

ON λ -DENDROIDS WITH THE ΩEP -PROPERTY

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ABSTRACT. A space X has the ΩEP -property if for each self-mapping the set of nonwandering points is contained in the closure of the set of eventually periodic points. A class of λ -dendroids is presented each member of which has the ΩEP -property for mappings satisfying certain additional conditions. These λ -dendroids are obtained as compactifications of some dendrites without finite sets. Some related problems are posed.

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1. INTRODUCTION

Let X be a (metric) space, and let $f : X \rightarrow X$ be a mapping of X to itself. For any $n \in \mathbb{N}$ let $f^n : X \rightarrow X$ denote the n -th composition of f . A point x of X is said to be:

- a *fixed* point of f if $f(x) = x$;
- a *periodic* point of f provided that there is $n \in \mathbb{N}$ such that $f^n(x) = x$; if, moreover, $f^k(x) \neq x$ for all integers k with $1 \leq k < n$, then x is called a periodic point of *period* n ;
- a *recurrent* point of f , provided that for each open set U containing x there is $n \in \mathbb{N}$ such that $f^n(x) \in U$;
- an *eventually periodic point of period* $n \in \mathbb{N}$ for f provided that there exists $m \in \{0\} \cup \mathbb{N}$ such that $f^m(x)$ is a periodic point of f of period n ;
- an *eventually periodic point for* f provided that there is $n \in \mathbb{N}$ such that x is an eventually periodic point of period $n \in \mathbb{N}$ for f ;
- a *nonwandering* point of f provided that for any open set U containing x there exist $y \in U$ and $n \in \mathbb{N}$ such that $f^n(y) \in U$.

Note that if the *orbit* of x is defined by $\text{orb}(f; x) = \{f^n(x) : n \in \{0\} \cup \mathbb{N}\}$, then x is eventually periodic if $\text{orb}(f; x)$ is finite, or equivalently if some element of $\text{orb}(f; x)$ is periodic.

For a mapping $f : X \rightarrow X$ the sets of fixed points, periodic points, recurrent points, eventually periodic points and nonwandering points of f will be denoted by $F(f)$, $P(f)$, $R(f)$, $EP(f)$ and $\Omega(f)$, respectively. The statement below is a consequence of definitions.

Statement 1.1. *For each compact space X and each self-mapping $f : X \rightarrow X$ we have*

$$(1.1.1) \quad F(f) \subset P(f) \subset R(f) \subset \Omega(f) \subset X, \quad P(f) \subset EP(f),$$

$$(1.1.2) \quad f(P(f)) = P(f), \quad f(R(f)) = R(f), \quad f(\Omega(f)) \subset \Omega(f),$$

$$(1.1.3) \quad \Omega(f) \text{ is closed.}$$

Further, for each $k \in \mathbb{N}$, we have

$$(1.1.4) \quad P(f^k) = P(f), \quad R(f^k) = R(f), \quad EP(f^k) = EP(f), \quad \Omega(f^k) \subset \Omega(f).$$

Remark 1.2. The inclusions in the last formulas of (1.1.2) and (1.1.4) cannot be replaced by the equalities, even for mappings of an arc onto itself. Namely, if $f : [0, 7] \rightarrow [0, 7]$ is a piecewise linear mapping defined by

$$f(0) = 3, \quad f(1) = f(2) = 4, \quad f(3) = 7, \quad f(4) = 7, \quad f(5) = 1, \quad f(7) = 3$$

(see [4, Example 8, p. 75]), then $1 \in \Omega(f) \setminus (f(\Omega(f)) \cup \Omega(f^2))$. In connection with this example a question arises.

Question 1.3. For what mappings $f : X \rightarrow X$ (a) does the equality $f(\Omega(f)) = \Omega(f)$ hold? (b) does the equality $\Omega(f) = \Omega(f^k)$ hold for all $k \in \mathbb{N}$?

Definition 1.4. A space X is said to have (a) the *periodic-recurrent property* (abbreviated *PR-property*), (b) the *nonwandering-eventually-periodic property* (abbreviated *ΩEP -property*) provided that each mapping $f : X \rightarrow X$ satisfies

$$(a) \quad \text{cl}_X(P(f)) = \text{cl}_X(R(f)),$$

$$(b) \quad \Omega(f) \subset \text{cl}_X(EP(f)),$$

respectively.

Topological dynamics on various spaces started with its study on the interval (see for example an expository paper [17] and references therein). In particular, it has been shown that the closed unit interval has the *PR-property*, [10, Theorem 1, p. 316], and the *ΩEP -property*, [3, Theorem C, p. 228]. These results have been extended to mappings of trees in [18, Theorem 2.6, p. 349] and in [11, Theorem 1.2, p. 36], and to ones of some dendrites in [13] and in [8, Corollary 5.14].

