A FUNCTIONAL APPROACH TO A TOPOLOGICAL ENTROPY IN BORNOLOGICAL LINEAR SPACES

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ABSTRACT. We extend the classical notion of topological entropy of non-compact sets, in metric spaces, to locally convex bornological spaces. The definition of this invariant is based on the dimensional approach of Pesin for $C-$structures in metric spaces and the limiting procedure of Almeida and Barreira for non-metrizable spaces. We present also a functional definition.

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1. INTRODUCTION

The study of dimensions of sets constitutes a relevant discipline in the area of dynamical systems. For instance, the *Hausdorff dimension* is a suitable tool to measure the complexity of sets in metric spaces. In dynamical systems is specially interesting to investigate the complexity of, for instance, the so called strange attractors. The knowledge of adequate "characteristic dimensions" for this kind of sets has been matter of study by pure and applied mathematicians, as well as physicists. Another (dynamic) invariant associated to subsets of a compact metric space is the *topological entropy.*

A general dimension theory in dynamical systems is due to Pesin. He considers a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a collection of subsets of X. This family satisfies some special conditions and it is called a $C-$*structure* (C after Carathéodory).

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Thus the above mentioned examples are particular cases which are obtained by considering special $C-$structures on metric spaces. For a comprehensive introduction in dimension theory we recommend the book of Pesin [4].

The Hausdorff dimension and the topological entropy definitions consider the metric structure of the space, it could be interesting the extension of these notions to study the complexity of sets and dynamics in non-metrizable spaces. There are important functional spaces which are, in general, non-metrizable, among these we can mention, for instance, certain spaces of holomorphic functions and spaces of tempered distributions. In [1] Almeida and Barreira introduced the concept of Hausdorff dimension in convex bornological spaces by mean of a limiting procedure, with limits running through all the bounded discs. This definition agrees with classical Hausdorff dimension in normed spaces.

In this article we continue with this analysis by presenting an entropy dimension defined from a structure originated by dynamics $f : X \to X$, where X is a locally convex bornological space and such that for a metrizable case our invariant be equivalent to the Bowen entropy for non-compact sets [2]. For this the approach of [1] is closely followed.

We also introduce an entropy with dynamics implemented by operators $\mathcal{D}$ acting on a space of functions $\mathcal{F}$, where $\mathcal{D}(1) = 1$ is not valid in general. This definition will be done by using partitions of the unity. A map $f : X \to X$ induces an operator $\mathcal{D}_f$, defined on a space $\mathcal{F}(X)$ of functions $\varphi : X \to \mathbf{R}$ by $\mathcal{D}_f(\varphi)(x) = \varphi(f(x))$. This functional approach, for non-unital operators, is new even in a metric case. We shall establish an equivalence between the entropy dimension associated to $(f, Z \subset X)$ and that defined for $(\mathcal{D}_f, \mathcal{F}(Z))$. A functional definition with dynamics implemented by positive and unital was presented in [3] One of the main motivations was the study of entropy of foliations to give a functional approach to measure-theoretic entropies earlier defined. Thus, we believe that is justified to extend this definition to non-metrizable functional spaces.

2. Preliminary definitions

We review some basic definitions and facts about bornological spaces. Let X be a set, a *bornology* on X is a covering $\mathcal{B}$ of X, such that $\mathcal{B}$ is closed under finite unions and where if $B \in \mathcal{B}$, $A \subset B$ then $A \in \mathcal{B}$. The members of $\mathcal{B}$ are the *bounded sets.* A *bornological linear space* is a $\mathbf{K}-$vectorial space($\mathbf{K}$ the field of the real or complex numbers) endowed with a bornology $\mathcal{B}$, and such that if $A, B \in \mathcal{B}$, $\lambda \in \mathbf{K}$ then $A + B \in \mathcal{B}$, $\lambda A \in \mathcal{B}$. A disc D is a convex subset of a linear space X, in which holds: $x \in D$, $|\lambda| \leq 1$ then $\lambda x \in D$. For a set $A \subset X$, the *convex hull* of A is $CA = \bigcap_{D \text{ disk} \supset A} D$. A *convex bornological linear space* is a bornological space in which the convex hull of any bounded set is also bounded.

