ALGEBRAIC GEOMETRY OF THE QUARK

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ABSTRACT. Using only the algebraic geometry defined by an irreducible representation of the Dirac ring we find an invariant tetrahedron that specifies the quadrupole shape occupied by nucleons with overwhelming probability. A simpler case is given by the cones occupied by spinning electrons. The Cayley cubic, which is invariant under the tetrahedral group, appears in Figure 2 and has 3 tangent cones representing quarks held together by an amorphous gluon condensate. This cubic is characterized by 9 lines emanating from the quarks and coincident with the edges of the tetrahedron. These quarks are short-lived because the representation is not time-invariant.

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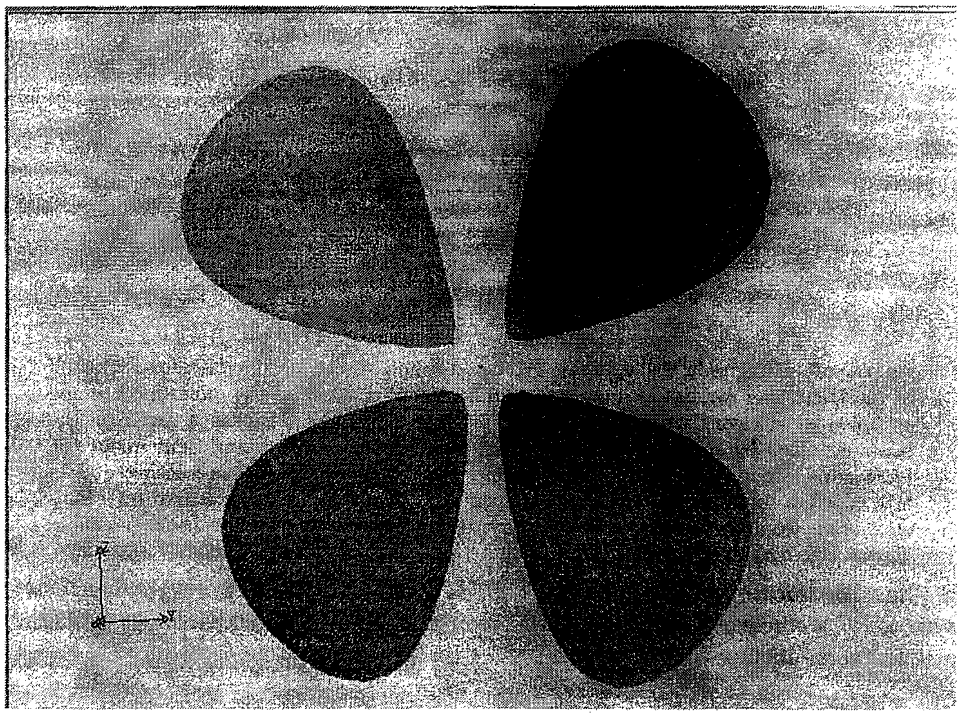
1. INTRODUCTION

Following on the ideas of Wigner [18], Dirac [8] and Eddington [9] the Author [4] found an irreducible representation (IR) of the quaternion or Dirac ring which was able to incorporate spin and isospin and was therefore an existence theorem for Wigner's supermultiplets [19]. This is equation (1) below which is an algebraic representation of a deuteron. However utilizing the work of Hudson [12], written a century ago, it is possible to turn this equation into a nuclear quadrupole surface, described by equation (5) and illustrated in Fig. 1, by using the techniques of algebraic geometry without any appeal to a Hamiltonian.

Similarly equation (1), without the isospin term E_{05}, is the algebraic representation of an electron and the corresponding algebraic surface is given by the quadric (17). The picture generated by a program due to Morris [16] appears in the bottom half of Fig. 1.

Essentially the quadrupole surface of Fig. 1 is the most general quartic that remains invariant under the Heisenberg group of Dirac rotations and reflections (3,4) of a fundamental tetrahedron in P^3 arising from equation (1). But another variety with tetrahedral symmetry, but no reflections, is the Cayley cubic C of equation (28) which is a section of the Segre cubic S_3 of equation (23) and is shown in Fig. 2. The tangent cones are believed to be occupied by quarks with spin fields similar to electrons [15].

