

FIXED POINTS AND FACTORING ITERATES OF THE LOGISTIC MAP

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ABSTRACT. The dynamics of the one-dimensional mapping $f(x) = rx(1-x)$ under iteration is investigated. The cycles one through four are discussed, and the onset of periodic points in the three and four-cycles are treated by solving for r and x at the values where the n -th iterate $f^{(n)}(x)$ intersects and is simultaneously tangent to the line $y = x$.

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1. INTRODUCTION

One-dimensional recursion equations have been used for some time to give a description of many mathematical and physical problems. Newton's method for numerically determining the roots of an equation is one example. As well, many chemical and biological processes can be modeled by one-dimensional iterative mappings. The work of Li and York [4] produced a resurgence of interest in this area [5]. Here, we will be concerned exclusively with the dynamics of the following function of a single variable

$$f(x) = rx(1-x), \quad (1)$$

under iteration. In particular, we would like to present a systematic, coherent study of the periodic points of $f(x)$ after n iterations, where n takes values from one to four, and to note several properties of the n -th iterate of $f(x)$, for $n = 1, \dots, 4$. In this paper, we will be primarily concerned with r positive and the domain of $f(x)$ and its iterates will be the interval $[0, 1]$.

Let $F : I \rightarrow I$. For $x \in I$, $F^0(x)$ denotes x and adopting the usual notation, $F^{n+1}(x)$ denotes $F(F^n(x))$ for $n = 0, 1, \dots$. Then p is said to be a periodic point with period n if $p \in I$, $p = F^n(p)$ and $p \neq F^k(p)$ for $1 \leq k \leq n$. Moreover, q is eventually periodic if for some positive integer m , $p = F^{(m)}(q)$ is periodic. The objective with respect to these one-dimensional maps is to understand situations in which iterates of a point are very irregular, or even chaotic [4]. It will be shown

that values of the parameter r in the logistic map (1) can be calculated, as well as values for x at which the equation $x = f^{(n)}(x)$ holds and $f^{(n)}(x)$ is simultaneously tangent to the line $y = x$ for the two, three and four-cycle in closed form. Some interest along these lines has been shown recently [6, 2, 3]. It should be stressed that although three-cycles have been treated in [6], there has appeared no work on four-cycles using the techniques presented here.

2. PERIOD ONE

With $f(x)$ defined on $[0, 1]$ by (1), we consider the case $r > 0$. Since the function $g(x) = x(1 - x)$ has a maximum at the point $x = 1/2$, the function $f(x)$ also has a maximum at $x = 1/2$ with $f(1/2) = r/4$. Suppose therefore that $r \in (0, 4)$.

To make the presentation self-contained, we begin by considering points with period one. These points are determined by the quadratic equation

$$x(rx + 1 - r) = 0.$$

This equation has two solutions, namely at $x = 0$, and another solution at

$$x_0 = 1 - r^{-1}.$$

This value of x is negative unless $r > 1$. Consequently, there is a nontrivial point of period one only if $r > 1$. Notice that $x_0 = 0$ when $r = 1$ and $f(x)$ is tangent to the line $y = x$ at zero.

3. PERIOD TWO

The period two equation $x = f^{(2)}(x)$ is somewhat more complicated. Define the function $g_2(x)$ as follows,

$$g_2(x) = f^{(2)}(x) - x. \quad (2)$$

Thus the existence of a root x' for $g_2(x)$ implies that $f^{(2)}(x') = x'$ and x' is a periodic point of period 2. This function has the following factorization,

$$g_2(x) = -x(rx + 1 - r)(r^2x^2 - (r^2 + r)x + r + 1). \quad (3)$$

Clearly, $x = 0$ is one solution of the equation $g_2(x) = 0$, and the equation for the one-cycle appears next in the factorization. The remaining third factor is a quadratic equation in x . This quadratic has real solutions as long as $r \geq 3$. When $r = 3$ the quadratic will factor to give

$$(9x^2 - 12x + 4) = (3x - 2)^2 \geq 0.$$

This has a minimum value of zero at $x = 2/3$ when equality holds. At this point, the quadratic is tangent to the x -axis.

