbf{B}$  and the following equality is valid

$$(10) \quad g\left(\bigcup_{n=1}^{\infty} A_n\right) = r - \lim_{n \rightarrow \infty} g(A_n).$$

Here  $r - \lim g(A_n)$  is a fuzzy limit in the sense of [6].

Because the difference between measures of any two sets from  $X$  cannot be larger than 1, we have the following result.

**Proposition 2.1** An arbitrary fuzzy measure  $g$  in  $X$  is 1-continuous.

**Definition 2.11** A fuzzy measure  $g$  in  $X$  is called uniformly fuzzy continuous if it is  $r$ -continuous for some non-negative real number  $r < 1$ .

**Definition 2.12** A fuzzy measure  $g$  in  $X$  is called fuzzy continuous if for any non-decreasing sequence  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$  of sets from  $\mathbf{B}$  there is a non-negative real number  $r < 1$  such that the union  $\bigcup_{n=1}^{\infty} A_n$  belongs to  $\mathbf{B}$  and the following equality is valid

$$(11) \quad g\left(\bigcup_{n=1}^{\infty} A_n\right) = r - \lim_{n \rightarrow \infty} g(A_n).$$

In addition to ordinary measures, fuzzy measures encompass many kinds of measures introduced and studied by different researchers. Here are some examples.

**Example 2.4** Beliefs play an important role in people's behavior. To study beliefs by methods of fuzzy set theory, the concept of belief measure was introduced and studied [3, 27].

**Definition 2.13** A belief measure in  $X$  is a partial function  $Bel : P(X) \rightarrow [0, 1]$  that is defined on a subset  $\mathbf{A}$  from  $P(X)$  and satisfies the following axioms:

(Be1)  $\emptyset, X \in \mathbf{A}$ ,  $Bel(\emptyset) = 0$ , and  $Bel(X) = 1$ ;

(Be2) For any system  $\{A_i; i = 1, 2, \dots, n\}$  of sets from  $\mathbf{A}$  and any  $n$  from  $\mathbf{N}$ ,  $Bel(A_1 \cup \dots \cup A_n) \geq \sum_{i=1}^n Bel(A_i) - \sum_{i < j} Bel(A_i \cap A_j) + \dots + (-1)^{n+1} Bel(A_1 \cap \dots \cap A_n)$ .

Axiom (Be2) implies that belief measure is a super-additive fuzzy measure as for  $n = 2$  and arbitrary subsets  $A$  and  $B$  from  $X$ , we have

$$(12) \quad Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B).$$

Axiom (Be2) also implies the following property

$$(13) \quad 0 \leq Bel(A) + Bel(\bar{A}) \leq 1$$

where  $\bar{A}$  is a complement of  $A$ . For each set  $A \in P(X)$ , the number  $Bel(A)$  is interpreted as the degree of belief (based on available evidence) that a given element  $x$  of  $X$  belongs to the set  $A$ .

Another interpretation treats subsets of  $X$  as answers to a particular question. It is assumed that some of the answers are correct, but we do not know with full certainty which ones they are. Then the number  $Bel(A)$  estimates our belief that the answer  $A$  is correct.

**Example 2.5** Related to belief measures are plausibility measures.

**Definition 2.14** A plausibility measure in  $X$  is a partial function  $Pl : P(X) \rightarrow [0, 1]$  that is defined on a subset  $\mathbf{A}$  from  $P(X)$  and satisfies the following axioms [3, 27]:

(Pl1)  $\emptyset, X \in \mathbf{A}$ ,  $Pl(\emptyset) = 0$ , and  $Pl(X) = 1$ ;

(Pl2) For any system  $\{A_i; i = 1, 2, \dots, n\}$  of sets from  $\mathbf{A}$  and any  $n$  from  $\mathbf{N}$ ,

$$(14) \quad Pl(A_1 \cap \dots \cap A_n) \leq \sum_{i=1}^n Pl(A_i) - \sum_{i < j} Pl(A_i \cup A_j) + \dots + (-1)^{n+1} Pl(A_1 \cup \dots \cup A_n).$$