Let D be a disc in a bornological space X, and X_D the subspace generated by D, recall that the *Minkowski semi-norm* in X_D is defined as $p_D(x) = \inf\{t > 0 : x \in tD\}$. If D, $\widetilde{D}$ are discs in X, with $D \subset \widetilde{D}$ then the embedding $i : X_D \hookrightarrow X_{\widetilde{D}}$ is continuous and $C\left(D \cup \widetilde{D}\right)$ is a disc containing D and $\widetilde{D}$.

Any convex bornological space X is the direct inductive limit in the category with objects the pairs (X_D, p_D) and the functors are maps which send bounded sets onto bounded sets.

3. Dimension theory in bornological spaces

Let $f : X \to X$ be a map such that $f\mid_{X_D}$ is continuous in the seminorm p_D for any bounded disc $D \subset X$. We also assume that for every D the set X_D is $f-$invariant and p_D-compact. In each (X_D, p_D) we consider the ball $B^D_{n,\varepsilon}(x) = \{y \in X_D : \max\{p^i_D(x-y)\} < \varepsilon,\ i = 0, 1..., n-1\}$ and the following, at most countable, collections $\mathcal{G}_D = \{B^D_{n_i,\varepsilon}(x_i)\}_{x_i \in X_D}$. Let $Z \subset X$, not necessarily $f-$invariant, we say that a collection $\mathcal{G}_D$ covers $Z \cap X_D$ if $Z \cap X_D \subseteq \bigcup_i B^D_{n_i,\varepsilon}(x_i)$, $x_i \in X_D$. Let

$$m_D(Z, N, \varepsilon, s) := \inf_{\mathcal{G}_D} \left\{ \sum_i \exp(-sn_i) \right\}, \tag{1}$$

where the infimum is taken over all the collections $\mathcal{G}_D$ which cover $Z \cap X_D$ with $n(\mathcal{G}_D) := \min_i\{n_i\} \geq N$. We set

$$m_D(Z, \varepsilon, s) := \lim_{N\to\infty} m_D(Z, N, \varepsilon, s). \tag{2}$$

This limit does exist, because $m_D(Z, N, \varepsilon, s)$ is not decreasing with respect to N, and there is a critical value $\overline{s}$ such that

$$\lim_{\widetilde{D}} \lim_{D} m_{\widetilde{D}}(Z \cap X_D, \varepsilon, s) = +\infty \text{ for } s < \overline{s}$$

and

$$\lim_{\widetilde{D}} \lim_{D} m_{\widetilde{D}}(Z \cap X_D, \varepsilon, s) = 0 \text{ for } s > \overline{s}. \tag{3}$$

The limits are direct inductive limits in the categories mentioned at the end of the precedent section.

We now define

$$\dim(f, Z, \varepsilon) = \overline{s} \tag{4}$$

and

$$\dim(f, Z) = \lim_{\varepsilon\to 0} \dim(f, Z, \varepsilon) \tag{5}$$

An equivalent definition can be done as follows: let $\mathcal{U}_D = \{U_1, ..., U_N\}$ be a finite covering of X_D and define a string as a sequence $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ such that $\{U_{\ell_0}, ..., U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, ..., N\}$. The length of the string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U}_D)$ the set of the strings L with length n from the covering $\mathcal{U}_D$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap ... \cap f^{-n+1}(U_{\ell_{n-1}}), \tag{6}$$

if $Z \subset X$ it says that a set of strings $\mathcal{G}_D$ covers $Z \cap X_D$ if $Z \cap X_D \subset \bigcup_{L\in\mathcal{G}_D} X(L)$.