The set $E_{23}; E_{14}; E_{05}$ of rotations in equation (1) is one of the 15 synthemes in the geometry of P^3 that specifies the fundamental tetrahedron (cf. [3] and [13]).In this way we



$$(x^2 + y^2 z^2) + (y^2 + x^2 z^2) + (z^2 + x^2 y^2) + 10 x y z = 0$$

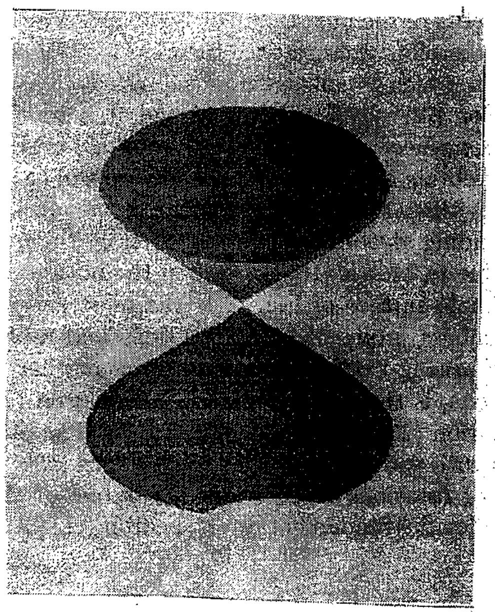


FIGURE 1. Nuclear Quadrupoles and Electron Spin Cones.

have come full circle from an IR (1) of the Dirac ring to a fundamental tetrahedron in every nucleon that in turn lies in a cubic C with 9 characteristic lines.

Equation (28) for the Cayley cubic is not invariant under the $H_{2,2}$ or Dirac reflections τ in (4). In particular it will not be invariant under time reversals which is the reason why quarks are short-lived. However the commuting operators E_{23},E_{14} and E_{05} of D are all rotations so (28) is indeed invariant under operations of the center.

The bottom picture of Fig. 2 is a rotation through $\pi/2$ and clearly shows a gluon condensate emanating from the fourth tetrahedral vertex which grows with its separation from a quark. Although it is not shown in Fig. 2, there are actually two invariant tetrahedra back to back as shown in Fig. 3 (cf. Hunt [13] p118). One of them corresponds to the neutron (d u d) and the other to the proton (u d u).

In section 2 we will discuss equations (1) and (5) and show how they lead to the surfaces in Fig. 1. We will also see how many particle representations do not change the shapes but lead to Hodge decompositions in a Calabi-Yau space.

Finally section 3 will cover the first generation of quarks.

2. The Shapes of a Deuteron and Electron Spin Cone

We shall employ the IR, or one-form

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{1}$$

of a minimum left ideal at the center D of the Dirac ring found by de Wet [4]. Here we use Eddington's transparent notation and associate the operators E_{23},E_{14} and E_{05} respectively with rotations in 3-space,4-space and isospace that correspond to the spin σ, parity π and charge T_3 carried by a single nucleon.But since the charge is double-valued, equation (1) more properly describes a deuteron characterized by the two invariant tetrahedra in Fig. 3. E_4 is the unit 4×4 matrix while ψ_2,ψ_3 and ψ_4 are half angles of rotation: e is a primitive idempotent. To see how E_{14} is related to parity we notice that a rotation through π about t will send x to $-x$ without inverting the time axis but instead changing to a left-handed coordinate system. Eddington's E-numbers are related to the Dirac γ matrices by

$$\gamma_\nu = iE_{0\nu}, E_{\mu\nu} = E_{\rho\mu}E_{\rho\nu} = -E_{\nu\mu}; E_{\mu\nu}^2 = -1;$$

$$E_{\mu\nu}E_{\sigma\tau} = E_{\sigma\tau}E_{\mu\nu} = iE_{\lambda\rho}, \mu < \nu = 1,\ldots,5 \tag{2}$$