The results saying that trees have the *PR-property* and the *ΩEP -property* cannot be extended to all *graphs* (i.e., continua that can be written as the union of finitely many arcs any two of which are either disjoint or intersect only at one or both of its end points) because the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ does not have the properties. Indeed, if $f : \mathbb{S} \rightarrow \mathbb{S}$ is an irrational rotation (i.e., a rotation by an angle α such that α/π is irrational), then $P(f) = \emptyset = EP(f)$, while $R(f) = \mathbb{S} = \Omega(f)$ (compare [5, Remarks 1.2, p. 132] and [11, Example 1.4, p. 44]).

However, as it was presented in [5] and in [2], the *PR-property* holds for each λ -dendroid which is obtained from a tree T by inserting (in a special way) finitely many λ -dendroids having the *PR-property* and satisfying some additional conditions. The aim of the present paper is to show that the same method can be applied

to obtain another class of λ -dendroids having the ΩEP -property. A number of related questions is also asked.

The paper consists of six sections. After Introduction and Preliminaries, λ -dendroids as compactifications are considered in Section 3. Some auxiliary results, known from earlier papers, are collected the next section. These results are used in Section 5 to prove the ΩEP -property for a class Δ of λ -dendroids. Section 6 contains a number of related questions and problems.

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2. PRELIMINARIES

All considered spaces are assumed to be metric, and a *mapping* means a continuous function. We denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the one of all reals, and by $\mathbb{C}$ the set of all complex numbers. For $A \subset X$ we denote $\text{cl}_X(A)$ the closure of A in X . A *continuum* means a compact connected space.

A mapping $r : X \rightarrow Y$ between continua X and Y is called a *retraction* if $Y \subset X$ and the partial mapping $r|_Y : Y \rightarrow Y$ is the identity. In this case Y is called a *retract* of X .

The following lemma on compositions of mappings is known (see [5, Lemma 3.1, p. 136] and compare [8, Lemma 2.9]).

Lemma 2.1. *Let X and Y be spaces with Y being a closed subset of X , and let $g : Y \rightarrow Y$ be a mapping. If $r : X \rightarrow Y$ is a retraction and $f = g \circ r : X \rightarrow Y$, then:*

$$(2.1.1) \quad f^n = g^n \circ r \text{ for each } n \in \mathbb{N};$$

$$(2.1.2) \quad P(f) = P(g);$$

$$(2.1.3) \quad R(f) = R(g);$$

$$(2.1.4) \quad Y \cap EP(f) = EP(g);$$

$$(2.1.5) \quad \Omega(f) = \Omega(g).$$

As a consequence of Lemma 2.1 we get a result below (see [5, Proposition 3.2, p. 136] and [8, Proposition 5.1]).

Corollary 2.2. *Both the PR -property and the ΩEP -property are preserved under retractions, i.e., if a space X having the PR -property (the ΩEP -property) contains a closed subspace Y which is a retract of X , then Y has the PR -property (the ΩEP -property), too.*

It is quite natural to ask what kinds of mappings (or what kinds of mappings between some special spaces) can be substituted in place of retractions in the above result. This question is discussed, and some results are obtained, in [6].

The following proposition is an immediate consequence of definitions.

Proposition 2.3. *Let A be a subspace of a space X , and let a mapping $f : X \rightarrow X$ be such that $f(A) \subset A$. Then*

$$(2.3.1) \quad P(f|A) = A \cap P(f);$$

$$(2.3.2) \quad R(f|A) = A \cap R(f);$$

$$(2.3.3) \quad EP(f|A) = A \cap EP(f);$$

$$(2.3.4) \quad \Omega(f|A) \subset A \cap \Omega(f).$$

Remark 2.4. The inclusion in (2.3.4) cannot be replaced by the equality, because if $f : [0, 7] \rightarrow [0, 7]$ is the mapping defined in Remark 1.2 and $A = \{1, 4\}$, then $f(A) \subset A$ and $A \cap \Omega(f) = A$, while $\Omega(f|A) = \{4\}$.

In connection with the above remark the next question is natural.

Question 2.5. For what mappings $f : X \rightarrow Y$ is the equality $\Omega(f|A) = A \cap \Omega(f)$ true for each $A \subset X$?

We use a concept of an *order* of a point p in a continuum X (in the sense of Menger-Urysohn), written $\text{ord}(p, X)$, as defined in [16, 9.3, p. 141] or [14, §51, I, p. 274].