It is straightforward to check all of these algebraic manipulations and factorizations using Maple [1].

4. PERIOD THREE

The third iterate of $f(x)$ is given by the equation

$$f^{(3)}(x) = -r^3x(x - 1)(rx^2 - rx + 1)(1 - r^2x + r^3x^3 + r^2x^2 + r^3x^4). \quad (4)$$

The factor $(rx^2 - rx + 1)$ has only complex roots for $r < 4$. All of the roots of $f^{(3)}(x)$ between 0 and 1 are determined by the remaining quartic polynomial.

Similarly, one can consider the function $g_3(x) = f^{(3)}(x) - x$ and investigate values of x for which $g_3(x) = 0$. This can be calculated and factored as in the previous

case. There are two simple factors, namely x as well as the one-cycle equation $(rx + 1 - r)$, and there remains a large sixth degree polynomial in the factorization which is given by

$$p_6(x) = r^6x^6 - (r^5 + 3r^6)x^5 + (r^4 + 4r^5 + 3r^6)x^4 - (r^3 + 3r^4 + 5r^5 + r^6)x^3 \\ + (r^2 + 3r^3 + 3r^4 + 2r^5)x^2 - (r + 2r^2 + 2r^3 + r^4)x + (1 + r + r^2). \quad (5)$$

For small values of the parameter r , the polynomial $p_6(x)$ is decreasing on $[0, 1]$. However, as r is slowly increased, minima gradually appear and start to approach the x -axis. The change in the behavior of $p_6(x)$ is very sensitive to changes in the parameter r . It is of interest to ask for which values of r and x the function $g_3(x)$ is tangent to the x -axis. This would imply the emergence of new real solutions to the equation $g_3(x) = 0$.

Let us write the equations for the three cycles in the following form [6],

$$y = rx(1 - x), \\ z = ry(1 - y) = f(y) = f^{(2)}(x), \\ x = rz(1 - z) = f(z) = f^{(3)}(x). \quad (6)$$

The symmetry of these equations suggests substitution of the variables

$$x = \frac{A}{r} + \frac{1}{2}, \quad y = \frac{B}{r} + \frac{1}{2}, \quad z = \frac{C}{r} + \frac{1}{2}. \quad (7)$$

If (A, B, C) is a solution, any permutation of them would be a solution as well. Thus A, B, C should be determined by the roots of the cubic equation,

$$(t - A)(t - B)(t - C) = t^3 - (A + B + C)t^2 + (AC + AB + BC)t - ABC.$$

In terms of these variables, the system of equations (6) is transformed into

$$R = \frac{r^2}{4} - \frac{r}{2} = A^2 + B = B^2 + C = C^2 + A. \quad (8)$$

The constraint supplied by the derivative takes the form

$$ABC = -\frac{1}{8}.$$

Define the variables $a = A + B + C$, $b = AB + BC + CA$ and $c = ABC$ as the coefficients of the cubic. Adding the three equations in (8) and using $A^2 + B^2 + C^2 = a^2 - 2b$ gives R in terms of a and b ,

$$R = \frac{1}{3}(a^2 + a - 2b).$$

Multiplying the first equation in (8) by A , the next by B , and the last by C , then adding produces another equation

$$aR = A^3 + B^3 + C^3 + AB + BC + AC.$$

Substituting $A^3 + B^3 + C^3 = a^3 - 3ab + 3c$ into this and using the definition of b , we obtain

$$2a^3 - 7ab - a^2 + 9c + 3b = 0. \quad (9)$$

Another independent equation can be obtained by multiplying two of the three factors for R presented in (8) to give

$$R^2 = (A^2 + B)(B^2 + C) = (AB)^2 + B^3 + A^2C + BC.$$

Repeating for the remaining two combinations and then adding gives the other required equation. Since $(AB)^2 + (BC)^2 + (CA)^2 = b^2 - 2ac$ one can show that $A^3 + B^3 + C^3 = A^2C + B^2A + C^2B$. The final independent equation can be written in the form