Referring to (1) the three commuting operators E_{23},E_{14} ,E_{05} each specify a pair of 'fix-lines' lying in the ± 1 or$\pm i$ eigenspaces determined by the relation $E_{\mu\nu}^2 = -1$.These are polar lines (eg.$1, \pm i$) defined by the fact that their two-dimensional vector spaces are orthogonal to each other.Thus the 6 fix-lines are the edges of a fundamental tetrahedron which may be inscribed in a cube as illustrated by Fig. 3. In this way equation (1) characterizes such a tetrahedron labelled by the coordinates X,Y,Z and T of its corners and if it is to be $H_{2,2}$ invariant under the Heisenberg group $H_{2,2}$ acting on P^3 then the required operations are the rotations

$$\sigma_1 : (X,Y,Z,T) \to (T,Z,Y,X)(E_{23}); \sigma_2 : (X,Y,Z,T) \to (Z,T,X,Y)(E_{05});$$

$$\sigma_1\sigma_2 = \sigma_2\sigma_1 : (X,Y,Z,T) \to (Y,X,T,Z)(iE_{14}) \tag{3}$$

and reflections

$$\tau_1 : (X,Y,Z,T) \to (X,Y,-Z,-T)\tau_2 : (X,Y,Z,T) \to (X,-Y,Z,-T) \tag{4}$$

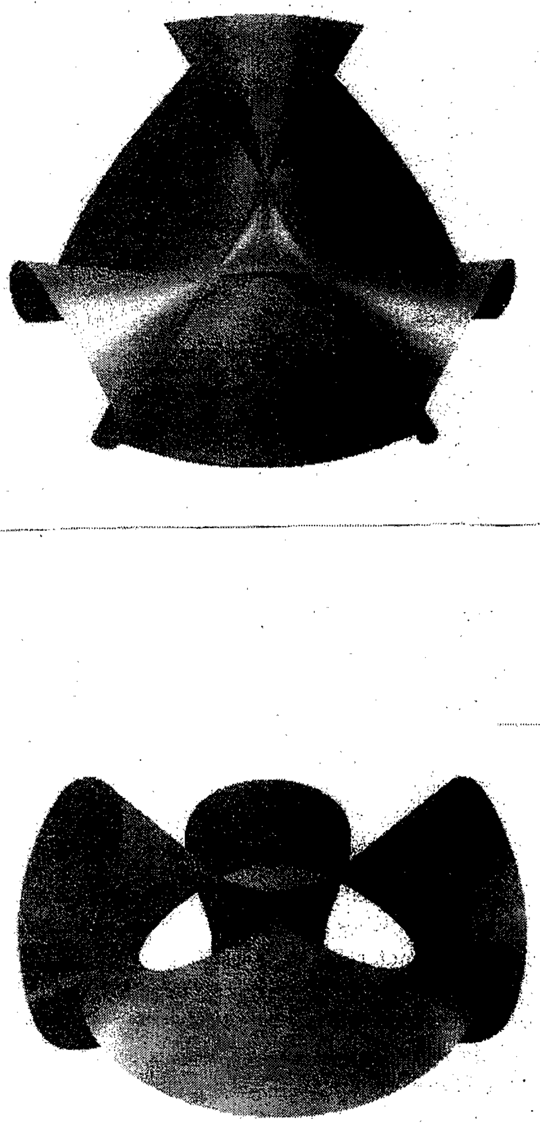


FIGURE 2. Quarks and Gluon Condensate.

