

EXACT SOLUTIONS FOR CHARGED SPHERE IN ROSEN'S BIMETRIC THEORY OF GRAVITATION

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ABSTRACT. Exact solution of the field equations of Bimetric General Relativity (BGR) are obtained for the case of static and spherically symmetric distribution of charged matter. The solutions are obtained by using the method of Patino and Rago (1989) for charged sphere in general relativity (GR). The physical plausability of the solutions is discussed and it is shown that all the solutions agree with Einstein general relativity for a physical system compared to the size of the universe such as solar system.

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1. INTRODUCTION

Exact solutions have played a crucial role in the development of general relativity, many areas of physics and Astrophysics. It is a well known fact that in any physical theory such as theory of gravitation the derivation of Exact solutions is helpful to get a real physical picture of space time and contains thereof.

Einstein was originally very sceptical that it would be possible to find an exact solution of the field equations because they constitute an extremely complicated set of non-linear partial differential equations. At the advantage of general relativity the first exact solutions leading to spherical symmetry i.e. of a particle gravitational field is due to Schwarzschild's, which is widely known as Schwarzschild's exterior solutions. Schwarzschild found an exact solution within a year of the field equations being published. In the ensuing years very few solutions were found until new invariant techniques were introduced in the mid 1960s. There are various attempts in obtaining the exact solutions having different symmetries in GR.

The basic difficulty which is inherent is that of solving non linear partial differential equations (p.d.e) of second order. Since, in the theory of solving non linear

p.d.e. is itself no definite procedure is prescribed one has to resort to another technic which may finally lead to the solutions may not be an exact solutions. The so called new technic has been adopted by [5, 1, 2, 7, 3, 8] in GR.

We thought that the same can be exploited to get another picture and the solutions thereof in Bimetric General Relativity (BGR). With this motivation our first attempt was directed towards getting exact solutions of Bimetric Relativity (BR) for uncharged fluid sphere [Khadekar and Kandalkar (2004)] .

In this paper we present a procedure to obtain exact analytical solutions of field equations for charged sphere in the context of BGR proposed by Rosen (1974, 1978) [9, 10]. The method is a generalization of one originally developed by Berger et al. (1987) for isotropic fluid sphere, and later applied to charged perfect fluid case by Pantino and Rago (1989) in GR and Khadekar and Kandalkar (2004)[6] representing neutral sphere in bimetric relativity. The approach of this work is based on the introduction of the generating function. This functions determine the relevant physical variables as well as the metrical coefficient.

The paper is organized as follows. In section 2 we present the field equations of BR and method used to obtain the solutions. In section 3 some examples are given for illustrating the method. In section 4 we present the conclusions.

2. FIELD EQUATIONS AND GENERATING FUNCTION

The Bimetric General Relativity (BGR) is a modification of Einstein's relativity theory in order to eliminate the singularities appearing in it., such as the singularity at the center of the Black hole or big bang in cosmology. In BGR there is a physical metric $g_{\mu\nu}$ as in conventional GR and there is also a background metric $\gamma_{\mu\nu}$ having a curvature tensor $P_{\lambda\mu\nu\sigma}$ given by

$$(1) \quad P_{\lambda\mu\nu\sigma} = \frac{1}{a^2}(\gamma_{\lambda\sigma}\gamma_{\mu\nu} - \gamma_{\mu\sigma}\gamma_{\lambda\nu}),$$

where a is a constant taken to be the order of the size of the universe, $a \sim 10^{28}$ cm. The field equations of BGR are taken to be the same as in GR, except for the fact that ordinary derivatives of $g_{\mu\nu}$ are replaced by covariant derivative with respect to $\gamma_{\mu\nu}$. Falik and Rosen (1980) [11] found that field equations in BGR can be written in the form of Einstein's field equations, but with an additional term on the right hand side

$$(2) \quad G_{\mu}^{\nu} = S_{\mu}^{\nu} - 8\pi T_{\mu}^{\nu}$$

where G_{μ}^{ν} is the Einstein tensor, T_{μ}^{ν} energy stress tensor and

$$S_{\mu}^{\nu} = \frac{3}{a^2}(\gamma_{\mu\alpha}g^{\alpha\nu} - \frac{1}{2}\delta_{\mu}^{\nu}g^{\alpha\beta}\gamma_{\alpha\beta})$$

where a is very large this term is usually neglected, so that for phenomena in solar system, BGR gives agreement with GR.

However, if one has a situation where according to GR, $g_{\mu\nu}$ has a singularity, then this term can not be neglected and the field equation of BGR may give results different from those of GR.