A point $p \in X$ is called an *end point* of X provided that $\text{ord}(p, X) = 1$, and it is called a *branch point* of X provided that $\text{ord}(p, X) \geq 3$. For a continuum X we denote the sets of end points of X and of branch points of X by $E(X)$ and $B(X)$, respectively.

Recall that a continuum X is said to be

- *hereditarily unicoherent* provided that the intersection of any two subcontinua of X is connected;
- *hereditarily decomposable* provided that every subcontinuum of X is the union of two of its proper subcontinua;
- a λ -*dendroid* if it is hereditarily unicoherent and hereditarily decomposable;
- a *dendroid* if it is hereditarily unicoherent and arcwise connected;
- a *dendrite* if it is locally connected and contains no simple closed curve;
- a *tree* if it is a graph containing no simple closed curve.

Let $\mathcal{T}$ denote the class of trees, $\mathcal{D}_0$ — the class of dendrites, $\mathcal{D}$ — the class of dendroids, $\lambda\mathcal{D}$ — of λ -dendroids, $\mathcal{TL}$ — of tree-like continua, and $\mathcal{HU}$ — the class of hereditarily unicoherent ones. Then we have the following inclusions.

$$\mathcal{T} \subset \mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{TL} \subset \mathcal{HU}.$$

The next result is an extension of the one formulated in [7, Proposition 4.1, p. 113] (compare also [2]).

Proposition 2.6. *Let $f : X \rightarrow X$ be a mapping of a continuum X into itself. Then $\{f^n(X)\}_{n=1}^\infty$ is a decreasing sequence of subcontinua of X and $M(X, f) = \bigcap \{f^n(X) : n \in \mathbb{N}\}$ is a subcontinuum of X such that*

- (2.6.1) $M(X, f) = \text{Lim } f^n(X)$;
- (2.6.2) $f|M(X, f) : M(X, f) \rightarrow M(X, f)$ is a surjection;
- (2.6.3) $P(f) \subset R(f) \subset M(X, f)$;
- (2.6.4) $P(f) = P(f|M(X, f))$ and $R(f) = R(f|M(X, f))$;
- (2.6.5) $EP(f) = EP(f|M(X, f))$ and $\Omega(f) = \Omega(f|M(X, f))$;
- (2.6.6) $M(X, f)$ is a maximal subcontinuum of X satisfying (2.6.2).

Proof. Indeed, (2.6.1)-(2.6.4) and (2.6.6) are proved in [7, Proposition 4.1, p. 113], and (2.6.5) follows from (2.6.2) and Proposition 2.3, equalities (2.3.3) and (2.3.4). $\square$

Remark 2.7. As it is indicated in [2], Proposition 2.6 shows that, when a mapping f from a continuum X into itself is investigated, the whole dynamics for f is on the subcontinuum $M(X, f)$ of X . So, in such situation, we can replace X by $M(X, f)$ and then assume that f is a surjection. However, we cannot reduce the study of the ΩEP -property of X to the study of this property on $M(X, f)$, since for a

given mapping f from X into itself, the sets X and $M(X, f)$ do not have to be homeomorphic and, in general, a given map g from $M(X, f)$ onto itself cannot be extended to a map from X into itself. In the next example we see that the ΩEP -property on some set $M(X, f)$ for some mapping f from X into itself, does not imply the ΩEP -property of X .

Example 2.8. *There exist a dendrite W which does not have the ΩEP -property, and a monotone retraction f from W into itself, such that the subcontinuum $M(W, f)$ has the ΩEP -property.*

Proof. The dendrite W is defined by

$$(2.8.1) \quad W = [0, 1] \times \{0\} \cup \bigcup \left\{ \left\{ \frac{1}{n} \right\} \times [0, \frac{1}{n}] : n \in \mathbb{N} \right\}.$$

So, any homeomorphic copy of W is a dendrite all branch points of which are of order 3, all lie in an arc so that one of the end points of the arc is the only accumulation point of the set of all branch points of W .

It is known that W does not have the ΩEP -property [8, Proposition 5.8]. Let $f : W \rightarrow Y = [0, 1] \times \{0\} \subset W$ be the monotone retraction. Then $M(X, f) = Y$ and since Y is an arc, it has the ΩEP -property. $\square$

The following question is related to the above example.

Question 2.9. Does there exist a continuum X having the ΩEP -property, and a mapping f from X into itself such that $M(X, f)$ does not have the ΩEP -property?

