

A FUNDAMENTAL GROUP FOR DYNAMICAL SYSTEMS IV: CONNECTED TOPOLOGIES

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ABSTRACT. This paper continues exploration of a recent homotopy construction for group-sets. This final paper considers the invariant on topological spaces which admit a universal covering space. The universal version is an extension of the profinite completion of the classical fundamental group.

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INTRODUCTION

This paper concludes computations of examples of a functorial invariant. Preceding papers [1, 2, 3] focused on spaces which did not support a classical fundamental group. We end the sequence by comparing π_G with π_1 on spaces for which both exist.

We continue in the language of the earlier papers. The following terminology is needed for the main theorem of this paper.

Recall that $\mathbf{HT}[\mathbb{Z}]$ is the category of topological $\mathbb{Z}$ -sets, whose underlying topology in Hausdorff. Let $\mathbf{Path}$ be the category of pairs (Y, y_0) in which

- (1.a) $Y \in \mathbf{HT}[\mathbb{Z}]$,
- (1.b) $y_0 \in Y$,
- (1.c) the underlying topology of Y is pathwise connected and locally simply connected.

A $\mathbf{Path}$ -morphism $f : (X, x_0) \longrightarrow (Y, y_0)$ is an $\mathbf{HT}[\mathbb{Z}]$ -morphism $f : X \longrightarrow Y$ for which $f(x_0) = y_0$. Composition and identity are induced from set theory.

As in earlier papers, we let $\pi_1(*)$ and $\pi_G(*)$ denote, respectively, the classical fundamental group functor (of point-set topology) and the adjusted version for $\mathbf{HT}[\mathbb{Z}]$ developed in [1] (and reviewed in [2]).

Throughout our discussion, we use the language of the *slice category*. Let $\mathcal{C}$ be a category, and let $A \in \mathcal{C}$. In the category $\mathcal{C}/A$, an object is a $\mathcal{C}$ -morphism into A ,

and C/A -morphism from $b : B \rightarrow A$ to $c : C \rightarrow A$ is a C -morphism $f : B \rightarrow C$ such that $c \circ f = b$.

Let **Prof** be the category of profinite groups. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$.

This paper proves

Theorem A. *Let E be any **Path**-object whose space consists of a single point. For each $(Y, y_0) \in \mathbf{Path}$, let ϵ_{Y, y_0} be the unique morphism $(Y, y_0) \rightarrow E$. In earlier papers, we set up a canonical **Prof**-isomorphism $\pi_G(E) \cong \widehat{\mathbb{Z}}$. There exists a functor from **Path** to the category of short exact sequences of groups which has the format*

$$\begin{array}{ccccccc} \text{On Objects: } (Y, y_0) \mapsto & & & & & & \\ 1 \longrightarrow \pi_1(Y, y_0) & \xrightarrow{\iota[Y, y_0]} & \Gamma(Y, y_0) & \xrightarrow{j[Y, y_0]} & \mathbb{Z} & \longrightarrow & 1, \end{array}$$

On Morphisms: $(f : (X, x_0) \rightarrow (Y, y_0)) \mapsto$

$$\pi_1(f) : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0), \quad \Gamma(f) : \Gamma(X, x_0) \rightarrow \Gamma(Y, y_0), \quad 1_{\mathbb{Z}},$$

where the third map is the identity function on $\mathbb{Z}$, such that

- (2.a) *The composition of the Γ assignments (on objects and morphisms) with the profinite completion functor on groups is naturally isomorphic to $\pi_G(*)$.*
- (2.b) *Define a functor $\mathbf{Path} \rightarrow \mathbf{Prof}/\widehat{\mathbb{Z}}$ by sending each $(Y, y_0) \in \mathbf{Path}$ to the composition of $\pi_G(\epsilon_{Y, y_0})$ with the isomorphism $\pi_G(E) \cong \widehat{\mathbb{Z}}$. This functor is naturally isomorphic to the composition of the profinite completion functor with $j[Y, y_0]$.*

1. NOTATION

Let **Top** be the category of topological spaces. We apply the slice category notation to it. That is, for $A \in \mathbf{Top}$, define the category **Top**/ A by

- (3.a) an object is a continuous function $b : B \rightarrow A$ into A , and
- (3.b) a morphism from function $b : B \rightarrow A$ to $c : C \rightarrow A$ is a continuous function $f : B \rightarrow C$ such that $c \circ f = b$.