In this case the *free energy* is defined as:

$$F(\mathcal{G}_D, s) = \sum_{L\in\mathcal{G}_D} \exp(-sn(L)). \tag{7}$$

Let

$$m_D(Z,N,\mathcal{U}_D,s)=\inf_{\mathcal{G}_D} F(\mathcal{G}_D,s), \tag{8}$$

where the infimum is taken over all the collections of strings $\mathcal{G}_D \subset \bigcup_{n\geq N} W_n(\mathcal{U})$ which cover $Z\cap X_D$. Next

$$m_D(Z,\mathcal{U}_D,s)=\lim_{N\to\infty} m_D(Z,N,\mathcal{U}_D,s), \tag{9}$$

and

$$m_D(Z,s)=\lim_{\Delta(\mathcal{U}_D)\to 0} m_D(Z,\mathcal{U}_D,s),$$

$(\Delta(\mathcal{U}_D)$ =diameter of $\mathcal{U}_D)$

There is an unique number $\overline{s}$ such that $\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)$ jumps from $+\infty$ to 0, now let

$$\dim(f,Z) = \overline{s}=\sup\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)=+\infty\right\}= \tag{10}$$

$$\inf\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)=0\right\}. \tag{11}$$

To justify the equivalence between the both two definitions it must be proved that $\lim_{\Delta(\mathcal{U}_D)\to 0} m_D(Z,\mathcal{U}_D,s)=\lim_{\varepsilon\to 0}\dim(f,Z,\varepsilon)$. This can be done in the following way: let $\mathcal{U}$ be a finite covering and δ its Lebesgue number, if $x\in X(L)$ for some $L\in W_n(\mathcal{U})$, then

$B_{n,\delta/2}(x)\subset X(L)\subset B_{n,2\Delta(\mathcal{U})}(x)$. Then the desired equality between the limits is achieved by following the proof of the theorem 11.1 (p. 69) of ??.

Remark: if Z is a bounded set, or more generally if Z is contained in X_D for some disc D is enough to consider $\lim_{\widetilde{D}} m_{\widetilde{D}}(Z,\varepsilon,s)$.

If there is a fundamental systems of discs (D_n), then by setting $X_n:=X_{D_n}$ the definitions can adopt the following form

$$m_n(Z,N,\varepsilon,s):=\inf_{\mathcal{G}_n}\left\{\sum_i \exp(-sn_i)\right\}, \tag{12}$$

where the infimum is taken over all the collections $\mathcal{G}_n$ which cover $Z\cap X_n$ with $n(\mathcal{G}_n):=\min_i\{n_i\}\geq N$. The critical value is found by

$$\lim_{k\to\infty}\lim_{n\to\infty} m_k(Z\cap X_n,\varepsilon,s)=+\infty \text{ for } s<\overline{s}$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty} m_k(Z\cap X_n,\varepsilon,s)=0 \text{ for } s>\overline{s}; \tag{13}$$

and the definitions follows similarly.

If Y is a subspace of X, with the induced bornology, and $Z\subset Y$, we denote the dimension of (f,Z) relative to Y by $\dim(f,Z,Y)$. We have that $\dim(f,Z,Y)=\dim(f,Z,X)=\dim(f,Z)$. The proof is similar to the analogue case for Hausdorff dimension (theor 1. of [1]). If X is a compact metric space and $Y\subset X$, with the induced metric, then , for $Z \subset X$, we write $h_{top}(f, Z, Y)$ for naming the topological entropy of (f, Z) relative to Y. Notice that we can write down, by using the definition of topological entropy from [2]:

$\dim(f, Z) = \lim_{\tilde{D}} \lim_{D} \dim(f, Z \cap X_D, X_{\tilde{D}}) = \lim_{\tilde{D}} \lim_{D} h_{top}(f, Z \cap X_D, X_{\tilde{D}})$.

To see that $\dim(f, Z)$ is really a characteristic dimension, in the sense of the Carathéodory-Pesin theory, it must be proved:

Lemma : *i) if* $Z_1 \subset Z_2$ *then* $\dim(f, Z_1) \leq \dim(f, Z_2)$

ii) $\dim\left(f, \bigcup_i Z\right) = \sup_i \{\dim(f, Z_i)\}$

Proof: i) follows immediately from the definition and from this is proved "$\geq$ " in *ii*), for the other inequality: let $\dim(f, Z_i) < t$, for every i, then for any disc D, $m_D(Z_i, \varepsilon, t) = 0$, for every i and so $m_D(\bigcup Z_i, \varepsilon, t) = 0$. Therefore $\dim\left(f, \bigcup_i Z, \varepsilon\right) \leq t$ and so $\dim\left(f, \bigcup_i Z, \varepsilon\right) \leq \sup_i \{\dim(f, Z_i, \varepsilon)\}$.