We consider the static spherically symmetric system we take a physical metric as

$$(3) \quad ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where λ and ν being functions of r alone. The energy momentum tensor for the charged matter can be expressed as

$$(4) \quad T_{\mu}^{\nu} = (\rho + p) U_{\mu} U^{\nu} - p \delta_{\mu}^{\nu} + \frac{1}{4} (-F^{\nu\lambda} F_{\lambda\mu} + \frac{1}{4} \delta_{\mu}^{\nu} F^{\alpha\beta} F_{\alpha\beta})$$

where ρ, p and U^{α} are respectively the energy density, the isotropic pressure and the unit time like four velocity of the fluid and $F^{\alpha\beta}$ is the Maxwell field tensor. If we take the back ground metric $\gamma_{\alpha\beta}$, de-Sitter universe of constant curvature

$$d\sigma^2 = (1 - \frac{r^2}{a^2}) dt^2 - (1 - \frac{r^2}{a^2})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

For $r \ll a$ this takes on the flat space form

$$(5) \quad d\sigma^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

By using equations (3) to (5) the field equations (2) following the procedure of Rosen (1980) can be written as:

$$(6) \quad e^{-\lambda} (\frac{\lambda'}{r} - \frac{1}{r^2}) + \frac{1}{r^2} = 8\pi\rho - \frac{3}{2a^2} e^{-\nu} + \frac{Q^2}{r^4}$$

$$(7) \quad e^{-\lambda} (\frac{\nu'}{r} + \frac{1}{r^2}) - \frac{1}{r^2} = 8\pi p - \frac{3}{2a^2} e^{-\nu} - \frac{Q^2}{r^4}$$

$$(8) \quad e^{-\lambda} (\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{\lambda'\nu'}{4}) = 8\pi p_e + \frac{Q^2}{r^4}$$

where we have defined

$$(9) \quad Q(r) = 4\pi \int_0^r \sigma r^2 e^{\frac{\lambda}{2}} dr$$

which represents the total electric charge obtain with in a sphere of radius r and σ is a charged density. Hereafter a prime denotes differentiation with respect to r . The consequences of energy momentum conservation $T_{\nu;\mu}^{\mu} = 0$ leads to

$$(10) \quad p' + (\rho + p) \frac{\nu'}{2} = \frac{1}{8\pi r^4} \frac{d(Q^2)}{dr}$$

It should be noted that the background metric does not appears in these field equations, although it is responsible for the presence of the first term on the right of each equation. One can define the effective density and pressures ρ_e, p_e (Harpaz and Rosen, 1985) [4] as

$$(11) \quad \rho_e = \rho - \frac{3}{16\pi a^2} e^{-\nu}, \quad p_e = p - \frac{3}{16\pi a^2} e^{-\nu}$$

So that equations (6) to (8) can be written as

$$(12) \quad e^{-\lambda} (\frac{\lambda'}{r} - \frac{1}{r^2}) + \frac{1}{r^2} = 8\pi\rho_e + \frac{Q^2}{r^4}$$

$$(13) \quad e^{-\lambda} (\frac{\nu'}{r} + \frac{1}{r^2}) - \frac{1}{r^2} = 8\pi p_e - \frac{Q^2}{r^4}$$

$$(14) \quad e^{-\lambda} (\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{\lambda'\nu'}{4}) = 8\pi p_e + \frac{Q^2}{r^4}$$

These equations look like the usual Einstein equations with ρ and p are replaced by ρ_e and p_e respectively. It is interesting that the equation (10) can also be written in terms of ρ_e and p_e as

$$(15) \quad p'_e + (\rho_e + p_e) \frac{\nu'}{2} = \frac{1}{8\pi r^4} \frac{d(Q^2)}{dr}$$

Integration of equation (12) gives

$$(16) \quad e^{-\lambda} = 1 - \frac{2m_e}{r}$$

where $m_e(r)$ is the mass function defined as

$$(17) \quad \begin{aligned} m_e(r) &= \int_0^r (4\pi \rho r^2 + \frac{QQ'}{r}) dr - \frac{3}{4a^2} \int_0^r r^2 e^{-\nu} dr \\ &= 4\pi \int_0^r (\rho r^2 + \frac{QQ'}{r}) dr - \frac{3}{4a^2} \int_0^r r^2 e^{-\nu} dr^2 \end{aligned}$$

Substitution of equations (15) and (16) into equation (13) gives

$$(18) \quad 8\pi p_e + \frac{1}{r^2} - \frac{Q^2}{r^4} = r \left[1 - \frac{2m_e(r)}{r} + \frac{Q^2}{r^2} \right] \left(\frac{1}{r^3} - 2 \frac{(p'_e - \frac{Q^2}{8\pi r^4})}{r^2(\rho_e + p_e)} \right)$$