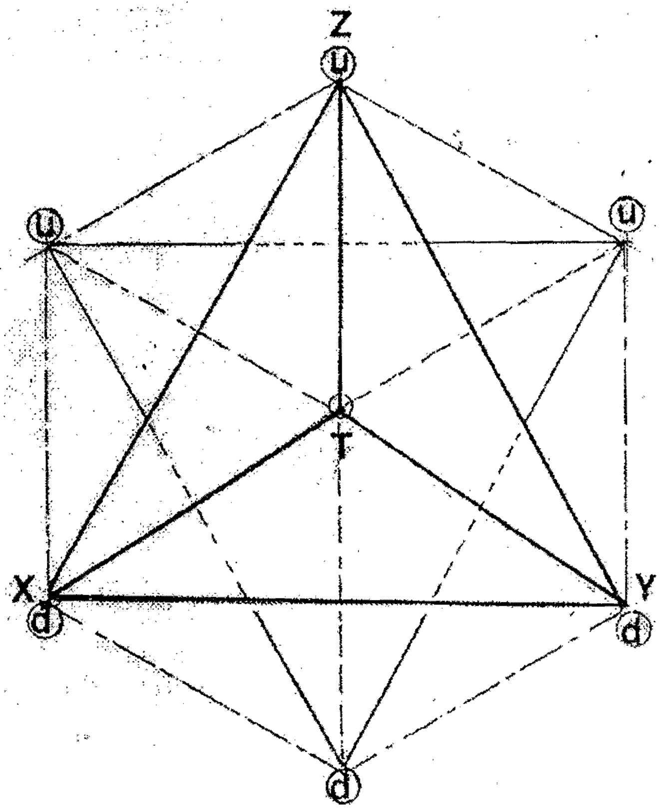


FIGURE 3. Tetrahedra Inscribed in a Cube.

(cf. [3]) These are of course Dirac operators and the most general quartic that remains invariant under the rotations and reflections is (ibid)

$$(x^2y^2 + z^2t^2) + (y^2z^2 + x^2t^2) + (x^2z^2 + y^2t^2) + kxyzt = 0, k > 6 \tag{5}$$

Fig.1 is the Lorentz invariant Kummer quadrupole drawn by a program written by Richard Morris [16] with t=1 and k=10 that characterizes many nuclei (cf. Refs. [1], [11])in full accord with experiments conducted by Heyde [11].In the case of the deuteron there are no bound excited states and thus no overlapping wave functions between deuterons so the many particle nuclear shape should not change (Ibid).

This may be shown algebraically by constructing the many particle wave functions which are the tensor product of (1) with itself and are de Broglie algebras with the $4^A \times 4^A$ basis elements (de Wet [4])

$$E^l_{\mu\nu} = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4 \tag{6}$$

with $E_{\mu\nu}$ in the l-th position.The de Broglie generators are

$$\Gamma^{(A)}_\nu = \frac{1}{2}(E^1_{0\nu} + E^2_{0\nu} + \cdots + E^A_{0\nu}), \nu = 1, \ldots, 5 \tag{7}$$

$$\sigma^{(A)}_{\mu\nu} = [\Gamma^{(A)}_\mu, \Gamma^{(A)}_\nu] = \frac{1}{2}(E^1_{\mu\nu} + \cdots + E^A_{\mu\nu}),\ \mu < \nu = 1, \ldots, 5 \tag{8}$$

$$\eta_\nu^{(A)} = E_{0\nu} \otimes \cdots \otimes E_{0\nu} = E_{0\nu}^1 E_{0\nu}^2 \ldots E_{0\nu}^A, \eta_{\mu\nu}^{(A)} = \eta_\mu^{(A)} \eta_\nu^{(A)} = E_{\mu\nu}^1 \ldots E_{\mu\nu}^A \tag{9}$$

where A=N+Z is the atomic number. Then the irreducible representations of the enveloping algebra$A(\gamma)$ of the Dirac ring are

$$\Psi^{(A)} = \sum C_{[\lambda]} P_{[\lambda]} \tag{10}$$

where$P_{[\lambda]}$ is a projection operator

(11)

$$P_{[\lambda]} = i^{-A}(i^A \psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} + \eta_{23}^{(A)}\psi_1^{\lambda_2}\psi_2^{\lambda_1}\psi_3^{\lambda_4}\psi_4^{\lambda_3})\epsilon_A + i^{-A}(\eta_{14}^{(A)}\psi_1^{\lambda_3}\psi_2^{\lambda_4}\psi_3^{\lambda_1}\psi_4^{\lambda_2} + \eta_5^{(A)}\psi_1^{\lambda_4}\psi_2^{\lambda_3}\psi_3^{\lambda_2}\psi_4^{\lambda_1})\epsilon_A$$

satisfying

$$P_{[\lambda]}^2 = P_{[\lambda]} \psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} \tag{12}$$

and $[\lambda] = [\lambda_1\lambda_2\lambda_3\lambda_4]$ labels a state, where $(\lambda_3 + \lambda_4)$ is the number of protons Z and$(\lambda_1 + \lambda_2)$ the number of neutrons N.Thus the first term in (11) characterizes a given nucleus where $(\lambda_2 + \lambda_3)$ is the number of nucleons with a negative spin s so that $(\lambda_1 + \lambda_4)$ is automatically the number with a positive s.The second bracket of (11) is the mirror image $T_3 = \frac{1}{2}(N-Z) \to -T_3$