The invariance conditions are deduced according to [4]: Let X, Y be two convex bornological spaces and $\chi : X \to Y$. We say χ *preserves structures* if for any discs D_X, D_Y in X, Y respectively, it holds:

i) for any ball $B^{D_X}_{n,\varepsilon}(x)$, there exist $y_1, y_2 \in D_Y$, and positive integers n_1, n_2 such that

$B^{D_Y}_{n_1,\varepsilon}(y_1) \subset \chi\left(B^{D_X}_{n,\varepsilon}(x)\right) \subset B^{D_Y}_{n_2,\varepsilon}(y_2)$.

ii) there is a constant $K \geq 1$, with $K^{-1}e^{-n_1} \leq e^{-n} \leq Ke^{-n_2}$.

If the definitions by coverings is considered, the above condition reads: for any covering $\mathcal{U}_{D_X}$ of X_{D_X} there is a covering $\mathcal{V}_{D_Y}$ of Y_{D_Y} such that

i) for every string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ of $\mathcal{U}_{D_X}$, there are strings L_1, L_2 of $\mathcal{V}_{D_Y}$ with

$X(L_1) \subset \chi(X(L)) \subset X(L_2)$

ii) there is a constant $K \geq 1$, with

$$K^{-1}\exp(-n(L_1)) \leq \frac{1}{n(L)} \leq K\exp(-n(L_2))$$

Now by rewriting and modifying conveniently the theorem 1.3 of [4] we obtain:

Theorem 1: Let $\chi : X \to Y$ be a bijective map between bornological spaces which preserves structures. If $f : X \to X$, $g : Y \to Y$ are maps with $f \circ \chi = \chi \circ g$, then for any $Z \subset Y$ holds $\dim(f, Z) = \dim(g, \chi(Z))$.

Example: We apply the above result to a space considered in [1]: let $X = H(C)$ the germen of holomorphic functions on C (non-empty, compact) $\subset \mathbf{C}$. For any natural n, let B_n be the $\frac{1}{n}$–neighborhood of C, thus we can consider a sequence of spaces $(X_n)_n$ consisting in all bounded holomorphic functions on B_n. Each X_n is a Banach space, with the supremum norm, and $H(C)$ is the inductive limit of the sequence $(X_n)_n$ [1]. Now, for a calculation in this space, we can use the definition given by Eqs. 12-13. For the special case $C = \{0\}$, we can identify $X = H(\{0\})$ with the space $\left\{(a_n)_n : a_n \in \mathbf{R},\ 1/\lim_{n\to\infty} \sup(|a_n|)^{\frac{1}{n}} > 0\right\}$, each sequence $(a_n)_n$ represents the coefficients of the Mac Laurin series of each function $f \in X$. In this case X_n consists of the functions f such that the radius of convergence of its Mac

Laurin series is $> \frac{1}{n}$, or equivalently of sequences which are coefficients of a series with radius of convergence $> \frac{1}{n}$. Let A be a bounded subset of $\mathbf{R}^+$ such that $0 \in \overline{A}$ and $Y = \{f_\varepsilon : \varepsilon \in A\}$. For $\varepsilon > \frac{1}{n}$ let us consider the map $\chi : A_n \to Y_n$ given by $\chi(\varepsilon) = f_\varepsilon(z) := \sum_{n=1}^{\infty} \left(\frac{z}{\varepsilon}\right)^n$, where $A_n = \left\{\varepsilon \in A : \varepsilon > \frac{1}{n}\right\}$, $Y_n = \left\{f_\varepsilon : \varepsilon > \frac{1}{n}\right\}$. The map χ is bi-Lipschitz [1] and from this is immediately checked that it preserves structure: we have

$C_1 |\varepsilon - \overline{\varepsilon}| \leq \|\chi(\varepsilon) - \chi(\overline{\varepsilon})\| \leq C_2 |\varepsilon - \overline{\varepsilon}|$ for some constants C_1, C_2, therefore if $g : A_n \to A_n$ and $T : Y_n \to Y_n$ with $T \circ \chi = \chi \circ g$, then