Axiom (Pl2) implies that plausibility measure is a subadditive fuzzy measure as for  $n = 2$  and arbitrary subsets  $A$  and  $B$  from  $X$ , we have:

$$(15) \quad Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B).$$

Transforming this inequality, we have:

$$(16) \quad Pl(A \cup B) \leq Pl(A \cup B) + Pl(A \cap B) \leq Pl(A) + Pl(B).$$

Belief measure and plausibility are dual measures as for any belief measure  $Bel(A)$ ,  $Pl(A) = 1 - Bel(\bar{A})$  is a plausibility measure and for any plausibility measure  $Pl(A)$ ,  $Bel(A) = 1 - Pl(\bar{A})$  is a belief measure.

Every belief measure and its dual plausibility measure can be expressed in terms of a function  $m : P(X) \rightarrow [0, 1]$  that satisfies the following axioms:

$$(1) \quad m(\emptyset) = 0;$$

$$(2) \quad \sum_{A \in \mathcal{A}} m(A) = 1.$$

Indeed, the function  $Bel(A) = \sum_{B \subset A} m(B)$  is a belief measure, while the function  $Pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$  is a plausibility measure.

This measure  $m(A)$  is usually interpreted either as the degree of evidence supporting the claim that a specific element of  $X$  belongs to the set  $A$  but not to any special subset of  $A$ , or as the degree to which we believe that such a claim is warranted.

**Example 2.6** When Axiom (Be2) for belief measures is replaced with a stronger axiom:

$$(17) \quad Bel(A \cup B) = Bel(A) + Bel(B) \text{ whenever } A \cap B = \emptyset,$$

we obtain a special type of belief measures, the classical probability measures (sometimes also referred to as Bayesian belief measures).

In a similar way, it is possible to obtain some special kinds of probability measures, for instance, those that are studied in [7].

Dynamical systems in spaces with belief, plausibility or possibility measure allow one to model mental processes, cognition, and information processing in intelligent systems. For example, it is possible to consider data- and knowledge bases as dynamical systems and study their behavior.

### 3. DYNAMICAL SYSTEMS WITH FUZZY MEASURES AND FUZZY TRANSFORMATIONS

We begin with the classical definition of a dynamical system.

A dynamical system in a space with a measure consists of a measurable space  $(X, \mathcal{A}, m)$ , some semigroup  $R$  with the operation  $\circ$ , which is called time in the space  $(X, \mathcal{A}, m)$ , and a parametric system  $\mathbf{T}_R = \{T^i : i \in R\}$  of endomorphisms of the space  $(X, \mathcal{A}, m)$ , i.e.,  $T^i(T^j(x)) = T^{i \circ j}(x)$ , for any  $i, j \in R$  and  $x \in X$  [19]. Such system may be fuzzy in any component. The basic set  $X$  can be a fuzzy set. The Borel field  $\mathcal{A}$  of sets from  $X$  can be fuzzy. The measure  $m$  can be fuzzy. The

temporal semigroup  $R$  can be fuzzy. Endomorphisms  $T^i$  in the space  $(X, \mathbf{A}, m)$  can be a fuzzy. And at last, the composition of these endomorphisms can be also fuzzy. As a result, we obtain different types of fuzzy dynamical systems.

Here we consider two types of fuzziness: dynamical systems in spaces with a fuzzy measure (fuzzy dynamical systems of the first type) and dynamical systems with fuzzy endomorphisms (fuzzy dynamical systems of the second type).

Let us take a set  $X$  with a fuzzy measure  $g$  and an algebra  $\mathbf{B}$  of its fuzzy measurable subsets.