$$a^4 - 4a^2b - 4a^3 + b^2 + 14ab + a^2 + 6ac - 18c - 3b = 0. \quad (10)$$

The constraint on the derivative simply gives $c = -1/8$. One way to solve this system of equations is to apply Gröbner basis or the resultant function, which can be found in Maple [1]. This gives the following possibilities,

$$\begin{aligned} 4b - 3 &= 0, & 2a + 3 &= 0, & (a, b) &= \left(-\frac{3}{2}, \frac{3}{4}\right), \\ 4b + 9 &= 0, & 2a - 1 &= 0, & (a, b) &= \left(\frac{1}{2}, -\frac{9}{4}\right), \end{aligned} \quad (11)$$

$$4b - 2a + 3 = 0, \quad 4a^2 - 6a + 9 = 0,$$

Substituting the first pair in (11) into the cubic, it is found that the cubic factors into $(t + \frac{1}{2})^3$ which implies that $R = -1/4$ and $r = 1$, with $g_3(x)$ tangent to the x -axis at $x = 0$. The last quadratic has only complex solutions for a . The middle pair is of most interest and gives the values of t in Table 1 with $R = 7/4$. Solving for r , one obtains that $r = 1 \pm 2\sqrt{2}$, and the positive case is $r = 1 + 2\sqrt{2}$. When this particular value of r is substituted into $p_6(x)$, it is found that this resulting sixth degree polynomial factors into a product of two cubics as follows

$$p_6(x) = \frac{1}{117649}(1593 + 1100\sqrt{2})(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2})^2. \quad (12)$$

The cubic has three real roots which are given in Table 2.

5. FOUR-CYCLES

Consider finally the case of four-cycles and the associated polynomial equation $g_4(x) = f^{(4)}(x) - x = 0$, and we would like to investigate $g_4(x) = 0$. This equation can be factored into a trivial linear factor x , the one-cycle factor $(rx + 1 - r)$, as well as a quadratic polynomial $(r^2x^2 + 1 - rx + r - r^2x)$, which is the equation obtained for the two cycle. It has two real roots when $r > 3$. There is a remaining polynomial factor, which is a large twelfth degree polynomial in x , written $p_{12}(x)$, which is given in Table 3. For relatively small r , it is found numerically that this large polynomial decreases monotonically as x goes from zero to one. However, as r increases, two minima appear which gradually approach the x -axis from above as r is slowly increased. At a particular value of r , two of these minima are tangent to the x axis. At a slightly larger value of r , three of the minima from above and a maximum from below are tangent to the x axis. The analysis which was done for the three-cycle can be extended to obtain the value of r and the values of x in terms of square roots at which the first two minima are tangent.

The equations for the four-cycle are given as follows

$$\begin{aligned} y &= rx(1-x), \\ z &= ry(1-y) = f^{(2)}(x), \\ w &= rz(1-z) = f^{(3)}(x), \\ x &= rw(1-w) = f^{(4)}(x). \end{aligned} \tag{13}$$

The constraint which arises from the derivative is given by

$$r^4(1-2w)(1-2z)(1-2y)(1-2x) = 1.$$

Symmetry of these equations suggests that the following substitution be carried out,

$$x = \frac{A}{r} + \frac{1}{2}, \quad y = \frac{B}{r} + \frac{1}{2}, \quad z = \frac{C}{r} + \frac{1}{2}, \quad w = \frac{D}{r} + \frac{1}{2}.$$

The system above can be written in the following form,

$$R = \frac{r^2}{4} - \frac{r}{2} = A^2 + B = B^2 + C = C^2 + D = D^2 + A, \quad 16ABCD = 1. \tag{14}$$

Note that the second equation in (14) implies that $d = 1/16$. If a given four-tuple is a solution, its cyclic variations should also be a solution. Hence the four-tuple should satisfy a single quartic equation of the form $(t-A)(t-B)(t-C)(t-D)$, where A, B, C and D are its four roots. Expanding this out, we are lead to introduce the following four new variables,

$$\begin{aligned} a &= A + B + C + D \\ b &= AB + AC + BC + AD + BD + CD, \\ c &= ABC + ABD + ACD + BCD, \\ d &= ABCD. \end{aligned} \tag{15}$$