For $A \in \mathbf{Top}$, let 1_A denote the identity function on A .

For the argument here, we name technical properties which are cited repeatedly.

- (4.a) Let $X \in \mathbf{HT}[\mathbb{Z}]$. A subset $U \subseteq X$ is a *weak $\mathbb{Z}$ -component*, if
 - with respect to the topology of X , U is both open and closed, and
 - U is closed under the action by $\mathbb{Z}$.
- (4.b) Let $X \in \mathbf{Top}$. We refer to a connected component of X as a *top-component*, to emphasize the distinction between it and either kind of $\mathbb{Z}$ -component.
- (4.c) Let $f : X \rightarrow Y$ be a continuous function. We refer to f as a *tiered projection* if there is a set F such that (with respect to discrete topology on F) f is **Top**/ V -isomorphic to the canonical second projection $F \times Y \rightarrow Y$. In addition, if V is a subset of Y , we say that f is a *tiered projection over V* if restriction of f to $f^{-1}V$ is a tiered projection to V .

(4.d) Let $b : B \longrightarrow A$ be a continuous function. We say that b has the *basic covering property* if

- A is locally simply connected, and
- for each $y \in A$, there is an open neighborhood V of y such that b is a tiered projection over V .

In the definition of tiered projection, we allow the set F to be empty. The tiered projections, that we consider, usually meet additional conditions which imply that F is non-empty.

The following remarks are trivial.

- (5.a) In early papers, a $\mathbb{Z}$ -component of some $X \in \mathbf{HT}[\mathbb{Z}]$ is a weak $\mathbb{Z}$ -component, as defined here. The former is defined only when X is a finite disjoint union of such subsets. The modified term makes no assumption on X . In practice, the “weak” property is useful only for objects whose topology fulfills other conditions.
- (5.b) An $\mathbf{HT}[\mathbb{Z}]$ -object X is $\mathbb{Z}$ -connected if and only if the only weak $\mathbb{Z}$ -components of X are $\emptyset$ and X .

The methods in this paper apply only to $\mathbf{HT}[\mathbb{Z}]$ -objects whose topology meets certain minimal conditions. We assign a name, which is only for the discussion here.

Definition 1. Let X be either a topological space of an $\mathbf{HT}[\mathbb{Z}]$ -object. We say that X is **-acceptable* if and only if

- (6.a) the topology of X is locally simply connected,
- (6.b) each top-component of X is open in X ,
- (6.c) each top-component of X is pathwise connected,
- (6.d) X is $\mathbb{Z}$ -connected.

In the context of the other three conditions, the last can be rephrased as: if U is a connected component of X , then

$$X = \bigcup_{n \in \mathbb{Z}} n \star U.$$

□

2. UNIVERSAL COVERING SPACES

We begin with an adjustment to the notion of universal covering space.

Definition 2. Let $W \in \mathbf{HT}[\mathbb{Z}]$. We say that W is *universally $\mathbb{Z}$ -connected* if and only if there is a non-empty subset $U \subseteq W$ such that

- (7.a) U is an open subset of W ,
- (7.b) as a topological space, U is pathwise connected, simply connected, and locally simply connected
- (7.c) $W = \bigcup_{n \in \mathbb{Z}} n \star U$, and
- (7.d) $\forall n \in \mathbb{Z}, (n \star U) \cap U \neq \emptyset \Rightarrow n = 0$.