**Definition 3.1** The system  $(X, \mathbf{B}, g)$  is called a fuzzy measurable space.

**Definition 3.2** A mapping  $T : X \rightarrow X$  is called an endomorphism of  $X$  if it is into  $X$ , for any  $A$  from  $\mathbf{B}$ ,  $T^{-1}(A)$  also belongs to  $\mathbf{B}$ , and

$$(18) \quad g(A) = g(T^{-1}(A))$$

where  $T^{-1}(A)$  is the inverse image of  $A$ .

**Example 3.1** If  $k$  is a real number, then the mapping  $T(x) = x + k$  is an endomorphism of the real line  $\mathbf{R}$ .

Let  $r$  be a non-negative real number.

**Definition 3.3** A mapping  $T : X \rightarrow X$  is called an  $r$ -endomorphism of  $X$  if it is into  $X$ , for any  $A$  from  $\mathbf{B}$ ,  $T^{-1}(A)$  also belongs to  $\mathbf{B}$ , and

$$(19) \quad |g(A) - g(T^{-1}(A))| \leq r$$

**Example 3.2** If  $r$  is a real number, then the mapping  $T(x) = (1 - r)x$  is an  $r$ -endomorphism of the interval  $[0, 1]$ .

**Remark 3.1** This construction is close to the concept of Ulam [31] of an approximate homomorphism of a universal algebra. For instance, in the category of groups, one can consider a mapping  $f$  from a group  $A$  to a group  $B$  such that  $f(xy)$  not necessarily equals to  $f(x)f(y)$ , but must be "close" to  $f(x)f(y)$ . Such approximate homomorphisms were also studied by Farah [12, 13].

This allows us to formulate the following problem related to the Ulam's problem for approximate homomorphisms [31].

**Problem 1** Given a class of spaces with measure  $m$ , is every  $r$ -endomorphism can be  $k_r$ -approximated by a strict endomorphism, where  $k_r$  is a constant absolute for the class of considered spaces.

**Definition 3.4** A mapping  $T : X \rightarrow X$  is called a fuzzy endomorphism of  $X$  if it is an  $r$ -endomorphism of  $X$  for some  $r \geq 0$ .

**Definition 3.5** A mapping  $T : X \rightarrow X$  is called an automorphism ( $r$ -automorphism, fuzzy endomorphism) of  $X$  if it is a one-to-one endomorphism ( $r$ -endomorphism, fuzzy endomorphism, correspondingly) of  $X$ .

**Example 3.3** The mapping  $T(x) = -x$  is an automorphism of the interval  $[-1, 1]$ .

Let  $(X, \mathbf{B}, g)$  be a fuzzy measurable space,  $R$  be some semigroup with the operation  $\circ$ , which we call time in the space  $(X, \mathbf{B}, g)$ , and  $\mathbf{T}_R = \{T^i; i \in R\}$  be a parametric system of endomorphisms of  $(X, \mathbf{B}, g)$ , i.e.,  $T^i(T^j(x)) = T^{i \circ j}(x)$  for any  $i, j \in R$  and  $x \in X$ .

**Definition 3.6** A structure  $((X, \mathbf{B}, g), R, \mathbf{T}_R)$  is called a fuzzy dynamical system of the first type.

Let  $(X, \mathbf{B}, m)$  be a measurable space,  $R$  be some semigroup with the operation  $\circ$ , which we call time in the space  $(X, \mathbf{B}, m)$ , and  $\mathbf{T}_R = \{T^i; i \in R\}$  be a parametric system of  $r$ -endomorphisms of  $(X, \mathbf{B}, m)$ , i.e.,  $T^i(T^j(x)) = T^{i \circ j}(x)$  for any  $i, j \in R$  and  $x \in X$ .

**Definition 3.7** A structure  $((X, \mathbf{B}, m), R, \mathbf{T}_R)$  is called a fuzzy dynamical system of the second type.