The quartic equation can be written in terms of these new variables as $t^4 - at^3 + bt^2 - ct + d$. Adding the set of equations in (14) and using (15) gives an equation for R in terms of a and b ,

$$R = \frac{1}{4}(a^2 + a - 2b). \tag{16}$$

Multiplying (14) by A, B, C and D respectively, gives

$$AR = A^3 + AB, \quad BR = B^3 + BC, \quad CR = C^3 + CD, \quad DR = D^3 + AD.$$

If these equations are added, the following result is obtained,

$$aR = A^3 + B^3 + C^3 + D^3 + AB + BC + AC + CD + BD - AC - BD. \tag{17}$$

For a quartic polynomial of the form given above, the formulas of Newton [7] can be used to express the sums of the powers of the roots in terms of a, b, c and d . In particular, if s_n is the sum of the n -th powers of the four roots of this polynomial, the following expressions hold $s_2 \equiv A^2 + B^2 + C^2 + D^2 = a^2 - 2b$, $s_3 \equiv A^3 + B^3 + C^3 + D^3 = a^3 - 3ab + 3c$ with s_4 obtained by solving the equation,

$$s_4 - as_3 + bs_2 - cs_1 + 4d = 0.$$

Sums of higher powers of the roots can be obtained by solving the equation,

$$s_5 - as_4 + bs_3 - cs_2 + ds_1 = 0.$$

The subscripts are increased by 1 on each s_m in this expression each time the equation for the next higher s_n is written down in order to be computed.

Substituting these results into (17), we can solve for $AC + BD$ as follows,

$$AC + BD = aR - a^3 + 3ab - 3c - b. \quad (18)$$

Another equation can be obtained from (14) by multiplying by B^2 , C^2 , D^2 and A^2 respectively, and then adding these results

$$(a^2 - 2b)R = b^2 - 2ac + 2d + a^3 - 3ab + 3c - (AC)^2 - (BD)^2. \quad (19)$$

Squaring the expression for $AC + BD$ in (18) gives $(AC)^2 + (BD)^2 = (a^3 - 3ab + 3c + b - aR)^2 - 2d$, and substituting this into (19) gives,

$$(a^2 - 2b)R = b^2 - 2ac + 4d + a^3 - 3ab + 3c - (a^3 - 3ab + 3c + b - aR)^2. \quad (20)$$

Another equation can be found by expanding out ab in (15) and obtaining an expression for the sum of the powers which appear. The result is

$$2ab + 2R^2 + 3aR - b - a^3 + \frac{1}{6}a^4 + 2d - \frac{2}{3}a(a^3 - 3ab + 3c) - b^2 - \frac{1}{2}a^2 = 0. \quad (21)$$

Equation (21) can be solved to give an equation which determines c in terms of a and b as follows

$$c = -\frac{1}{16a}(3a^4 - 12a^2b + 4b^2 - 3a^2 + 8b - 1). \quad (22)$$

Substituting R from (16) into (20) and (21), we can eliminate c between these two equations to give an equation which relates a and b ,

$$\begin{aligned} &9 - 48a - 144b + 480ab + 22a^2 + 504b^2 - 168a^3 + 195a^4 - 184a^5 + 40a^4b \\ &+ 672a^3b - 152a^2b^2 - 576ab^2 - 712ba^2 + 128a^5b - 32a^3b^2 - 24a^6b - 56a^4b^2 \\ &+ 96b^3a^2 - 26a^6 - 24a^7 + 9a^8 + 144b^4 + 576b^3 - 384b^3a = 0. \end{aligned} \quad (23)$$