$C_1 \left|g^i(\varepsilon) - g^i(\overline{\varepsilon})\right| \leq \left\|\chi\left(g^i(\varepsilon)\right) - \chi\left(g^i(\overline{\varepsilon})\right)\right\| \leq C_2 \left|g^i(\varepsilon) - g^i(\overline{\varepsilon})\right|, i = 0, 1, ..., n-1$, so

$C_1 \left|g^i(\varepsilon) - g^i(\overline{\varepsilon})\right| \leq \left\|T^i(\chi(\varepsilon)) - T^i(\chi(\overline{\varepsilon}))\right\| \leq C_2 \left|g^i(\varepsilon) - g^i(\overline{\varepsilon})\right|, i = 0, 1, ..., n-1$, therefore for $\delta > 0$, $\overline{\varepsilon} \in B_{n,\delta}(\varepsilon)$

$$B_{n,\delta/C_2}(\chi(\varepsilon)) \subset \chi(B_{n,\delta}(\varepsilon)) \subset B_{n,\delta/C_1}(\chi(\varepsilon)),$$

and thus it is proved that χ preserves structure. Now by the above theorem

$\dim(g, A_n) = \dim(T, Y_n)$, for any dynamics g, T such that $T \circ \chi = \chi \circ g$, therefore

$$\dim(g, A) = \lim_{n\to\infty} \dim(g, A_n) = \lim_{n\to\infty} \dim(T, Y_n) = \dim(T, Y)$$

We shall see that in seminormed spaces the dimension here defined agrees with the classical Bowen notion of topological entropy for non-compact sets [2]. Now, let X be a seminormed space and B_n the ball with centre in 0 and radius n, so for every n it holds $X_{B_n} \equiv X$. The identity

$i : \left(X_{B_n}, p_{B_n}\right) \to (X, \|\|)$, $p_{B_n}(x) = \frac{\|x\|}{n}$, preserves structures. This is immediately justified from $\|x - y\| = n p_{B_n}(x - y)$. We use that

$\dim(f, Z, X_{B_n}) = h_{top}(f, Z, X_{B_n})$, and therefore for $Z \subset X$

$$\dim(f, Z) = \lim_{n\to\infty} \dim(f, Z, X_{B_n}) = \lim_{n\to\infty} h_{top}(f, Z, X_{B_n}) = \lim_{n\to\infty} h_{top}(f, Z, X),$$

because the invariance property. So that

$$\dim(f, Z) = h_{top}(f, Z).$$

Therefore in the above example we have

$$\dim(T, Y) = h_{top}(g, A)$$

4. A FUNCTIONAL APPROACH

We present here a definition of a dynamical dimension where the transformations are operators on functional spaces. This functional dimension will be expressed in terms of partitions of the unity and the approach to define it is based on [3]. Let $\mathcal{F}$ be a space of functions $\varphi : X \to \mathbf{R}$, where X is a compact metric space, and $\mathcal{D}$ an operator defined on the space of piecewise continuous functions on X. Let $\Phi = \{\varphi_1, ..., \varphi_s\}$ be a partition of the unity by uniformly piecewise continuous in the interiors of the members of a partition of X. Let the string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$, and in this case we set

$$X(L) := \varphi_{\ell_0} \mathcal{D} \varphi_{\ell_1} ... \mathcal{D}^{n-1} \varphi_{\ell_{n-1}}, \tag{14}$$

and let call $W_n(\Phi)$ the set of the strings L with length n from the partition of the unity Φ. A subset Γ of $W_n(\Phi)$ is partition of the unity if $\sum_{L \in \Gamma} X(L) = 1$.

Let $R(L) := \sup_i \left\{ \sup_{x \in X(L)} \varphi_i(x) \right\}$ and $\mathcal{H} \subset \mathcal{F}$, not necessarily $\mathcal{D}$–invariant, we set

$$m(\mathcal{H}, N, \Phi, s) = \inf_{\Gamma_\Phi} \sum_{L \in \Gamma_\Phi} \exp(-sn(L)) R(L), \tag{15}$$

where the infimum is taken over all the strings $\Gamma_\Phi \subset W_n(\Phi), n \geq N$, which form a partition of the unity by functions in $\mathcal{H}$.