We define a function $G(r)$ as

$$(19) \quad G(r) = - \frac{r[1 - \frac{2m_e(r)}{r} + \frac{Q^2}{r^2}]}{8\pi p_e r^2 + \frac{1}{r^2} - \frac{Q^2}{r^4}}$$

In terms of $G(r)$, equation (18) takes the form

$$(20) \quad \begin{aligned} &8\pi p'_e + \frac{(r^3 + G)(r^2 + G')}{G(G - r^3)} - 8\pi p_e \\ &+ \frac{(r^3 + G')}{r^3 G(G - r^3)} \left[r^3 + G'r - 2G - Q^2 r - \left[\frac{GQ^2}{r^4} \right]' r^3 \right] + \frac{2Q^{2'}}{(G - r^3)r} = 0 \end{aligned}$$

It should be noted that given function G and the charge distribution as a known function of r , the solution of the linear differential equation (20) takes the general form

$$(21) \quad 8\pi p_e = e^{-\int B(r) dr} [p_0 + \int C(r) e^{\int B(r) dr} dr]$$

where p_0 is a constant of integration and $B(r)$ and $C(r)$ are given by

$$\begin{aligned} B(r) &= \frac{(G + r^3)(G' + r^2)}{G(G - r^3)} \\ C(r) &= - \frac{(G + r^3)}{Gr^3(G - r^3)} \left[r^3 + G'r - 2G - Q^2 r - \left[\frac{GQ^2}{r^4} \right]' r^3 \right] - \frac{2Q^{2'}}{(G - r^3)r} \end{aligned}$$

After obtaining effective pressure p_e the expression for effective density ρ_e is easily calculated from equation (17) and (19) as

$$(22) \quad 8\pi \rho_e = \frac{1}{r^2} \left[1 + G'(8\pi p_e + \frac{1}{r^2} - \frac{Q^2}{r^4}) + G(8\pi p'_e - \frac{2}{r^3} - \left[\frac{Q^2}{r^4} \right]' + \frac{Q^2}{r^2}) \right]$$

Finally taking into account equations (15) - (17) and (19) the metric coefficient in terms of G and Q can be expressed as

$$(23) \quad e^{-\lambda} = -\frac{G}{r} \left(8\pi p + \frac{1}{r^2} - \frac{Q^2}{r^4} - \frac{3}{2a^2} e^{-\nu} \right)$$

$$(24) \quad e^{\nu} = \frac{A^2}{r} e^{-\int \frac{r^2}{G} dr}$$

where A is a constant.

3. THE SOLUTIONS

The function G must satisfy some general requirement if its associate solution is to be physically meaningful. Thus at the origin $r = 0$ [$\frac{m_e(r)}{r}$ and $\frac{Q^2(r)}{r^2} \rightarrow 0$ as $r \rightarrow 0$] imply that $\lim_{r \rightarrow 0} \frac{G(r)}{r^3} = -1$ assuming the non divergent effective pressure at origin.

For instant, if $G(r) = -r^3$ and $Q(r) = 0$ one obtains Minkowski flatspace-time as a trivial solution. The following choice

$$(25) \quad G(r) = -\frac{r^3 \left[1 - \frac{2M}{r} + \frac{e^2}{r^2} \right]}{\left(1 - \frac{e^2}{r^2} \right)}$$

$$(26) \quad Q(r) = e$$

where M and e are constant.

Equation (21) gives a nonvanishing effective pressure

$$(27) \quad p_e = 0$$

$$(28) \quad \Rightarrow p = \frac{3}{2a^2} e^{-\nu}$$

By using (24)

$$(29) \quad \Rightarrow p = \frac{3r}{2a^2 A^2} e^{\int \frac{r^2}{G} dr}$$

and from equation (22)

$$(30) \quad 8\pi\rho = \frac{1}{r^2} \left[1 + G' \left(\frac{1}{r^2} - \frac{e^2}{r^4} \right) - G \left[\frac{2}{r^3} + \left(\frac{e^2}{r^4} \right)' \right] + \frac{e^2}{r^4} \right] + \frac{3}{2a^2} e^{-\nu}$$

From (23)

$$(31) \quad e^{-\lambda} = -\frac{G}{r^3} \left(1 - \frac{e^2}{r^2} \right)$$

We now turn to the example:

3.1. **Case (i).** As a first case, consider

$$(32) \quad G(r) = \alpha r^3$$

where α is constant.