Let Y be a $\mathbb{Z}$ -connected $\mathbf{HT}[\mathbb{Z}]$ -object. A *universal $\mathbb{Z}$ -covering map into Y* is an $\mathbf{HT}[\mathbb{Z}]$ -morphism $\zeta : W \longrightarrow Y$ such that

- (8.a) W is universally $\mathbb{Z}$ -connected, and
- (8.b) ζ satisfies the basic covering property.

□

Suppose that W is universally $\mathbb{Z}$ -connected. Easy consequences are

(9.a) W is $\mathbb{Z}$ -connected.

(9.b) Every top-component has the properties of U in (7).

We can extend the mapping properties of universal (that is, simply connected) covering spaces to covering maps from universally $\mathbb{Z}$ -connected objects.

Proposition 3. *Let Y be a $*$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object. Then there exists a universal $\mathbb{Z}$ -covering map into Y .*

Proof. Let Y_0 be any top-component of Y . The standard covering space theory implies that there exists a simply connected covering space $\zeta_0 : W_0 \rightarrow Y_0$. Define W to be the topological product $W_0 \times \mathbb{Z}$, in which $\mathbb{Z}$ is assigned the discrete topology. Define $\zeta : W \rightarrow Y$ by

$$\forall z \in W, \forall k \in \mathbb{Z}, \zeta(x, k) = k \star \zeta_0(x) .$$

Define a binary operation $\mathbb{Z} \times W \rightarrow W$ by

$$\forall x \in X, \forall k, m \in \mathbb{Z}, m \star (x, k) = (x, m + k) .$$

Proof of the following assertions are left to the reader.

(10.a) With respect to this action, W becomes an $\mathbf{HT}[\mathbb{Z}]$ -object. We regard it as a member in this category.

(10.b) The function ζ is an $\mathbf{HT}[\mathbb{Z}]$ -morphism.

(10.c) W is universally $\mathbb{Z}$ -connected.

The basic covering property requires a short argument.

Suppose that V is an open, non-empty, connected subset of Y , F is a set, and σ is a $\mathbf{Top}/V$ -isomorphism from projection $F \times V \rightarrow V$ to the restriction of ζ_0 to $\zeta_0^{-1}V \rightarrow V$. We claim that there is a set F' such that the restriction of $\zeta^{-1}V \rightarrow V$ is $\mathbf{Top}/V$ -isomorphic to projection $F' \times V \rightarrow V$.

Essentially, ζ is a disjoint union of functions on components $W_0 \times \{n\}$, where n varies over $\mathbb{Z}$. Fix $n \in \mathbb{Z}$, and let f be the restriction of ζ to $W \times \{n\}$. It suffices to find an analogy to F for f .

The standard theory of covering spaces implies that there exists a continuous function $\nu : W_0 \rightarrow W_0$ such that

$$\forall x \in W_0, (\zeta_0 \circ \nu)(x) = (-n) \star \zeta_0(x) .$$

(Actually, we can index the choices of ν by members of any fiber over any point in Y .) Any choice is a homeomorphism. Consider the function σ^* on $F \times V$ given by

$$\sigma^*(t, x) = \nu(\sigma(t, x)) .$$

The established properties of σ and ν easily implies that σ^* is a $\mathbf{Top}/V$ -isomorphism from projection $F \times V \rightarrow V$ to the restriction of f from $f^{-1}V \rightarrow V$.

We conclude

(11) The function ζ has the basic covering property.

□

We list two elementary observations.

Lemma 4. *Let $\zeta : W \rightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Then Y is $*$ -acceptable.*

Lemma 5. *Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Let W_0 be a top-component of W . Note that there is a top-component Y_0 of Y which contains $\zeta(W_0)$. Then restriction of ζ to $W_0 \longrightarrow Y_0$ is a universal covering space map.*