Let

$$m(\mathcal{H}, \Phi, s) = \lim_{N \to \infty} m(\mathcal{H}, N, \Phi, s), \tag{16}$$

There is an unique number $\overline{s} = \overline{s}(\Phi)$ such that $m(\mathcal{H}, \Phi, s)$jumps from $+\infty$ to 0, now let

$$\dim(\mathcal{D}, \mathcal{H}, \Phi) = \overline{s} = \sup\{s : m(\mathcal{H}, \Phi, s) = +\infty\} = \tag{17}$$

$$\inf\{s : m(\mathcal{H}, \Phi, s) = 0\}. \tag{18}$$

We similarly obtain a number $\overline{s}_k = \overline{s}_k(\Phi)$, k a positive integer, by considering the sets

$X_k(L) := \varphi_{\ell_0} \mathcal{D}^k \varphi_{\ell_1} ... \mathcal{D}^{k+n-1} \varphi_{\ell_{n-1}}$, i.e. using $\mathcal{D}^k$ instead of $\mathcal{D}$. We can write $\dim(\mathcal{D}^k, \mathcal{H}, \Phi) = \overline{s}_k(\Phi)$. Now $\dim(\mathcal{D}, \mathcal{H})$is defined as the infimum of the numbers d with the following property: for any partition Φ there is a k such that $\overline{s_k}(\Phi) \leq kd$.

It is particularly interesting the following case: let $f : X \to X$ be a continuous map, and X a compact metric space. Let $\mathcal{D}_f$ be the operator defined by $\mathcal{D}_f(\varphi)(x) = \varphi(f(x))$, and let $Z \subset X$, not necessarily f–invariant, we can consider $\dim(\mathcal{D}_f, \mathcal{F}(Z))$, where $\mathcal{F}(Z)$ is a set of functions defined on Z.

For invariant sets and unital operators, i.e. $\mathcal{D}(1) = 1$, the functional definition of entropy introduced in [3] is recovered.

We give now the definition for convex bornological spaces following the approach of the beginning of this section: let X be a convex bornological space and as before X_D the set generated by the bounded disc X_D. If Φ_D is a partition of the unity by functions of $\mathcal{F}(X_D)$, and $W_n^D(\Phi)$ is the set of the strings L with length n from the partition of the unity Φ_D, then we set

$$m_D(\mathcal{F}(Z), N, \Phi_D, s) = \inf_{-\Phi_D} \sum_{L \in -\Phi_D} \exp(-sn(L)) R(L), \tag{19}$$

where the infimum is taken over all the strings $\Gamma_{\Phi_D} \subset W_n^D(\Phi), n \geq N$, which form a partition of the unity by functions in $\mathcal{F}(Z \cap X_D)$.

Let

$$(20)\qquad m_D\left(\mathcal{F}(Z),\Phi,s\right)=\lim_{N\to\infty} m\left(\mathcal{F}(Z),N,\Phi,s\right),$$

We define the number $m_D^k\left(\mathcal{F}(Z),\Phi_D,s\right)$ by considering as above the sets $X_k(L):=\varphi_{\ell_0}\mathcal{D}_{f^k}\varphi_{\ell_1}...\mathcal{D}_{f^{k+n-1}}\varphi_{\ell_{n-1}}$, i.e. using the operators $\mathcal{D}_f^k=\mathcal{D}_{f^k}$. In a similar way as before we define $m_D\left(\mathcal{F}(Z),s\right)$ as the infimum of the numbers h_D with the following property: for any partition Φ_D there is positive integer k, enough large, such that $m_D^k\left(\mathcal{F}(Z),\Phi_D,s\right)\le kh_D$.

. There is an unique number $\overline{s}$ such that $\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}\left(\mathcal{F}(Z\cap X_D),s\right)$ jumps from $+\infty$ to 0, now let

$$(21)\ \dim\left(\mathcal{D}_f,\mathcal{F}(Z)\right)=\overline{s}=\sup\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}\left(\mathcal{F}(Z\cap X_D),s\right)=+\infty\right\}=$$

$$(22)\qquad \inf\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}\left(\mathcal{F}(Z\cap X_D),s\right)=0\right\}.$$

It can be proved, in a similar way as in the non-operational case, the equivalence of these definitions when the functions are defined on semi-normed spaces.

We now establish an equivalence between $\dim(f,Z)$ and $\dim\left(\mathcal{D}_f,\mathcal{F}(Z)\right)$, which is new even in a metric case.