The charge distribution will be choose so that $p_e = 0$

$$(33) \quad \Rightarrow p = \frac{3}{2a^2} e^{-\nu}$$

Making use of (20) we obtain for the $Q(r)$ the expression

$$(34) \quad Q^2(r) = r^{\frac{(\alpha+1)}{\alpha}} \left[C_1 + \frac{\alpha+1}{(\alpha-1)^2} r^{\frac{(\alpha-1)}{\alpha}} \right]$$

where C_1 is the constant of integration to be calculated from (26).

The metric coefficients and the energy density can be obtained from the equations (22) - (24)

$$(35) \quad e^{-\lambda} = -\alpha \left[1 - C_1 r^{\frac{(1-\alpha)}{\alpha}} - \frac{\alpha+1}{(\alpha-1)^2} \right]$$

$$(36) \quad e^{\nu} = A^2 r^{-(\frac{\alpha+1}{\alpha})}$$

$$(37) \quad 8\pi\rho = -\frac{1}{r^2} \left[2C_1 r^{\frac{(1-\alpha)}{\alpha}} + \frac{4\alpha(\alpha+1)}{(\alpha-1)^2} \right] + \frac{3 r^{\frac{(\alpha+1)}{\alpha}}}{2a^2 A^2}$$

Using equations (34), (35) with $C_1 = 0$ and equation (9), we obtain for the charge density the expression

$$(38) \quad 8\pi\sigma = \pm \frac{4\alpha(\alpha+1)}{r^2(\alpha-1)}$$

Equation (37) can be expressed as

$$(39) \quad 8\pi\rho = \frac{1}{r^2} \left[\frac{-\alpha(\alpha+1)}{(\alpha-1)^2} \right] + \frac{3 r^{\frac{(\alpha+1)}{\alpha}}}{2a^2 A^2}$$

From this Eq. we observe that ρ is everywhere positive for $-1 < \alpha < 0$. If $C_1 \neq 0$, Eq. (34) shows that $\frac{Q^2}{r^2}$ diverges as $r \rightarrow 0$.

3.2. **Case (ii).** As a second case consider $G(r)$ as given by Eq. (32) and consider that the proper density is constant, then Eq. (9) implies

$$(40) \quad Q(r) = e \left[\frac{r}{r_0} \right]^3$$

where $\frac{e}{r_0^3} = \frac{4\pi\sigma(\alpha-1)}{\alpha}$ The matter variables calculated from Eq. (21) and (22) are found to be

$$(41) \quad 8\pi p = p_0 r^{-A_0} + \frac{B_0}{A_0 - 2} r^{-2} + \frac{D_0}{A_0 + 2} r^2 + \frac{3}{2a^2} A^2 r^{-E_0}$$

$$(42) \quad 8\pi\rho = \left[5D_0\alpha - \frac{4e^2\alpha}{r_0^6} \right] r^2 - \frac{B_0}{A_0 - 2} \left[\frac{4\alpha}{\alpha+1} + 1 \right] r^{-2} - \left[\frac{2\alpha A_0}{\alpha+1} \right] p_0 r^{-A_0} + \frac{3}{2a^2} A^2 r^{-E_0}$$

where the constants A_0, B_0, D_0 and E_0 are given by

$$A_0 = \frac{(\alpha+1)(3\alpha+1)}{\alpha(\alpha-1)}, \quad B_0 = -\frac{(\alpha+1)}{\alpha(\alpha-1)}$$

$$D_0 = \frac{e^2(5\alpha-6\alpha+1)}{r_0^6 \alpha(\alpha-1)}, \quad E_0 = \frac{\alpha+1}{\alpha}$$

A is already defined after Eq.(24).

The neutral sphere is obtained for $D_0 = 0$.

4. CONCLUSION

In the above procedure, we have presented a static analytic solutions for spherically symmetric distributions of charge matter in the context of bimetric theory of gravitation by introducing a arbitrary function $G(r)$. In case (i) we observed that for $C_1 = 0$ the energy density is positive everywhere for $-1 < \alpha < 0$ and for $C_1 \neq 0$, $\frac{Q^2}{r^2}$ diverges as $r \rightarrow 0$. For $Q(r) = 0$, we recover a previously known solution of Khadekar and Kandalkar (2004) for a neutral matter. For a physical system which is small compare to the size of the universe for example a solar system, the term $\frac{3}{2\alpha^2}e^{-\nu}$ in the field equations (6) -(8) is negligible. The present result reduced to the Einstein general relativity. It is interesting to note that the solution obtained for case (ii) be the uniformly charged version of the Tolman V solution [12] [with a slight change in notation, his n corresponding to $-\frac{(\alpha+1)}{2\alpha}$]. For a neutral sphere (i.e. $D_0 = 0$), Tolman's results are recovered.

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