Proof. Trivial. $\square$

Proposition 6. *Let Y be a $\ast$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object, and let $\phi : X \longrightarrow Y$ be an $\mathbf{HT}[\mathbb{Z}]$ -morphism with the basic covering property. Let $g : W \longrightarrow Y$ be an $\mathbf{HT}[\mathbb{Z}]$ -morphism from a universally $\mathbb{Z}$ -connected object W . Let $z_0 \in W$ and x_0 such that $\zeta(z_0) = \phi(x_0)$. Then there is a unique $\mathbf{HT}[\mathbb{Z}]$ -morphism $\gamma : W \longrightarrow X$ such that*

$$(12.a) \quad \phi \circ \gamma = g, \text{ and}$$

$$(12.b) \quad \gamma(z_0) = x_0.$$

Proof. Let W_0 be the top-component of z_0 in W and let X_0 be the top-component of x_0 in X . Put $y_0 = \zeta(z_0) = \phi(x_0)$. Let Y_0 be the top-component of y_0 in Y . By standard theory, there is a unique continuous function $\gamma_0 : W_0 \longrightarrow X_0$ such that $\gamma_0(z_0) = x_0$ and $\phi \circ \gamma_0 = g$.

Define $\tau : W_0 \times \mathbb{Z} \longrightarrow W$ by $\tau(z, n) = n \star z$. By hypothesis, τ is a homeomorphism: Define a binary function $\mathbb{Z} \times (W_0 \times \mathbb{Z}) \longrightarrow W_0 \times \mathbb{Z}$ by

$$m \star (x, k) = (x, m + k).$$

With this function, $W_0 \times \mathbb{Z}$ becomes an $\mathbf{HT}[\mathbb{Z}]$ -object. With respect to this structure, τ is an $\mathbf{HT}[\mathbb{Z}]$ -isomorphism.

Define $\gamma_t : W_0 \times \mathbb{Z} \longrightarrow X$ by $\gamma_t(z, k) = k \star \gamma_0(z)$. By inspection,

$$(13.a) \quad \gamma_t \text{ is an } \mathbf{HT}[\mathbb{Z}]\text{-morphism,}$$

$$(13.b) \quad \phi \circ \gamma_t = g \circ \tau,$$

$$(13.c) \quad \gamma_t(z_0, 0) = x_0 \text{ and } \tau(z_0, 0) = z_0.$$

It follows that $\gamma = \gamma_t \circ \tau^{-1}$ has the property of the γ described in the conclusion.

Uniqueness is an easy consequence. $\square$

Corollary 7. *Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering space map. Let ν be an $\mathbf{HT}[\mathbb{Z}]/Y$ -morphism from ζ into itself. If there is $z_0 \in W$ such that $\nu(z_0) = z_0$, then ν is the identity morphism.*

Proof. This is a special case of Proposition 6. $\square$

Corollary 8. *Let Y be a $\ast$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object. Suppose that $\zeta_X : X \longrightarrow Y$ and $\zeta_W : W \longrightarrow Y$ are two universal $\mathbb{Z}$ -covering space maps. Let ν be an $\mathbf{HT}[\mathbb{Z}]/Y$ -morphism from ζ_X to ζ_W . Then ν is an $\mathbf{HT}[\mathbb{Z}]$ -isomorphism.*

Proof. Pick any $x_0 \in X$, and put $z_0 = \nu(x_0)$. By Proposition 6, there must be an $\mathbf{HT}[\mathbb{Z}]/Y$ -morphism $\mu : \zeta_W \longrightarrow \zeta_X$ such that $\mu(z_0) = x_0$. Corollary 7 implies that $\nu \circ \mu$ and $\mu \circ \nu$ are identity morphisms. Consequently, ν and μ are inverses to one another. $\square$

Lemma 9. *Let Y be a $\ast$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object. Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Let G be the group of $\mathbf{HT}[\mathbb{Z}]/Y$ -automorphisms of ζ .*

(A) *The action by G on W is locally free.*

(B) *For each $y_0 \in Y$, the action by G on $\zeta^{-1}\{y_0\}$ is regular*

(C) *The function ζ is a quotient map, in the sense of the category $\mathbf{HT}[\mathbb{Z}]$, for the action by G .*

By “a quotient map, in the sense of the category $\mathbf{HT}[\mathbb{Z}]$,” we mean a quotient map $q : W \longrightarrow Q$ for equivalence modulo the action by G on the set W , with the further assignment

- (14.a) Q receives the quotient topology induced from W , and
- (14.b) each $n \in \mathbb{Z}$ acts on Q in the unique manner that commutes with q and the action by n on W .