Theorem 2: *Let X be a convex bornological space and $f:X\to X$. Let us assume that for any bounded disc $D\subset X$ the space X_D is $f-$invariant, and each X_D admits a decomposition $\mathcal{P}_D=\{C_{1,...,}C_s\}$ with $intC_i\cap intC_j=\emptyset$, $i\neq j$. If the partitions of the unity are in functions uniformly piecewise continuous in the interiors of the C_i, then for any $Z\subset X$, not necessarily $f-$invariant,* $\dim(f,Z)=\dim\left(\mathcal{D}_f,\mathcal{F}(Z)\right)$.

Proof: We define a new dimension which will serve to link $\dim(f,Z)$ and $\dim\left(\mathcal{D}_f,\mathcal{F}(Z)\right)$. Let $\mathcal{P}_D=\{C_{1,...,}C_s\}$ be a partition with $intC_i\cap intC_j=\emptyset$, $i\neq j$, the strings considered are sequences $L=(\ell_0,\ell_1,...,\ell_{n-1})$ such that $\left\{C_{\ell_0},...,C_{\ell_{n-1}}\right\}\subset\mathcal{P}_D$, $\ell_i\in\{1,2,...,s\}$. Let us call $S_n(\mathcal{P}_D)$ the set of the strings L with length n from the partition $\mathcal{P}_D$. Like in the earlier definitions we set

$$(23)\qquad m_D\left(Z,\mathcal{P}_D(j),s\right),$$

and we say that a collection of strings $\mathcal{S}_D\subset S_n(\mathcal{P}_D)$ is a partition of $Z\cap X_D$ if $Z\cap X_D=\bigcup_{L\in\mathcal{S}_D}X(L)$, with $intX(L_1)\cap intX(L_2)=\emptyset$, $L_1\neq L_2$. The free energy is

$$(24)\qquad F(\mathcal{S}_D,s)=\sum_{L\in\mathcal{S}_D}\exp\left(-sn(L)\right).$$

Let

$$(25)\qquad m_D\left(Z,N,\mathcal{P}_D,s\right)=\inf_{\mathcal{S}_D}F(\mathcal{S}_D,s),$$

where the infimum is taken over all the collections of strings $\mathcal{S}_D\subset\bigcup_{n\ge N}S_n(\mathcal{P}_D)$ which form a partition of $Z\cap X_D$. Thus

$$m_D(Z,\mathcal{P}_D,s)=\lim_{N\to\infty} m_D(Z,N,\mathcal{P}_D,s), \tag{26}$$

for $j=1,2,...$, let $\mathcal{P}_D(j)$ be a the partition obtained from $\mathcal{P}_D$ by successive refinements. The partition $\mathcal{P}_D(j)$ is formed by a collection of sets obtained by splitting the members of $\mathcal{P}_D$ in smaller sets. The numbers of these sets is j^p, for a fixed p. For instance if the member of $\mathcal{P}_D$ are standard cubes or simplices, then p can be taken as the dimension of these standard sets. In this way we have the number

$m_D(Z,\mathcal{P}_D(j),s)=\lim_{N\to\infty} m_D(Z,N,\mathcal{P}_D(j),s)$.

We define the number $m_D^k(Z,\mathcal{P}_D(j),s)$ by considering the sets $X_k(L)=C_{\ell_0}\cap f^{k-1}(C_{\ell_1})\cap...\cap f^{k-n+1}(C_{\ell_{n-1}})$, and $m_D(Z,s)$ is defined as the infimum of the numbers b_D with the following property: for any j there is positive integer k, enough large, such that

$m_D^k(Z,\mathcal{P}_D(j),s)\le kb_D$.