**Example 3.4** Let  $(X, \mathbf{B}, m)$  be the interval  $[0, 1]$  with the Lebesgue measure  $m$ . Then the mapping  $T(x) = x^2$  is a  $\frac{1}{4}$ -endomorphism and  $((X, \mathbf{B}, m), \mathbf{Z}, \mathbf{T}_Z)$  is a fuzzy dynamical system of the second type with time  $\mathbf{T}_Z = \{T^i; i \in \mathbf{Z}\}$ .

#### 4. POINCARÉ RECURRENCE THEOREM FOR FUZZY DYNAMICAL SYSTEMS

Let  $A \subseteq X$  and  $T: X \rightarrow X$  be a mapping.

**Definition 4.1** A point  $a$  from  $X$  is called recurrent with respect to  $A$  and  $T$  if  $T^n(a)$  belongs to  $A$  for some natural number  $n > 0$ .

Let  $A \subseteq X$ ,  $R$  be a partially ordered semigroup with 0, and  $\mathbf{T}_R = \{T^i; i \in R\}$  be a parametric system of endomorphisms of  $(X, \mathbf{B}, g)$ .

**Definition 4.2** A point  $a$  from  $X$  is called recurrent with respect to  $A$  and  $\mathbf{T}_R$  if  $T^i(a)$  belongs to  $A$  for some  $i \in R$  with  $i > 0$ .

**Remark 4.1** There is a close connection between recurrent points, fuzzy/approximate fixed points [9, 5], and fuzzy/approximate periodic points. This connection is explicated by introduction of recurrence time for a point  $x$  with respect to a set  $A$  that contains this point. Namely, recurrence time  $TR(x, A)$  of  $x$  with respect to  $A$  is a sequence of integers  $(t_1, t_2, \dots, t_i, \dots)$  such that  $T^{t_i}(x) \in A$  for all  $i = 1, 2, \dots$ . For any point in a dynamical system with discrete positive time, there is the largest recurrent time.

Let us denote  $d_x(A) = \max\{d(x, z); z \in A\}$ .

**Proposition 4.1** A point  $x \in X$  is a  $k$ -fuzzy/approximate fixed point for a mapping  $T^n$  if there is a set  $A \subseteq X$  such that  $n$  belongs to some periodic time of  $x$  with respect to  $A$  and  $d_x(A) \leq k$ .

It is possible to prove a similar result for fuzzy/approximate periodic points.

For fuzzy dynamical systems of the first type with discrete countable time, the Poincaré recurrence theorem is not valid, in general, as it is demonstrated by the following example.

**Example 4.1** Let  $X = [-1, 1]$ ,  $\mathbf{A}$  be a Borel field of sets in  $X$ ,  $m$  be the Lebesgue measure on  $X$ , and  $D = \{\pm \frac{1}{n}; n = 2, 3, \dots\}$ . As a countable set,  $D \in \mathbf{A}$ . We define the fuzzy measure  $g$  by the following rules:

(a) if  $A \in \mathbf{A}$  and  $A \cap D \neq \emptyset$ , then  $g(A) = \frac{10}{21}(m(A) + \frac{1}{10})$ ;

(b) if  $A \in \mathbf{A}$  and  $A \cap D = \emptyset$ , then  $g(A) = \frac{10}{21}m(A)$ .

These rules imply that  $g(D) = g(C) = \frac{1}{21}$  for any  $C \subseteq D$ . In addition, all conditions 1-3 from Definition 2.2 are satisfied. In particular, multiplication by the fraction  $\frac{10}{21}$  makes the measure  $g$  normed, i.e.,  $g(X) = 1$ . As a result, we obtain a fuzzy measurable space  $(X, \mathbf{A}, g)$ .

We define on  $X$  the following transformation:

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = -\frac{1}{2} \\ x & \text{if } x \notin D \\ \frac{1}{n+1} & \text{if } x = \pm \frac{1}{n} \text{ \& } x \neq -\frac{1}{2} \end{cases}$$

As the transformation  $T$  maps  $D$  into itself,  $T$  is an endomorphism of  $X$ . As a result, we obtain a fuzzy dynamical system  $((X, \mathbf{A}, g), Z, \mathbf{T}_Z)$  of the first type. This system has discrete time  $Z$ .