3. λ -DENDROIDS AS COMPACTIFICATIONS

Given a λ -dendroid X we denote by $\mathcal{P}(X)$ the family of all subcontinua S of X such that for each finite cover of X the elements of which are subcontinua of X , the continuum S is contained in a member of the cover. A (transfinite) well-ordered sequence (numbered with ordinals α) of nondegenerate subcontinua X_α of a λ -dendroid X is said to be *normal* provided that the following conditions are satisfied:

$$\begin{aligned} X_1 &= X; \\ X_{\alpha+1} &\in \mathcal{P}(X_\alpha); \\ X_\beta &= \bigcap \{X_\alpha : \alpha < \beta\} \quad \text{for each limit ordinal } \beta. \end{aligned}$$

The *depth* $k(X)$ of a λ -dendroid X is defined as the minimum ordinal number η such that the order type of each normal sequence of subcontinua of X is not greater than η . The reader is referred to [12] and [15] for an additional information related to this concept. The following three assertions concerning the depth are known, [12, Theorems 1, 2 and 3, p. 94 and 95].

Statement 3.1. *Let X and Y be λ -dendroids.*

(3.1.1) *If $Y \subseteq X$, then $k(Y) \leq k(X)$.*

(3.1.2) *X is locally connected (i.e., it is a dendrite) if and only if $k(X) = 1$.*

(3.1.3) *If Y is a continuous image of X , then $k(Y) \leq k(X)$.*

A subcontinuum Q of a continuum X is said to be *terminal* provided that for every subcontinuum K of X if $K \cap Q \neq \emptyset$ then either $K \subset Q$ or $Q \subset K$. We need the following result, see [1, Theorem, p. 35] and [7, Theorem 3.1, p.111].

Theorem 3.2. *If M is a locally compact, noncompact metric space, then each continuum is a remainder of M in some compactification of M as a terminal subcontinuum of the compactification.*

To make the paper self-contained and to formulate the needed results we recall a construction described in [9, Section 3].

Let a dendrite D and points $q_1, \dots, q_n$ of D be given for some positive integer n . Let $Q_1, \dots, Q_n$ be continua. Choose in D closed connected and mutually disjoint neighborhoods $U_1, \dots, U_n$ of points $q_1, \dots, q_n$. Then for each $i \in \{1, \dots, n\}$ the sets $U_i \setminus \{q_i\}$ are locally compact and noncompact, thus applying Theorem 3.2 to each of them we construct in a standard way a compactification

$$(\gamma) \quad \gamma: (D \setminus \{q_1, \dots, q_n\}) \rightarrow \gamma(D \setminus \{q_1, \dots, q_n\})$$

such that:

- ($\gamma.1$) $X = \text{cl}_X(\gamma(D \setminus \{q_1, \dots, q_n\}))$ is a continuum;
- ($\gamma.2$) the remainder $X \setminus \gamma(D \setminus \{q_1, \dots, q_n\})$ consists of n components $Q_1, \dots, Q_n$;
- ($\gamma.3$) for each index $i \in \{1, \dots, n\}$ the continuum Q_i is a terminal subcontinuum of X .

The next quoted result is in [9, Observation 3.3].

Observation 3.3. *If the inserted continua Q_i are λ -dendroids, then the resulting continuum X satisfying ($\gamma.1$)-($\gamma.3$) is a λ -dendroid, too.*

Thus the concept of the depth $k(X)$ is well defined for such X (and for all subcontinua of X). We say that the inserted continua Q_i have the same finite depth provided that there is $d \in \mathbb{N}$ such that for each $i \in \{1, \dots, n\}$ we have $k(Q_i) = d$.

4. KNOWN AUXILIARY PROPERTIES

To make formulation of the forthcoming results shorter, accept the following definition (introduced in [9, Definition 3.4]).

Definition 4.1. Let Δ denote the class of all λ -dendroids which are obtained from a dendrite D by replacing finitely many its points $q_1, \dots, q_n$ by λ -dendroids $Q_1, \dots, Q_n$ of the same finite depth using a compactification γ with ($\gamma.1$)-($\gamma.3$). For a λ -dendroid $X \in \Delta$ we put

$$(4.1.1) \quad A = \bigcup \{Q_i : i \in \{1, \dots, n\}\} \quad \text{and} \quad H = X \setminus A.$$

Now we recall the main results concerning the members of the class Δ obtained in Section 3 of [9].