There is an unique number $\overline{s}$ such that $\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)$ jumps from $+\infty$ to 0, now let

$$D(f,Z) = \overline{s}=\sup\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)=+\infty\right\}= \tag{27}$$

$$\inf\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)=0\right\}. \tag{28}$$

We shall prove that $D(f,Z)\le \dim(f,Z)$, firstly notice that, for any string L, each member of $X(L)$ is contained in a neighborhood formed by an union of at most $m-$members of $X(L)$, this origins a covering $\mathcal{U}$. By repeating this argument for each refinement j, we obtain a covering $\mathcal{U}_j$. Thus

$\sum_{L\in\mathcal{S}_D(j)}\exp(-sn(L))\le\sum_{L\in\mathcal{G}_D(j)}\exp(-sn(L))\,m^{n(L)}=$
$\sum_{L\in\mathcal{G}_D(j)}\exp((-s+\log m)\,n(L)),$

here $\mathcal{G}_D(j)$ means the strings for the covering $\mathcal{U}_j$ which cover $Z\cap X_D$. Therefore $m_D(Z,\mathcal{P}_D(j),s)\le m_D(Z,\mathcal{U}_D(j),s^*)$, $s^*=s+\log m$.

If $\overline{s}=\sup\left\{s:\lim_{\widetilde{D}}\lim_{D} m_{\widetilde{D}}(Z\cap X_D,s)=+\infty\right\}$ and $\overline{s^*}=\overline{s}+\log m$, then $m_D\left(Z,\mathcal{U}_D(j),\overline{s^*}\right)=+\infty$. We now obtain

$D(f,Z)=\overline{s}\le\overline{s^*}\le\dim(f,Z)$.

We shall establish next that $\dim(\mathcal{D}_f,\mathcal{F}(Z))\le D(f,Z)$, let $\Phi_D=\{\varphi_1,...,\varphi_s\}$ be a partition of the unity and $\mathcal{A}_D=\{A_\ell\}$ a partition by sets such that $\max_{i,\ell}\{osc_\ell(\varphi_i)\}<\frac{\varepsilon}{s}$, for a given $\varepsilon>0$, where $osc_\ell(\varphi_i):=\max_{A_\ell}\varphi_i-\min_{A_\ell}\varphi_i$. Let $\overline{\varphi}_{i,\ell}:=\dfrac{\sup_{A_\ell}\{\varphi_i I_{A_\ell}\}}{\sum_i\sup_{A_\ell}\{\varphi_i I_{A_\ell}\}}$ (I_A the characteristic function of A), so that $\{\overline{\varphi}_{i,\ell}\}$ is a partition of the unity. We have

$\varphi_i(x)\le(1+\varepsilon)\,\overline{\varphi}_{i,\ell}(x)$ and so $\sup\{\varphi_i\}\le\dfrac{1+\varepsilon}{s}$, for any i. Let $L=(\ell_0,\ell_1,...,\ell_{n-1})$ be a string with $X(L_{\Phi_D}):=\varphi_{\ell_0}\mathcal{D}\varphi_{\ell_1}...\mathcal{D}^{n-1}\varphi_{\ell_{n-1}}$, being a partition of the unity, so

$$R(L):=\sup_i\left\{\sup_{x\in X(L)}\varphi_i(x)\right\}\le\frac{(1+\varepsilon)^{n(L)}}{card\{L_{\Phi_D}:X(L_{\Phi_D})\text{ is a partition of the unity}\}},$$

therefore
$\sum\limits_{L} \exp(-sn(L))\, R(L) \leq$

$\sum\limits_{L} \dfrac{(1+\varepsilon)^{n(L)}}{card\{L_{\Phi_D} : X(L_{\Phi_D}) \text{ is a partition of the unity}\}} \exp(-sn(L))$,
and thus $\dim(\mathcal{D}_f, \mathcal{F}(Z)) \leq D(f, Z)$.
The inequality $\dim(\mathcal{D}_f, \mathcal{F}(Z)) \geq D(f, Z)$ is obtained immediately by considering characteristic functions of the sets $\mathcal{P}_D(j)$. To finish the demonstration of the theorem we must prove that $\dim(f, Z) \leq D(f, Z)$. To do this, let j such that $diam\mathcal{P}_D(j) \leq \alpha$ =Lebesgue number for a given covering $\mathcal{U}$. Therefore, if $\mathcal{S}_D \subset S_n(\mathcal{P}_D)$ is a partition of $Z \cap X_D$ and $\mathcal{G}_D \subset W_n(\mathcal{U}_D)$ covers $Z \cap X_D$, then $\mathcal{S}_D \subset \mathcal{G}_D$. Now the desired inequality is obtained.

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