The term $\epsilon_A = e \otimes \cdots \otimes e$ is idempotent so that (11) has the same form in configuration space $A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ as (1) has in 4-space with the essential difference that the operators E_{23}, E_{14}, E_{05} are replaced by their tensor products $\eta_{23}, \eta_{14}, \eta_5$. However this will not affect the fix-lines because $E_{\mu\nu}^l, E_{\mu\nu}^{l+1}$ commute. Therefore each nucleon in the ensemble A will belong to the same tetrahedron and the ensemble will obey (5)as found by Heyde [11].

The nuclear physics lies in the three-form entanglement operator [7]

$$C_{[\lambda]} = i^{\lambda_1} \sum (E_{23}^1 \ldots E_{23}^{\lambda_2} E_{14}^{\lambda_2+1} \ldots E_{14}^{\lambda_2+\lambda_3} E_{05}^{\lambda_2+\lambda_3+1} \ldots E_{05}^{A-\lambda_1}) \tag{13}$$

which in the case of the ground state $[\Lambda] = [\Lambda_1\Lambda_2\Lambda_3\Lambda_4]$ can be written as a Hodge decomposition in

$$\sigma_0 = 2\sigma_1 = (E_{23}^1 + \cdots + E_{23}^A) = 2is, \pi_0 = 2\pi_1 = (E_{14}^1 + \ldots E_{14}^A) = 2ip,$$

$$T_0 = 2\Gamma_5^{(A)} = (E_{05}^1 + \cdots + E_{05}^A) = i(Z-N) = 2T_3 \tag{14}$$

where the spin s and parity p for any given state are

$$s = \frac{1}{2}(A - 2(\Lambda_2 + \Lambda_3)), p = \frac{1}{2}(2(\Lambda_2 + \Lambda_4) - A) \tag{15}$$

This state is unique because $\Lambda_1 = \Lambda_4 \geq (\Lambda_2, \Lambda_3)$ otherwise there is no decomposition. This means that all of the other states$[\lambda]$, that label the rows of the matrix $C_{[\Lambda]}$, are Zeeman sublevels of the ground state and not the so-called 'excited' states determined by decay products of neighboring nuclei. However if the 'pouches' of Fig.1 are deemed to contain subnuclei then it is possible to envisage bands in a rotating quadrupole which would also appear to be nearly spherical especially as the subnuclei are entangled by [7]

Several papers have appeared on nuclear structure and string theory based on (13)(cf.eg.[6]) and a synopsis is given in [7].But another summary would take us too far afield so we will move on to the dipole of Fig.1 which is based on equation (1) without isospin E_{05} (and consequently parity $E_{14} = iE_{05}E_{23}$) and governs spinning electrons. The basic equation is therefore

$$\Psi = 2(i\psi_1 + \epsilon_{23}\psi_2)e_1 \tag{16}$$

Here $\epsilon_{23} = i\sigma_1$, if σ_1 is a Pauli spin matrix, and e_1 is a primitive idempotent with the eigenvalues 0,1.

There is now only one pair of fix-lines determined by $\epsilon_{23}^2 = -1$ which specify the Plucker quadric surface

$$x^2 + y^2 - z^2 = 0 \tag{17}$$

that is the double conic of Fig.1.Again the r-th tensor product of (16) with itself has the same form as (16), namely a sum of terms

$$C_{[\lambda]}(\psi_\lambda + (-i)^r \eta_1^{(r)} \psi_{r-\lambda}) e_1^{(r)} \tag{18}$$

where

$$\eta_1^{(r)} = \epsilon_{23}^1 \epsilon_{23}^2 \dots \epsilon_{23}^r \tag{19}$$

if

$$\epsilon_{23}^l = E_2 \otimes \cdots \otimes E_2 \otimes \epsilon_{23} \otimes E_2 \otimes \cdots \otimes E_2 \tag{20}$$