Theorem 4.2. *Let a λ -dendroid X belong to the class Δ , and let Y be a nondegenerate subcontinuum of X such that, for some $i \in \{1, \dots, n\}$,*

$$(4.2.1) \quad Y \cap Q_i \neq \emptyset \neq Y \setminus Q_i.$$

Then

$$(4.2.2) \quad k(Y) \geq 1 + k(Q_i).$$

Proposition 4.3. *Let a λ -dendroid X be in Δ , and let A and H be defined by (4.1.1). If $f: X \rightarrow X$ is a surjection, then*

$$(4.3.1) \quad A \subset f(A);$$

(4.3.2) there are no indices $i, j \in \{1, \dots, n\}$ such that $f(Q_i) \cap Q_j \neq \emptyset \neq f(Q_i) \cap (X \setminus Q_j)$;

(4.3.3) there is no index $i \in \{1, \dots, n\}$ such that $f(Q_i) \subset H$.

Corollary 4.4. *Let a λ -dendroid X be in Δ . If $f : X \rightarrow X$ is a surjection, then for each $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $f(Q_i) = Q_j$. Furthermore, the correspondence between indices i and j is one-to-one and it maps $\{1, \dots, n\}$ onto itself. In other words, each surjective mapping $f : X \rightarrow X$ permutes the continua Q_i for $i \in \{1, \dots, n\}$, as well as the components of the set H , and consequently we have*

$$(4.4.1) \quad f(H) = H.$$

For a λ -dendroid $X \in \Delta$ let D be the dendrite as in Definition 4.1, and let $p : X \rightarrow D$ be the natural projection, i.e., such a mapping that $p(Q_i) = \{q_i\}$ for each $i \in \{1, \dots, n\}$ and that $p|_H = \gamma^{-1} : H \rightarrow (D \setminus \{q_1, \dots, q_n\})$ is a one-to-one mapping. Thus $p|_H$ is a homeomorphism. Let $f : X \rightarrow X$ be a surjective mapping. Define a mapping $g : D \rightarrow D$ by

$$(*) \quad g(z) = p(f(p^{-1}(z))) \quad \text{for each point } z \in D.$$

In particular, since $p^{-1}(q_i) = Q_i$ for each $i \in \{1, \dots, n\}$, hence $g(q_i) = p(f(p^{-1}(q_i)))$. Therefore g is well defined, and its continuity is a consequence of the definition. Further, statement 4.3 implies similar statement for the mapping g . More precisely, we have the following corollary.

Corollary 4.5. *Let a λ -dendroid X be a member of the class Δ , a mapping $f : X \rightarrow X$ be a surjection, $p : X \rightarrow D$ be the natural projection, and let $g : D \rightarrow D$ be defined by (*). Then the following assertions hold.*

(4.5.1) $p|_H : H \rightarrow D \setminus \{q_1, \dots, q_n\}$ is a homeomorphism;

(4.5.2) $g(\{q_1, \dots, q_n\}) = \{q_1, \dots, q_n\}$.

(4.5.3) $g(D \setminus \{q_1, \dots, q_n\}) = D \setminus \{q_1, \dots, q_n\}$.

(4.5.4) The mapping g permutes the points q_i for $i \in \{1, \dots, n\}$.

(4.5.5) $q_i \in P(g)$ for each $i \in \{1, \dots, n\}$.

(4.5.6) $p(f(x)) = g(p(x))$ for each point $x \in X$ and, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ p \downarrow & & \downarrow p \\ D & \xrightarrow{g} & D \end{array}$$

(4.5.7) $p(f^m(x)) = g^m(p(x))$ for each point $x \in X$ and for each $m \in \mathbb{N}$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f^m} & X \\ p \downarrow & & \downarrow p \\ D & \xrightarrow{g^m} & D \end{array}$$

(4.5.8) for each $m \in \mathbb{N}$

$$f^m(A) = A \quad \text{and} \quad f^m(H) = H.$$

5. ΩEP -PROPERTY FOR THE λ -DENDROIDS OF THE CLASS Δ

We apply notation introduced in Sections 3 and 4.

Proposition 5.1. *For each point $x \in H$ we have the following two equivalences:*

(5.1.1) $x \in EP(f)$ is equivalent to $p(x) \in EP(g)$;

(5.1.2) $x \in \Omega(f)$ is equivalent to $p(x) \in \Omega(g)$.

Proof. To show (5.1.1) take $x \in EP(f)$. This means that there are $n \in \mathbb{N}$ and $m \in \{0\} \cup \mathbb{N}$ such that for each $j \in \mathbb{N}$ with $j \geq m$ we have $f^{n+j}(x) = f^j(x)$. Since $x \in H$ and $p|_H$ is a homeomorphism (thus one-to-one), this last equality is equivalent to $p(f^{n+j}(x)) = p(f^j(x))$, which in turn is equivalent to $g^{n+j}(p(x)) = g^j(p(x))$ by the commutativity of diagram (4.5.7). Using the definition of $EP(g)$ we see that the equivalence (5.1.1) is shown.