Proof. Proposition 6 implies assertion (B). It follows that ζ is a quotient map, with respect to the category of sets. The basic covering property implies (A). Part (C) follows. $\square$

Corollary 10. *Let Y be a $*$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object. Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Let G be the group of $\mathbf{HT}[\mathbb{Z}]/Y$ -automorphisms of ζ . Let $\mathcal{L}$ be the set of subgroups of G . To each $H \in \mathcal{L}$, assign*

- (15.a) *a quotient map $q_H : W \longrightarrow Q_H$ for the action by H on W , in the category $\mathbf{HT}[\mathbb{Z}]$, and*
- (15.b) *the factoring $s_H : Q_H \longrightarrow Y$ of ζ through q_H .*

The following assertions are true.

- (A) *For each $H \in \mathcal{L}$, q_H has the basic covering property.*
- (B) *Let $H, L \in \mathcal{L}$. Put*

$$K = \{ g \in G : g \cdot H \cdot g^{-1} \subseteq L \}.$$

Let J be the subset of $g \in G$ for which $q_L \circ g$ factors through q_H . For each $g \in J$, let $j[g]$ be the unique morphism such that $q_L \circ g = j[g] \circ q_H$. Let M be the set of $\mathbf{HT}[\mathbb{Z}]/Y$ morphisms $s_H \longrightarrow s_L$. Then

- (16.a) $K = J$,
- (16.b) K is closed under left multiplication by L , and
- (16.c) $g \mapsto j[g]$ induces a bijection $L \backslash K \longrightarrow M$.

Proof. Part (A) follows from Lemma 9.(A).

Most of the assertions in part (B) follow from the fact that G acts regularly on fibers. In particular, elementary consequences are

- (17.a) $j[g]$ exists if and only if $g \in K$,
- (17.b) K is closed under left action by L , and
- (17.c) j induces an injection $L \backslash K \longrightarrow M$.

Suppose $f \in M$. Observe that q_L has basic covering property. Consequently, there is $g : W \longrightarrow W$ such that $q_L \circ g = f \circ q_H$. We may apply Corollary 8 to g , using ζ for the ζ_X and ζ_W of that proposition. Therefore, $g \in G$. It follows that $f = j[g]$. $\square$

Corollary 11. *Let Y be a $*$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object. Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Let G be the group of $\mathbf{HT}[\mathbb{Z}]/Y$ -automorphisms of ζ . Let $\phi : X \longrightarrow Y$ be a separable $\mathbf{HT}[\mathbb{Z}]$ -morphism. Then there is a subgroup $H \subseteq G$ and a function $q : W \longrightarrow X$ such that*

- (18.a) $\phi \circ q = \zeta$, and
- (18.b) q is a quotient map, with respect to $\mathbf{HT}[\mathbb{Z}]$, of equivalence $\text{mod}(H)$.

Moreover, ϕ is Galois if and only if H is a normal subgroup of G . Furthermore, if ϕ is Galois, then the group H is uniquely determined by ϕ .

Proof. Proposition 6 yields existence of a morphism $q : W \longrightarrow X$ such that $\phi \circ q = \zeta$. By assumption, ϕ has the basic covering property, and each fiber is finite. It follows that each point in Y has an open, connected, neighborhood such that both ϕ and ζ are tiered projections over V . From this observation, it follows that q has the basic covering property.

The assumption that ϕ is separable implies that X is $*$ -acceptable. Lemma 9 allows us to characterize q as a quotient, which places it in the framework of Corollary 10. That proposition implies the comments about the Galois property. $\square$

Let Y be a $*$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -object, let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map and let G be the group of $\mathbf{HT}[\mathbb{Z}]/Y$ -automorphisms of ζ . Let $y_0 \in Y$. Our first conclusion is that $\pi_G(Y, y_0)$ is the profinite completion of group G .