However, for any interval  $[\frac{1}{k}, \frac{1}{2}]$  with  $k > 2$ , the measure  $g$  of non-recurrent points is not zero but is equal to  $\frac{1}{21}$ .

At the same time, when we take fuzzy dynamical systems with  $k$ -additive fuzzy measure and discrete countable time, the result of the Poincaré recurrence theorem remains true.

Let  $R$  be the additive group  $\mathbf{Z}$  of all integer numbers,  $k$  be a non-negative real number, and  $\mathbf{D} = ((X, \mathbf{B}, g), R, \mathbf{T}_Z)$  be a fuzzy dynamical system of the first type with discrete time  $\mathbf{Z}$ .

**Theorem 4.1** (Generalized Poincaré Recurrence Theorem) For any set  $A$  from  $\mathbf{B}$  in a fuzzy dynamical system  $\mathbf{D}$  of the first type with a  $k$ -additive fuzzy measure  $g$ , the measure of the set of non-recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$  is equal to zero, or, in other words, almost all points in  $A$  are recurrent.

*Proof.* Given a subset  $A$  of a set  $X$ , let us consider the set  $Y$  of all non-recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$ . Then by the definition  $Y = A \cap (\bigcap_{n=1}^{\infty} T^{-n}(X - A))$ . Really, if  $x \in \bigcap_{n=1}^{\infty} T^{-n}(X - A)$  and  $x$  is recurrent with respect to  $A$  and  $\mathbf{T}_Z$ , then  $T^m(x) \in A$  for some  $m > 0$ . Besides,  $x \in T^{-m}(X - A)$ . Consequently,  $T^m(x) \in T^m(T^{-m}(X - A)) = X - A$ . This contradicts to our supposition that  $T^m(x) \in A$  and thus, the point  $x$  is not recurrent and  $x \in Y$ . As  $x$  is an arbitrary point from  $\bigcap_{n=1}^{\infty} T^{-n}(X - A)$ , we have  $\bigcap_{n=1}^{\infty} T^{-n}(X - A) \subseteq Y$ . If  $x \notin T^{-m}(X - A)$  for some  $m > 0$ , then  $x \in T^{-m}(A)$ . Consequently,  $T^m(x) \in A$ , i.e.,  $x$  is a recurrent point. Thus,  $x \in T^{-m}(X - A)$  for all  $m > 0$  and we have proved that  $Y = A \cap (\bigcap_{n=1}^{\infty} T^{-n}(X - A))$ .

In addition,  $Y \cap T^{-n}(Y) = \emptyset$  for all  $n > 0$ . Really, if  $x \in Y$ , then  $T^n(x) \notin A$  for all  $n > 0$ . However, if  $x \in T^{-n}(Y)$ , then  $T^n(x) \in Y$  and  $Y \subseteq A$ , i.e.,  $T^n(x) \in A$ .

Moreover,  $T^{-n}(Y) \cap T^{-m}(Y) = \emptyset$  when  $m \neq n$ . Really, if  $0 < m < n$ , then  $T^{-n}(Y) \cap T^{-m}(Y) = T^{-m}(Y \cap T^{-(n-m)}(Y)) = \emptyset$  as  $Y \cap T^{-k}(Y) = \emptyset$  for all  $k > 0$ .