To show (5.1.2) take $x \in \Omega(f)$, and let U be an open neighborhood of $p(x)$ in D . Therefore $p^{-1}(U)$ is an open neighborhood of x in X . By the definition of $\Omega(f)$ there are a point $y \in p^{-1}(U)$ and a number $m \in \mathbb{N}$ such that $f^m(y) \in p^{-1}(U)$. Therefore $p(f^m(y)) \in p(p^{-1}(U)) = U$, which gives $g^m(p(y)) \in U$ by the commutativity of diagram (4.5.7). Thus one implication of (5.1.2) follows from the definition of $\Omega(g)$.

To see the other one, take $p(x) \in \Omega(g)$ and let V be an open neighborhood of x in X . Since H is open in X by its definition, and since $p|_H : H \rightarrow D \setminus \{q_1, \dots, q_n\}$ is a homeomorphism, the set $p(V \cap H)$ is an open neighborhood of $p(x)$. Since $p(x) \in \Omega(g)$, there are $m \in \mathbb{N}$ and $y \in p(V \cap H)$ such that $g^m(y) \in p(V \cap H)$. Further, since $p|_H$ is a homeomorphism, there is exactly one point $x' \in V \cap H$ such that $y = p(x')$. Again by the commutativity of diagram (4.5.7) we have $g^m(y) = g^m(p(x')) = p(f^m(x')) \in p(V \cap H)$, whence $p^{-1}(p(f^m(x')))) \in p^{-1}(p(V \cap H)) = V \cap H \subset V$, which shows that $x \in \Omega(f)$ by the definition of $\Omega(f)$. Thus the proof is complete. $\square$

The next theorem corresponds to [9, Theorem 4.2] (where the PR -property is considered).

Theorem 5.2. *Let a λ -dendroid X be a member of the class Δ . If*

(5.2.1) *the dendrite D has the ΩEP -property, and*

(5.2.2) *the union $A = Q_1 \cup \dots \cup Q_n$ has the ΩEP -property,*

then for each surjective mapping $f : X \rightarrow X$ such that

(5.2.3) $\Omega(f|A) = A \cap \Omega(f)$

the inclusion $\Omega(f) \subset \text{cl}_X(EP(f))$ holds, that is, X has the ΩEP -property for surjections that satisfy (5.2.3).

Proof. Let a surjective mapping $f : X \rightarrow X$ satisfying (5.2.3) be given. According to Definition 1.4 we have to show that

$$\Omega(f) \subset \text{cl}_X(EP(f)).$$

So, take $x \in \Omega(f)$. Since $X = A \cup H$, consider two cases.

CASE 1. $x \in A$.

By (5.2.3) $x \in \Omega(f|A)$, whence by (5.2.2) there is a sequence of points $x_k \in EP(f|A)$ such that $x = \lim x_k$. But since $EP(f|A) = A \cap EP(f)$ by (2.3.3) of Proposition 2.3, we get $x_k \in EP(f)$, and consequently $x \in \text{cl}_X(EP(f))$.

CASE 2. $x \in H$.

This is equivalent to the assertion that $p(x) \in D \setminus \{q_1, \dots, q_n\}$. Since D has the ΩEP -property by (5.2.1), the inclusion $\Omega(g) \subset \text{cl}_D(EP(g))$ is satisfied, and thus there is a sequence of points $y_k \in EP(g)$ having $p(x)$ as its limit. Since the set $\{q_1, \dots, q_n\}$ is finite, we may assume that all points y_k are in $D \setminus \{q_1, \dots, q_n\}$. Since $p|_H : H \rightarrow D \setminus \{q_1, \dots, q_n\}$ is one-to-one, for each $k \in \mathbb{N}$ there is a unique point $x_k \in H$ such that $p(x_k) = y_k \in EP(g)$, whence it follows by (5.1.1) of Proposition 5.1 that $x_k \in EP(f)$. Since the sequence $\{p(x_k)\}$ tends to $p(x)$ and $p|_H$ is a homeomorphism, we conclude that $\{x_k\}$ tends to x . Therefore $x \in \text{cl}_X(EP(f))$, as required, and thus the proof is complete. $\square$

As regards assumption (5.2.2) of the above theorem, note the following proposition.