We turn to the functorial aspects of universal $\mathbb{Z}$ -covering maps.

Suppose that Y_1 and Y_2 are two $*$ -acceptable $\mathbf{HT}[\mathbb{Z}]$ -objects. Let $\zeta_1 : W_1 \longrightarrow Y_1$ and $\zeta_2 : W_2 \longrightarrow Y_2$ be universal $\mathbb{Z}$ -covering maps. Let G_1 and G_2 be, respectively, automorphism groups of ζ_1 and ζ_2 . Let $f : Y_1 \longrightarrow Y_2$ be an arbitrary $\mathbf{HT}[\mathbb{Z}]$ -morphism. If $\phi : W_1 \longrightarrow W_2$ has the property that $\zeta_2 \circ \phi = f \circ \zeta_1$, then there is a unique group homomorphism $\sigma : G_1 \longrightarrow G_2$ which satisfies

$$(19) \quad \forall g \in G_1, \quad \phi \circ g = \sigma(g) \circ \phi.$$

Existence and uniqueness follow from Proposition 6, after we choose a point. Let $z_1 \in W_1$. Choose $g' = \sigma(g)$ by the conditions

$$(20.a) \quad g' \in G_2,$$

$$(20.b) \quad g'(\phi(z_1)) = \phi(g(z_1)).$$

This g' will satisfy the identity (19).

The problem with this construction is that σ changes with ϕ . To avoid ambiguity, imagine the construction on pointed objects (Y_1, y_1) and (Y_2, y_2) . We restrict "morphism" to functions f with the additional constraint that $f(y_1) = (y_2)$. In selection of ζ_1 and ζ_2 , we also choose points $z_1 \in \zeta_1^{-1}\{y_1\}$ and $z_2 \in \zeta_2^{-1}\{y_2\}$. Then there is a unique choice of ϕ that satisfies the additional condition $\phi(z_1) = z_2$.

In this manner, we construct a functor Γ from $\mathbf{Path}$ to groups. This the Γ in Theorem A.

3. REFINEMENTS

The standing hypothesis for the next discussion is

- (21) Let Y be an $\mathbf{HT}[\mathbb{Z}]$ -object whose underlying topology is pathwise connected and locally simply connected. Let $\zeta : W \longrightarrow Y$ be a universal $\mathbb{Z}$ -covering map. Let G be the group of $\mathbf{HT}[\mathbb{Z}]/Y$ -automorphisms of ζ . Let W_0 be a top-component of W . Let ζ_0 be the restriction of ζ to W_0 . Let G_0 be the group of $\mathbf{Top}/Y$ -automorphisms of ζ_0 . Regard $W_0 \times \mathbb{Z}$ as an $\mathbf{HT}[\mathbb{Z}]$ -object by assigning it the product topology (with the discrete topology on $\mathbb{Z}$) and the action

$$\forall z \in W_0, \forall n, k \in \mathbb{Z}, \quad n \star (z, k) = (z, k + n).$$

Define $\tau : W_0 \times \mathbb{Z} \longrightarrow W$ by $\tau(z, k) = k \star z$.

The axioms on the topology of W easily imply that τ is an $\mathbf{HT}[\mathbb{Z}]$ -isomorphism. Observe that

$$\forall z \in W_0, \forall k \in \mathbb{Z}, \quad \zeta(\tau(z, k)) = k \star \zeta(z).$$

For each $y_0 \in Y$, there is a standard identification of $\pi_1(Y, y_0)$ with G_0 .

Let H be the set of pairs (ν, n) in which $n \in \mathbb{Z}$ and $\nu : W_0 \rightarrow W_0$ is a continuous function such that

$$\forall z \in W_0, \zeta_0(\nu(z)) = (-n) \star \zeta_0(z).$$

Define a binary operation on H by

$$\forall (\nu, n), (\mu, m) \in H, (\nu, n) \cdot (\mu, m) = (\nu \circ \mu, n + m).$$

Then

- (22.a) with respect to this operation, H is a group,
- (22.b) the identity element is $(1_{W_0}, 0)$,
- (22.c) the function $(\nu, n) \mapsto n$ defines a surjective group homomorphism $H \rightarrow \mathbb{Z}$ whose kernel is $G_0 \times \{0\}$.