Now let us suppose that  $g(Y) = a > 0$ . As  $g$  is a  $k$ -additive fuzzy measure, the transformation  $T$  preserves measure, and  $Y \cap T^{-1}(Y) = \emptyset$ , we have  $g(Y \cup T^{-1}(Y)) \geq (1/2)(1+k)(g(Y) + g(T^{-1}(Y))) = (1+k)g(Y) = (1+k)a$ . This implies  $g(T^{-2}(Y) \cup T^{-3}(Y)) = g(T^{-2}(Y \cup T^{-1}(Y))) = g(Y \cup T^{-1}(Y)) \geq (1+k)a$ . Then by the same arguments, we have  $g(Y \cup T^{-1}(Y) \cup T^{-2}(Y) \cup T^{-3}(Y)) = (1+k)^2a$ . By induction, we prove that  $g(\bigcup_{n=0}^r T^{-n}(Y)) \geq (1+k)^m a$  when  $r = 2^m - 1$ .

As  $k > 0$  and  $a > 0$ , there is such  $m$  for which  $g(\bigcup_{n=0}^r T^{-n}(Y)) > 1$ . Because  $\bigcup_{n=0}^r T^{-n}(Y) \subseteq X$  and  $g(X) = 1$ , monotonicity of  $g$  yields a contradiction. Thus, our assumption cannot be true and  $g(Y) = 0$ .

Theorem is proved.

Any additive fuzzy measure  $g$  is a  $k$ -additive fuzzy measure with  $k = 1$ . This implies the following result.

**Corollary 4.1** For any measurable set  $A$  in a fuzzy dynamical system  $\mathbf{D}$  of the first type with an additive fuzzy measure, the measure of the set of non-recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$  is equal to zero or almost all points from  $A$  are recurrent.

**Remark 4.2** For fuzzy dynamical systems of the first type in which the measure is not additive, two conclusions of Corollary 4.1 are not always equivalent as it is demonstrated by the following example.

**Example 4.2** Let  $X = [-1, 1]$ ,  $\mathbf{A}$  be a Borel field of sets in  $X$ ,  $m$  be the Lebesgue measure on  $X$ , and  $D = \{\pm \frac{1}{n}; n = 2, 3, \dots\}$ . As a countable set,  $D \in \mathbf{A}$ . We define the fuzzy measure  $g$  by the following rules:

(c) if  $A \subseteq (\frac{1}{3}, \frac{1}{2})$ , then  $g(A) = m(A) - \frac{1}{10}$ ;

(d) for all other sets  $A \in \mathbf{A}$ ,  $g(A) = m(A)$ .

All conditions from Definition 2.3 are satisfied. As a result, we obtain a fuzzy measurable space  $(X, \mathbf{A}, g)$ .

We define on  $X$  the following transformation:

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = -\frac{1}{2} \\ x & \text{if } x \notin D \\ \frac{1}{n+1} & \text{if } x = \pm \frac{1}{n} \text{ \& } x \neq -\frac{1}{2} \end{cases}$$

As the transformation  $T$  maps  $D$  into itself,  $T$  is an endomorphism of  $X$ . As a result, we obtain a fuzzy dynamical system  $\mathbf{D} = ((X, \mathbf{B}, g), R, \mathbf{T}_Z)$  of the first type. This system has discrete time  $\mathbf{Z}$ .

In addition, for any set  $A \in \mathbf{A}$ , the measure  $g$  of non-recurrent points is equal to zero, but the measure  $g$  of all recurrent points in the segment  $[\frac{1}{3}, \frac{1}{2}]$  is not equal to  $g([\frac{1}{3}, \frac{1}{2}]) = \frac{1}{6}$ .

**Corollary 4.2** (Poincaré Recurrence Theorem). For any measurable set  $A$  in a dynamical system with a  $\sigma$ -additive measure, the measure of the set of non-recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$  is equal to zero or almost all points from  $A$  are recurrent.

**Definition 4.3** A point  $a$  from  $X$  is called infinitely (finitely) recurrent with respect to  $A$  and  $\mathbf{T}_R$  if  $T_i(a)$  belongs to  $A$  for an infinite (finite) number of  $i \in R$  with  $i > 0$ .