Proposition 5.3. *Let continua $Q_1, \dots, Q_n$ be mutually disjoint. If the union $A = Q_1 \cup \dots \cup Q_n$ has the ΩEP -property, then each of the continua Q_i for $i \in \{1, \dots, n\}$ has the ΩEP -property.*

Proof. Assume that A has the ΩEP -property. Then for a fixed i_0 and for a mapping $f_{i_0} : Q_{i_0} \rightarrow Q_{i_0}$ one can define $f : A \rightarrow A$ by $f(x) = f_{i_0}(x)$ for $x \in Q_{i_0}$, and $f(x) = x$ for $x \in Q_i$ with $i \neq i_0$. Then the ΩEP -property for Q_{i_0} follows from Proposition 2.3. $\square$

Remark 5.4. Note that the assumption that the continua Q_i are pairwise disjoint is essential in the above proposition, because the circle, which does not have the ΩEP -property, is the union of two arcs having their end points in common only, each of which has the property.

It is interesting to know whether the converse implication to that of Proposition 5.3 is true.

Question 5.5. Let continua $Q_1, \dots, Q_n$ be mutually disjoint and let each of them have the ΩEP -property. Does it follow that the union $A = Q_1 \cup \dots \cup Q_n$ has the ΩEP -property?

As regards assumption (5.2.3) in Theorem 5.2 the next two questions are natural.

Question 5.6. Is the assumption (5.2.3) necessary in Theorem 5.2?

Question 5.7. Under what conditions about the union A does the equality in (5.2.3) hold?

Accept the following definition.

Definition 5.8. A compact space C is said to have the ΩEP -property *hereditarily* provided that each subcontinuum of C has the ΩEP -property.

Note that each tree has the ΩEP -property hereditarily. Other known examples of continua having the ΩEP -property have the property hereditarily. Thus the following question is of some importance.

Question 5.9. Let a continuum (or a compact space) have the ΩEP -property. Does it follow that it has the ΩEP -property hereditarily?

Let W be (as previously) the dendrite defined by (2.8.1). In connection with assumption (5.2.1) of Theorem 5.2 recall the following characterization of dendrites having the ΩEP -property is known, see [8, Corollary 5.14].

Theorem 5.10. *A dendrite X has the ΩEP -property if and only if X does not contain any copy of W .*

Since each subcontinuum of a dendrite X with no copy of W in X also has the same property, we have a corollary.

Corollary 5.11. *If a dendrite has the ΩEP -property, then it has the property hereditarily.*

The next theorem is the main result of the paper.

Theorem 5.12. *Let a λ -dendroid X be a member of the class Δ . If*

- (5.2.1) *the dendrite D has the ΩEP -property, and*
 - (5.12.1) *the union $A = Q_1 \cup \dots \cup Q_n$ has the ΩEP -property hereditarily,*
- then X has the ΩEP -property for mappings $f : X \rightarrow X$ that satisfy (5.2.3).*

Proof. Let a mapping $f : X \rightarrow X$ satisfying (5.2.3) be given, and let the continuum $M(X, f)$ be as in Proposition 2.6. Since $M(X, f)$ is the maximal subcontinuum of X such that $f|M(X, f)$ is a surjection by (2.6.2) and (2.6.6), the whole dynamics for f is on the subcontinuum $M(X, f)$ (compare Remark 2.7). Thus the following three cases have to be considered.

CASE 1. $M(X, f) \subset H = X \setminus A$.

Since H is homeomorphic to $D \setminus \{q_1, \dots, q_n\}$ according to (4.5.1), it follows that $M(X, f)$ is homeomorphic to a subcontinuum of D , so it has the ΩEP -property by (5.2.1) and Corollary 5.11, that is $\Omega(f|M(X, f)) \subset \text{cl}_X(EP(f|M(X, f)))$. Therefore the conclusion follows from (2.6.5) of Proposition 2.6.

CASE 2. $M(X, f) \subset A$.

Let $x \in \Omega(f)$. Since $\Omega(f) = \Omega(f|M(X, f)) \subset M(X, f) \subset A$ by (2.6.5), and since A has the ΩEP -property hereditarily by assumption, the subcontinuum $M(X, f)$ of A has the ΩEP -property. Consequently, there is a sequence of points $x_k \in EP(f|M(X, f))$ such that $x = \lim x_k$, and thus $x \in \text{cl}_{M(X, f)}(EP(f|M(X, f))) \subset \text{cl}_{M(X, f)}(M(X, f) \cap EP(f)) = \text{cl}_X(EP(f))$ according to (2.3.3) of Proposition 2.3.

CASE 3. $M(X, f) \cap A \neq \emptyset \neq M(X, f) \cap H$.

Then $X' = M(X, f)$ is a λ -dendroid which is a member of the class Δ , and the partial mapping $f|X' : X' \rightarrow X'$ is a surjection. Thus X' has the ΩEP -property by Theorem 5.2.