There is a binary operation

$$H \times W \rightarrow W,$$

characterized by

$$\forall (\nu, n) \in H, z \in W_0, \forall k \in \mathbb{Z}, (\nu, n) \wedge \tau(z, k) = \tau(\nu(z), k + n).$$

Straightforward algebra shows that

- (23.a) this is a continuous left action by the group H ,
- (23.b) the H -stabilizer of any $z \in W$ is trivial, and
- (23.c) for each $h \in H$, the function $z \mapsto h \wedge z$ on W is an **HT** $[\mathbb{Z}]/Y$ -automorphism of ζ .

Let $\psi : H \rightarrow G$ be the induced group homomorphism. Then (23.b) implies that ψ is injective.

Let $y_0 \in Y$. Fix $z_0 \in \zeta_0^{-1}\{y_0\}$. Suppose that $(z_1, k_1) \in (\zeta \circ \tau)^{-1}\{y_0\}$. Then

$$k_1 \star \zeta(z_1) = y_0.$$

The standard theory of covering spaces implies that there is a continuous function $\nu : W_0 \rightarrow W_0$ such that

- (24.a) $\nu(z_0) = z_1$ and
- (24.b) $\forall z \in W_0, \zeta(\nu(z)) = (-k_1) \star \zeta(z)$.

It follows that $(\nu, k_1) \in H$ and this element takes $\tau(z_0, 0)$ to $\tau(z_1, k_1)$.

It follows that ψ is a group isomorphism.

All that remains is to convert our structure theory in the language of Theorem A.

Define $\iota : G_0 \rightarrow H$ by $\zeta \mapsto (\zeta, 0)$ and $j : H \rightarrow \mathbb{Z}$ by $(\nu, n) \mapsto n$. Then

$$1 \rightarrow G_0 \xrightarrow{\iota} H \xrightarrow{j} \mathbb{Z} \rightarrow 1,$$

is a short exact sequence. We have functorial identifications of G_0 with $\pi_1(Y, y_0)$ and of H with $\Gamma(Y, y_0)$. Now, we define the natural transformations $\iota[Y, y_0]$ and $j[Y, y_0]$ to be the ι and j constructed here, filtered through the appropriate isomorphisms. All but one of the assertions of Theorem A has been established.

Put $\epsilon = \epsilon_{Y, y_0}$, as defined in Theorem A. The final detail is to identify $\pi_G(\epsilon)$ with the profinite completion of j

Since ψ is an isomorphism, we can use it and (22.c) to define a canonical group homomorphism $G \rightarrow \mathbb{Z}$. We can characterize this in terms of π_G .

Let $n \in \mathbb{N}$, and let C stand for $\mathbb{Z}/(n)$ as an $\mathbf{HT}[\mathbb{Z}]$ -object. We use overbars to indicate equivalence mod (n) . Now the unique morphism $C \longrightarrow E$ is a Galois morphism; furthermore, every Galois morphism into E is isomorphic to some such C for some $n \in \mathbb{N}$. One choice for the pullback of this morphism along ϵ is the projection $Y \times C \longrightarrow Y$. This projection is Galois.

Now

$$\delta(z, n) = (\zeta(z), \bar{n})$$

represents a canonical factoring $W \longrightarrow (Y \times C)$. Unwinding formulations, it follows that the induced map

$$\pi_G(Y, y_0) \longrightarrow \widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}/(n)$$

corresponds to the profinite completion of

$$(\nu, n) \mapsto n \text{ from } H \longrightarrow \mathbb{Z}$$

composed with the canonical projection $\widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}/(n)$. This completes the proof of Theorem A.

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