**Corollary 4.3** For any set  $A$  from  $\mathbf{B}$  in a fuzzy dynamical system  $\mathbf{D}$  of the first type with a  $k$ -additive continuous fuzzy measure  $g$ , the measure of the set of non-infinitely recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$  is equal to zero.

**Proof.** Given a subset  $A$  of a set  $X$ , let us consider the set  $Y$  of all points from  $A$  that are not infinitely recurrent with respect to  $A$  and  $\mathbf{T}_Z$  points in  $A$ . If  $x$  belongs to  $Y$ , then there is some number  $p$  such that for all  $k > p$ ,  $T^k(a)$  does not belong to  $A$ . By the definition, it means that  $x$  is non-recurrent with respect to  $A$  and  $\mathbf{T}_Z^p$ . By the Generalized Poincaré recurrence theorem the measure of the set  $Y_p$  of all such points is equal to zero. As  $Y = \bigcap_{p=1}^{\infty} Y_p$ , for all  $n$ ,  $Y_n \subseteq Y_{n+1}$ , and the measure in  $\mathbf{D}$  is continuous, the measure of the set  $Y$  is equal to zero.

Corollary 4.3 is proved.

**Corollary 4.4** For any measurable set  $A$  in a fuzzy dynamical system  $\mathbf{D}$  of the first type with an additive continuous fuzzy measure, almost all points in  $A$  are infinitely recurrent with respect to  $A$  and  $\mathbf{T}_Z$ .

Let us consider a space  $X$  with a measure (fuzzy measure)  $m$ , a non-negative real number  $k$ , some property  $P$  of elements from  $X$ , and a subset  $A$  of  $X$ .

**Definition 4.4** We say that  $k$ -almost all elements from  $A$  have property  $P$  if  $m(A) - m(A_P) \leq k$  where  $A_P = \{x \in A; x \text{ has the property } P\}$ .

Let  $R$  be the additive group  $\mathbf{Z}$  of all integer numbers and  $\mathbf{D} = ((X, A, g), R, T_Z)$  be a fuzzy dynamical system of the second type with an additive fuzzy measure  $g$  and discrete time  $\mathbf{Z}$ .

**Theorem 4.2** (Fuzzy Poincaré Recurrence Theorem) For any set  $A$  from  $\mathbf{A}$  in a fuzzy dynamical system  $\mathbf{C}$  of the second type with an  $r$ -endomorphism  $T$ , the measure of the set of non-recurrent with respect to  $A$  and  $T_Z$  points in  $A$  is less than or equal to  $r$ , or, in other words,  $r$ -almost all points in  $A$  are recurrent.

*Proof.* Let us consider a subset  $A$  of the set  $X$  and the set  $Y$  of all non-recurrent with respect to  $T_Z$  points in  $A$ . As it is demonstrated in the proof of Theorem 4.1,  $Y = A \cap (\bigcap_{n=1}^{\infty} T^{-n}(X - A))$ . In addition,  $Y \cap T^{-n}(Y) = \emptyset$  for all  $n > 0$ . Moreover,  $T^{-n}(Y) \cap T^{-m}(Y) = \emptyset$  when  $m \neq n$ . Really, if  $0 < m < n$ , then  $T^{-n}(Y) \cap T^{-m}(Y) = T^{-m}(Y \cap T^{-(n-m)}(Y)) = \emptyset$  as  $Y \cap T^k(Y) = \emptyset$  for all  $k > 0$ . The measure  $g$  is additive, the transformation  $T$  is an  $r$ -endomorphism, and  $Y \cap T^{-1}(Y) = \emptyset$ . So, we have  $g(Y \cup T^{-1}(Y)) \geq 2g(Y) - r$  because  $g(T^{-1}(Y)) \geq g(Y) - r$ . By induction,  $g(\bigcup_{n=0}^p T^{-n}(Y)) \geq (p+1)g(Y) - pr$  for an arbitrary  $p = 1, 2, \dots$ . Now let us assume that  $g(Y) = a > r$ . Then  $g(\bigcup_{n=0}^p T^{-n}(Y)) \geq (p+1)a - pr > p(a-r)$ . As  $a-r$  is a fixed positive number and  $p$  tends to infinity, for some  $p$  we have  $g(\bigcup_{n=0}^p T^{-n}(Y)) > p(a-r) > 1$ . This is a contradiction because all considered sets are subsets of  $X$ ,  $g(X) = 1$ , and  $g$  satisfies condition (2) from Definition 2.3. Equivalence of the two conclusions of the theorem follows from Lemma 2.1. Theorem is proved.

