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Osculating-type curves with Myller configuration in Euclidean 4-space

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Abstract

In this study, we introduce generalized osculating-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. This study especially improves the theory of curves with respect to the osculating-type curves and gives the relationships between the special cases and general form related to the osculating-type curves. We examine some characterizations of the first kind and second kind osculating-type curves. Then, we obtain some correlations between curvatures and invariants of the generalized osculating-type curves, as well. The osculating curves with Frenet frame in E_4 are one of the natural special cases for the generalized osculating-type curves with Myller configuration in E_4 since the geometry of versor fields along a curve with Myller configuration in E_4 is a generalization of the usual theory of curves in E_4. Also, we give some new theorems that are not valid for osculating curves with Frenet frame in E_4. Then, we give a numerical example with respect to the studied concept.

Subject Classification: *53A04.*

Keywords: *Myller configuration, Versor field, Generalized osculating-type curves, Euclidean 4-space.*

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1. Introduction

The geometry of curves is one of the most fundamental and attractive concepts of classical differential geometry, and several researchers have completed studies concerning this framework; also, investigations are ongoing. Lots of special type curves have been studied in literature, such as osculating curves, rectifying curves, and normal curves, which are examined in Euclidean 3-space E_3 and Minkowski space in differential geometry, are also studied in Euclidean 4-space E_4 and some characterizations are given [1-4, 8, 13, 16-20, 46]. In [17, 20], the first kind of osculating curves and the second kind of osculating curves in Euclidean and Minkowski space are introduced by İlarslan and Nešović.

The curve $\mathcal{C}: I \to E_3$ for which the position vector of the curve $\mathcal{C}$ always lie in their rectifying plane, are called rectifying curves [3, 4, 17]. With the same logic, provided that the position vector of the curve $\mathcal{C}$ always lies in its normal plane, the curve is named a normal curve [16, 17, 19], and also provided that the position vector of the curve $\mathcal{C}$ always lies in its osculating plane, the curve is named an osculating curve [17, 20].

In Euclidean 3-space E_3, one describes some versor fields such as; tangent, principal normal, and binormal, as well as some plane fields such as; rectifying, osculating, and normal planes along the curve $\mathcal{C}$. Generally, a versor field[1] is denoted by $(\mathcal{C}, \overline{\zeta})$, and a plane field is denoted by $(\mathcal{C}, \pi)$. The couple $\{(\mathcal{C}, \overline{\zeta}), (\mathcal{C}, \pi)\}$ where $\overline{\zeta} \in \pi$ is named as a *Myller configuration* which is denoted by the notation $\mathcal{M}(\mathcal{C}, \overline{\zeta}, \pi)$ in Euclidean 3-space E_3. In 1960, some studies were done with respect to the Myller configuration by Miron [34]. If the plane π is tangent to the curve $\mathcal{C}$, we get a *tangent Myller configuration* which is denoted by $\mathfrak{M}_t(\mathcal{C}, \overline{\zeta}, \pi)$ [29, 33]. It should be noted that the geometry of tangent Myller configuration $\mathcal{M}_t(\mathcal{C}, \overline{\zeta}, \pi)$ is a special case of the geometry of general Myller configuration $\mathcal{M}(\mathcal{C}, \overline{\zeta}, \pi)$ [33, 34]. The versor field $(\mathcal{C}, \pi)$ can be represented in an invariant frame, which is an orthonormal and positively oriented frame. The moving equations, invariants, and fundamental theorem of the Frenet frame of the versor field are examined in [33]. Miron examined the Frenet-type frame in Myller configuration, and curvatures of this frame are written [33]. One can see that the derivative of the position vector $\overline{r}(s)$ of the curve $\mathcal{C}$ is expressed as a linear combination of the Frenet-type frame elements in Myller configuration. Also, the constructed functions are called invariants of the Frenet-type frame in Myller configuration [7, 33]. The geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a

[1] The notion of the *versor field* is a vector field which is determined along a curve and associated with a presented plane at each point. Also, a Myller configuration is the combination of a curve $\mathcal{C}$, a versor field $\overline{\zeta}$, and a plane π at each point. Each $\overline{\zeta}$ belongs to the corresponding plane π (see [33]).

generalization of the usual theory of curves with Frenet frame in classical Euclidean space[2].

Provided that the curve $\mathcal{C}$ is a curve on the surface $\mathcal{S}$, the geometry of the versor field $(\mathcal{C},\overline{\zeta})$ on the surface $\mathcal{S}$ is the geometry of the associated Myller configurations $\mathcal{M}_t(\mathcal{C},\overline{\zeta},\pi)$ [33]. Then, the Darboux frame and invariants are also defined for a Myller configuration $\mathcal{M}(\mathcal{C},\overline{\zeta},\pi)$ [33]. When literature is scrutinized, a great number of fundamental studies have been completed on this general quite special work-frame related to the geometry of curves and surfaces. The concept of parallelism of the versor field $(\mathcal{C},\overline{\zeta})$ in the plane field $(\mathcal{C},\pi)$ developing a generalization of Levi-Civita parallelism on the curved surfaces was studied by Myller in 1922 (see also [28]). Also, Mayer examined these investigations, which have some new invariants for $\mathfrak{M}(\mathcal{C},\overline{\zeta},\pi)$ [33]. We want to refer to some fundamental studies, which include different spaces and applications with respect to the Myller configuration. Myller configuration in nonholonomic manifolds E_3^2 in Euclidean space E_3 [10, 36, 38], Mark Krien's formula and Gauss-Bonnet formula [39], Riemannian geometry [37, 40-42], symplectic geometry [47], infinite-dimensional Riemannian manifolds [6], Finsler space [5], and Minkowski space [32, 45] were studied, respectively. Additionally, the applications and geometry of versor fields of Myller configuration in hydromechanics in Euclidean space E_3 [11, 14] were examined. Then, in four-dimensional Lorentz space, versor fields along a curve [15] were determined. Also, we can refer to the studies for the fundamental concepts of Myller configuration [10, 27, 33, 35, 43, 44].