with ϵ_{23} in the l-th position (cf. [5]).Because $[\epsilon_{23}^l, \epsilon_{23}^{(l+1)}] = 0$ the fix-lines will not be affected. In other words the spin vectors can be found anywhere on the surfaces of the cone subject to the requirement that paired spins $\pm\frac{1}{2}$ occupy positions on the upper and lower cones. Each term in (18) labels a possible spin state $[(r-\lambda), \lambda]$ of $r = \lambda + (r - \lambda)$ where λ is the number of spins with the same sign. In particular when $\lambda = r/2 = even$, the ground state is

$$C_{r/2} = \frac{(2i)^{r/2}}{(r/2)!}(\Gamma_1^2 + 1)(\Gamma_1^2 + 9) \dots (\Gamma_1^2 + (r/2 - 1)^2)) \tag{21}$$

where

$$\Gamma_1 = \frac{1}{2}(\epsilon_{23}^1 + \epsilon_{23}^2 + \cdots + \epsilon_{23}^r) \tag{22}$$

andϵ_{23} can assume all of the possible spins$\pm i$.

Actually because (16) is also a representation of $SU(2)$ the double cone is already well-known from standard angular momentum theory and makes an angle of $cos^{-1}(1/\sqrt{3})$ with the vertical axis (cf.eg.[2]),so this also underlines the power of algebraic geometry in atomic theory.

3. Fingerprints of the Quark

We come now to the kernel of this contribution which is Fig.2. Because equation (28) below, governing the quark surface, is not invariant under reflections we can no longer appeal to equation (5). However, in a very comprehensive review of projective embeddings Hunt [13] Ch. 3 covers related algebras and in particular on p118 finds a double tetrahedron in pentahedral coordinates.

The starting point is the Segre cubic

$$S_3 = \sum_{i=0}^{5} x_i = 0; \sum_{i=0}^{5} x_i^3 = 0 \tag{23}$$

which has ten nodes or singular points.Then the Nieto quintic N_5

$$x_0x_1x_2x_3x_4 + \cdots + x_1x_2x_3x_4x_5 = 0 \tag{24}$$

is the invariant Hessian of S_3,($Hess(S_3) = \det(\partial^2 S_3/\partial x_i \partial x_j)$) and the moduli interpretation or space which is also $H_{2,2}$ invariant is the quartic (5). The Cayley cubic

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0 \tag{25}$$

is a section of S_3 and can be written in tetrahedral coordinates

$$x = x_0/v, y = x_1/v, z = x_2/v \tag{26}$$

together with a cone at infinity

$$v = \frac{5}{2}(x_0 + x_1 + x_2 + 2x_3) \tag{27}$$

which can be used to substitute for x_3. Then dividing (25) by v^3 we find

$$-5(xy^2 + x^2y + xz^2 + x^2z + yz^2 + y^2z + xyz) + 2(xy + yz + xz) = 0 \tag{28}$$

which is Fig.2 compiled by a program due to Morris [16]. Here the invariant tetrahedron is clearly visible with tangent cones at the 3 vertices. In view of the fact that quarks are also spin half fermions with negligible size it is natural to identify the tangent cones with the 3 quarks associated with a proton or neutron (cf.eg.[15]).

In fact Hunt ([13],p118) finds two invariant tetrahedra lying back to back as shown in Fig.3 and because equation (1) describes a deuteron , one triangle corresponds to a neutron (d u d) and its mate to a proton (u d u) as illustrated. The remaining two vertices are evidently the sources of a gluon condensate appearing in the bottom half of the figure which is a rotation through $\pi/2$.

Actually there are 6 lines emanating from the 3 tangent cones which intersect the surface of the condensate via edges of the tetrahedron. These correspond to up and down quark representations of $SU(3)$ and their color reflections which are gluons (cf. [17],Table 21).

Because we are very unlikely to ever see a quark or electron, the cones of Fig.2 must be thought of as quark 'footprints' or signatures and similarly the gluon condensate is more properly a manifold where gluons of the first generation of quarks may be found.Nevertheless lattice calculations by Ji, Boucard et.al.[14] appear to estimate a condensate of the right order although our analysis does not specifically cover high energy quarks.

Finally, if we insist on invariance under all of the operations (3,4) then equation (28) reduces to (5) and the quark fingerprints are hidden. Although, of course, (28) is invariant under the transformations (3) of the center of the Dirac ring.

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