Although it would be preferable to obtain another equation in a and b to use with equation (23), it seems another constraint equation is difficult to find and may be impossible, so a complete solution is not presented. The solution in terms of a , b , c and d would be used to determine R and then r . There is not enough information above to determine a and b uniquely in general. However, we can proceed in the other direction, that is, begin with a value of r . The following example can be carried through to the end. As mentioned above, as the parameter r increases, there is a particular value of r at which two minima of the twelfth degree factor are tangent to the x -axis. for x values inside $(0, 1)$. This is easy to show since by substituting $r = 1 + \sqrt{6}$ into $p_{12}(x)$ gives an expression which factorizes into an eighth degree polynomial, which is found to have only complex roots, and the square of a quadratic factor of the form

$$(25x^2 - (20 + 5\sqrt{6})x + 2 + 3\sqrt{6})^2.$$

This polynomial has two real roots inside the interval $(0, 1)$ given by

$$x_{1,2} = \frac{1}{10}(\sqrt{6} - 1)(2 \pm \sqrt{2} + \sqrt{6}).$$

These evaluate to the approximate values $x_1 = 0.439960$ and $x_2 = 0.84993$. For the case $r = 1 + \sqrt{6}$, a value for R is found from (14) to be given by $R = (\tau^2/4) - \tau/2 =$

5/4. Hence from (16), a and b are related by $(a^2 + a - 2b) = 5$. Thus, b can be eliminated from (23) to give a polynomial equation only in terms of the variable a . This polynomial is of degree eight in a and given by

$$a^8 + 4a^7 - 36a^6 - 54a^5 + 513a^4 - 446a^3 - 651a^2 + 180a + 36 = 0.$$

This expression has six real roots which are given by $a = -6.515964, -5.15966, -0.884660, -0.138373, 0.3700864$, and 3.4318800 . From these, we can calculate values for b from (16) and values for c from (22). This gives the full picture for this case.

As r is increased further, there occur four points of tangency, one from below. This occurs for $r = 3.9601018826$ and the polynomial $p_{12}(x)$ is tangent for the following x values : $x_1 = 0.0392645830$, $x_2 = 0.1493864306$, $x_3 = 0.5032106413$ and $x_4 = 0.9899846490$.

6. DISCUSSION

It is remarkable that the three minima of the function $p_6(x)$ approach and then touch the x -axis simultaneously at three distinct x values when $r = 2\sqrt{2} + 1$. As r increases slightly each minima generates two additional roots of the polynomial. At $r = 2\sqrt{2} + 1$, the polynomial $p_6(x)$ factors into a product of identical cubics.

It is also seen that the polynomial $p_{12}(x)$ which is embedded in $g_4(x)$ has two minima which are tangent to the x -axis at $r = 1 + \sqrt{6}$. As r is increased, each minima gives rise to two additional roots of $g_4(x)$. It is found that, as r is increased further, four additional extrema appear and become tangent to the x -axis, one a maximum and the other three minima. As r is increased a little further beyond this value, such extrema give rise to two new roots, the minima drop below the x -axis and the maximum rises above, creating more solutions.

REFERENCES

- [1] Char B. W., Geddes K. O., Gonnet G. H., Leong B. L., Monagen M., Watt S., Maple V Language Reference Manual, *Springer-Verlag* (1991).
- [2] Devaney R. L., An Introduction to Chaotic Dynamical Systems, 2nd Edition, Addison-Wesley, (1989).
- [3] Grebogi C., Ott E., Yorke J. A., Chaos, strange attractors, and fractal basin boundaries in nonlinear dynamics, *Science*, **238**, (1987) 632 .
- [4] Li T. Y. and Yorke J. A., Period three implies chaos, *American Mathematical Monthly*, **82**, (1975) 985 .
- [5] Lorenz E. N., Deterministic nonperiodic flow, *J. Atmos. Sci.* **20**, (1963) 130 .
- [6] Saha P. and Strogatz S., *Mathematics Magazine*, **68**, (1995) 42 .
- [7] Serrat J. A., Cours D'Algèbre Supérieure, Vol. 1, *Gauthier-Villars* (1885).