For  $r = 0$ , we have Poincaré Recurrence Theorem for the case when the measure is only additive.

**Corollary 4.5** For any measurable set  $A$  in a dynamical system, the measure of the set of non-recurrent with respect to  $A$  and  $T_Z$  points in  $A$  is equal to zero or almost all points from  $A$  are recurrent.

However, the statement of Corollary 4.4 about existence of infinite recurrent points for dynamical systems of the first type, as well as for classical dynamical systems, is not true for dynamical systems of the second type. It is demonstrated by the following example.

**Example 4.3** Let  $X = [0, 1]$ ,  $\mathbf{A}$  be a Borel field of sets in  $X$ ,  $m$  be a Lebesgue measure on  $X$ , and  $T(x) = (\frac{9}{10})x$ . The mapping  $T(x)$  is a  $(\frac{1}{10})$ -endomorphism of  $X$  as  $m(A) - m(TA) \leq \frac{1}{10}$ . As a result, we obtain a fuzzy dynamical system  $((X, \mathbf{A}, m), \mathbf{Z}, T_Z)$  of the second type with the discrete time  $\mathbf{Z}$ .

Let us consider the set  $A = [\frac{8}{10}, 1]$  and the set  $NRA$  of all non-recurrent points from  $A$ . Then  $m(NRA) = \frac{8}{9} - \frac{8}{10} = \frac{4}{45} < \frac{1}{10}$ . This correlates with the statement of the Fuzzy Poincaré Recurrence Theorem. However, any point from  $A$  returns to  $A$  at most two times. Moreover, for any set  $B = [a, b]$  with  $0 < a < b < 1$ , there are no infinitely recurrent points.

## 5. CONCLUSION

Thus, we have extended the scope of application of the Poincaré recurrence theorem, proving its more advanced versions for fuzzy dynamical systems. This allows us to explain various phenomena in social and individual practice. For instance, if we analyze history of science as the dynamics of scientific knowledge, we can see many recurrent ideas and approaches. Let us look at the idea of atomic structure of matter [22]. The beginning was in ancient Greece. Two Attic philosophers Democritus (460-370 BCE) and Leucippus (fifth century BCE) suggested that the whole

universe consists of a void and a very large number of invisible and indivisible particles that were called *atoms*.

Then for a long time the majority of philosophers and then physicists refused to accept the atomic idea at all. Only in the 17th century, this idea returned to the scientific discourse. The main protagonists of this theory were Pierre Gassendi and Robert Boyle. Then the 18th century was barren of further progress in the domain of the atomic theory, so far as physics is concerned. Only in the 19th century this theory, due to many supporting discoveries, became domineering, although disagreement continued, with eminent physicists like Ernst Mach and Wilhelm Ostwald denouncing the atomic theory.

Examples of recurrent ideas in physics are the concept of void (vacuum), corpuscular structure of light, and idea of relativity. Examples of other recurrent ideas in mathematics are reduction of geometry to arithmetic/algebra, axiomatic approach, and integration.

Thus, we can observe recurrence predicted by fuzzy versions of the Poincaré recurrence theorem not only in physics but also in other domains.

To conclude, it is necessary to mention that it is possible to develop many directions of classical theory of dynamical systems, for example, symbolic dynamics and chaos theory [26], in the fuzzy context constructed in this paper.

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