Taking once more $x \in \Omega(f) \subset M(X, f)$ we get $x \in M(X, f) \cap \Omega(f) = \Omega(f|X')$. By the ΩEP -property for X' we conclude that there is a sequence of points $x_k \in EP(f|X') \subset X'$ having x as its limit, whence $x \in \text{cl}_{X'}(EP(f|X')) = \text{cl}_X(EP(f))$ by (2.3.3) of Proposition 2.3.

The proof is complete. □

Remark 5.13. Observe that if a λ -dendroid X is a member of the class Δ , then each of subcontinua of X is also a member of Δ . Further, if X satisfies the assumptions of Theorem 5.12, then each subcontinuum of X does. Therefore we have the following corollary.

Corollary 5.14. *Let a λ -dendroid X be a member of the class Δ . If (5.12.1) holds, then X has the ΩEP -property hereditarily for mappings f that satisfy condition (5.2.3).*

Question 5.15. Can the assumption in (5.12.1) of Theorem 5.12 that A has the ΩEP -property *hereditarily* be relaxed to having the ΩEP -property only, that is, to the assumption (5.2.1)? (Compare Question 5.9).

6. FINAL REMARKS AND QUESTIONS

Taking $D = [0, 1]$ and applying the construction (see Section 3) by replacing the point $q_1 = 0$ by a dendrite Q_1 which does not contain any copy of the dendrite W we get a corollary.

Corollary 6.1. *A compactification X of a ray having a dendrite Q_1 without any copy of W as the remainder has the ΩEP -property hereditarily for mappings $f : X \rightarrow X$ such that $\Omega(f|_{Q_1}) = Q_1 \cap \Omega(f)$.*

In particular we obtain the following result.

Corollary 6.2. *The $\sin(1/x)$ -curve S defined by*

$$S = \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\} \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 : x \in (0, 1]\}$$

has the ΩEP -property hereditarily for mappings f satisfying (5.2.3) for $A = \{0\} \times [-1, 1]$.

Consider the following very particular but important version of Question 5.5.

Question 6.3. Let $Q_1, \dots, Q_n$ be mutually disjoint dendrites no one of which has the ΩEP -property (equivalently, no one of which contains a copy of the dendrite W). Does it follow that the union $A = Q_1 \cup \dots \cup Q_n$ has the ΩEP -property?

Observe that if the above question has a positive answer, then the union A has the ΩEP -property hereditarily, so (5.12.1) in Theorem 5.12 is satisfied, and therefore if D is as in (5.2.1), then the resulting λ -dendroid X has the ΩEP -property hereditarily by Corollary 5.14.

We do not know if all the assumptions made in the results proved in the previous sections of the paper are essential, and/or if the results can be extended or generalized in a way. So, we close this section with several questions related to the concept of the class Δ and to Theorem 5.12. Similar questions (but connected with the PR -property) were already asked in the last section of [9].

Question 6.4. Is the assumption that all inserted continua Q_i are of the same (finite) depth (see Definition 4.1 of the class Δ) essential in Theorems 5.2 and 5.12?

Question 6.5. Can Theorems 5.2 and/or 5.12 be generalized to λ -dendroids X in which the depth of some (of all) inserted continua Q_i is infinite?

Questions 6.6. Is the condition that the number n of the continua Q_i is finite an essential assumption in Theorems 5.2 and/or 5.12? Under what conditions the result can be generalized to λ -dendroids X in which a) the number of the considered continua Q_i is countable? b) the union A of the considered continua Q_i is closed in X ?

As a particular case of the above questions we have the following.

Questions 6.7. Can Theorems 5.2 and/or 5.12 be extended to some continua X obtained as compactifications of complements of closed countable subsets of (a) trees (b) dendrites?

All of the above questions are very particular cases of more general problems, which can be treated as a research program in the area, and which (at the present moment) seem to be rather far from any final solution.

Problem 6.8. What λ -dendroids have the ΩEP -property?

Problems 6.9. Let an upper semicontinuous decomposition $\mathcal{D}$ of a continuum X into continua (possibly degenerate) be given. Consider the following three conditions:

- (6.9.1) the continuum X has the ΩEP -property;
- (6.9.2) all (or some) members of the decomposition $\mathcal{D}$ have the ΩEP -property;
- (6.9.3) the decomposition space $X/\mathcal{D}$ has the ΩEP -property.

What are interrelations between these conditions?

In particular, the following question related to Theorems 5.2 and 5.12 is specially interesting.

Question 6.10. Under what assumptions conditions (6.9.2) and (6.9.3) imply (6.9.1)?

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