Table 1. Roots of the cubic $t^3 - \frac{1}{2}t^2 - \frac{9}{4}t + \frac{1}{8} = 0$ and corresponding values of x for $r = 2\sqrt{2} - 1$ in the three cycle using equations for x, y, z in terms of A, B and C .

t	$-\frac{1}{3}\sqrt{7}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) - \frac{1}{3}\sqrt{21}\sin(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{1}{6}$	
x	$-\frac{r}{3\sqrt{7}}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) - \frac{r}{\sqrt{21}}\sin(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{r}{42} + \frac{1}{2}$	0.1599288
t	$-\frac{1}{3}\sqrt{7}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{1}{3}\sqrt{21}\sin(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{1}{6}$	
x	$-\frac{r}{3\sqrt{7}}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{r}{\sqrt{21}}\sin(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{r}{42} + \frac{1}{2}$	0.5143552
t	$\frac{2}{3}\sqrt{7}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{1}{6}$	
x	$\frac{2r}{3\sqrt{7}}\cos(\frac{1}{3}\tan^{-1}(3\sqrt{3})) + \frac{r}{42} + \frac{1}{2}$	0.9563178

Table 2. Real roots of the cubic equation obtained from $p_6(x)$ at $r = 2\sqrt{2} + 1$,

$$343x^3 - 490x^2 - 49\sqrt{2}x^2 + 91x + 112\sqrt{2}x + 31 - 41\sqrt{2} = 0,$$

calculated from this cubic.

$$a = \frac{11}{441}\sqrt{6} - \frac{25}{882}\sqrt{3}$$

$$b = \frac{1}{3}\tan^{-1}(3\sqrt{3})$$

$$c = \frac{59}{9261} - \frac{1100}{250047}\sqrt{2}$$

$$d = \cos(\frac{1}{3}\tan^{-1}(3\sqrt{3}))$$

$$r_1 = -\frac{1}{2}c^{1/6}\cos(b) + c^{-1/6}\left(\frac{2}{63}\sqrt{2} - \frac{1}{44}\right)\cos(b) + \frac{10}{21} + \frac{\sqrt{2}}{21} + \frac{\sqrt{3}}{2}(c^{1/6} - c^{-1/6}\left(\frac{4}{63}\sqrt{2} - \frac{1}{7}\right))\sin(b)$$

$$r_2 = -\frac{1}{2}c^{1/6}\cos(b) + c^{-1/6}\left(\frac{2}{63}\sqrt{2} - \frac{1}{44}\right)\cos(b) + \frac{10}{21} + \frac{\sqrt{2}}{21} - \frac{\sqrt{3}}{2}(c^{1/6} - c^{-1/6}\left(\frac{4}{63}\sqrt{2} - \frac{1}{7}\right))\sin(b)$$

$$r_3 = (1593 - 1100\sqrt{2})^{1/3}\sqrt{7}d - 4\sqrt{14}d + 9\sqrt{7}d + (10 + \sqrt{2})(1593 - 1100\sqrt{2})^{1/6}$$

Table 3. The twelfth degree polynomial $p_{12}(x) = \sum_{s=0}^{12} b_s x^s$ obtained from factoring the polynomial $g_4(x)$.

$$\begin{aligned}
 & r^{12}x^{12} - 6r^{12}x^{11} + (15r^{12} + 3r^{11})x^{10} - (20r^{12} + 15r^{11} + r^9)x^9 \\
 & + (15r^{12} + 30r^{11} + 3r^{10} + 6r^9)x^8 - (6r^{12} + 30r^{11} + 12r^{10} + 14r^9 + 2r^8)x^7 \\
 & + (r^{12} + 15r^{11} + 18r^{10} + 17r^9 + 10r^8 + r^6)x^6 - (3r^{11} + 12r^{10} + 12r^9 + 18r^8 + r^7 + 4r^6)x^5 \\
 & + (3r^{10} + 5r^9 + 14r^8 + 4r^7 + 6r^6 + 2r^5)x^4 - (r^9 + 4r^8 + 5r^7 + 4r^6 + 5r^5 + r^3)x^3 \\
 & + (2r^7 + r^6 + 4r^5 + r^4 + 2r^3)x^2 - (r^5 + r^4 + r^3 + r^2)x + 1 + r^2
 \end{aligned}$$