Recently, the rectifying-type curves [29], Bertrand curves [31], and some special curves and characterizations [30] with Frenet-type frame in E_3 with Myller configuration were defined by Macsim et al. Since the geometry of versor fields along a curve with Myller configuration in E_3 is a generalization of the usual theory of curves in classical Euclidean space, one can say that the rectifying-type curves with Frenet-type frame in Myller configuration in E_3 are a generalization of rectifying curves with Frenet frame in E_3. Similarly, the Bertrand curves with Frenet-type frame in Myller configuration in E_3 are a generalization of Bertrand curves with Frenet frame in E_3. In [25], rectifying-type curves and rotation minimizing frame are studied by Keskin and Yaylı. Additionally, osculating-type curves with Frenet-type frame in Myller configuration for Euclidean space E_3 [22] and rectifying-type curves in Myller configuration for Euclidean space E_4 [21] are determined by İsbilir and Tosun. Then, İşbilir and Tosun introduce the generalized Smarandache curves with Frenet-type frame in Myller configuration for E_3 [23]. An investigation of the geometry of Myller configurations $\mathcal{M}(\mathcal{C},\overline{\zeta},\pi)$ and $\mathcal{M}_t(\mathcal{C},\overline{\zeta},\pi)$ are examined in detail in [33]. In addition to these, a new type exhaustive frame, which is called the generalized Frenet-type frame in 3-dimensional Lie groups with Myller configurations, includes several special and classical type frames for Euclidean

[2] See generalization [15, 33].

3-space and 3-dimensional Lie groups in [24] is investigated by İşbilir et al. Also, Doğan Yazıcı and Tosun determine the quasi-type frame and quasi-type osculating curves in Myller configuration [7].

At the beginning of this study, we started with the question "What kind of conclusions do we have if we study osculating curves with Frenet-type frame in Myller configuration for Euclidean space E_4?". In order to answer this question, we intend to study this notion. We absolutely believe that this paper contributes to the literature and leads future studies with respect to the theory of curves and Myller configuration. This paper particularly polishes up the theory of curves with Myller configuration and includes a general construction for osculating curves. Further, it gives some relationships between this general concept for osculating curves and special cases like usual osculating curves with Frenet frame.

In this paper, we introduce a new perspective for osculating curves, which can also be named generalized osculating-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. According to the natural properties and construction of the Frenet-type frame in Myller configuration for E_4, we give relations between the osculating curves with Frenet frame in E_4 and generalized osculating-type curves with Frenet-type frame in Myller configuration for E_4. We examine some characterizations of the first kind and second kind osculating-type curves. We obtain some correlations between curvatures and invariants of generalized osculating-type curves. Then, we construct a numerical example for generalized osculating-type curves with Frenet-type frame in Myller configuration for E_4 to improve the studied argument.

2. Frenet-Type Frame with Myller Configuration in Euclidean 4-Space

In this section, we remind some necessary and fundamental concepts and notations, which we use throughout this article, with respect to the Frenet-type frame with Myller configuration in E_4. Heroiu studied the Frenet-type frame with Myller configuration in Lorentz 4-space [15]. By using and inspiring the study [15] and [33], the followings hold for Frenet-type frame with Myller configuration in Euclidean 4-space (see also [21]):

Let $(\mathcal{C},\overline{\zeta})$ be a versor field in E_4 and $\overline{r}(s)$ is a position vector of the curve $\mathcal{C}$ where s is the arc-length on the curve $\mathcal{C}$. One can construct a Frenet-type frame $\mathcal{R}_F=\{P,\overline{\zeta}_1(s),\overline{\zeta}_2(s),\overline{\zeta}_3(s),\overline{\zeta}_4(s)\}$ of the versor field $(\mathcal{C},\overline{\zeta})$. If $\overline{\zeta}_1'(s)\neq 0$, then $\left\langle\overline{\zeta}_1'(s),\overline{\zeta}_1(s)\right\rangle=0$. Hence, we can take $\overline{\zeta}_2(s)$ in the direction of $\overline{\zeta}_1'(s)$. Since $\frac{\overline{\zeta}_1'(s)}{\mathrm{P}\overline{\zeta}_1'(s)\mathrm{P}}=\overline{\zeta}_2(s)$, $\overline{\zeta}_2(s)$ is then the normalized vector field corresponding to $\overline{\zeta}_1'(s)$. Then, we have $\overline{\zeta}_1'(s)=\mathcal{K}_1(s)\overline{\zeta}_2(s)$ where $\mathcal{K}_1(s)=\mathrm{P}\overline{\zeta}_1'(s)\mathrm{P}$. The versor field $\overline{\zeta}_3(s)$ is in the direction of the normal component of $\overline{\zeta}_2'(s)$ with according to plane $\{\overline{\zeta}_1(s),\overline{\zeta}_2(s)\}$. Therefore, $\overline{\zeta}_3(s)$ is the normalized vector field of the normal component and $\overline{\zeta}_4(s)$ is the unique unit vector field perpendicular to the subspace $\{\overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\}$ is 3-dimensional. In that case, Frenet-type frame derivative formulas are given by:

$$\begin{cases} \overline{\zeta}_1'(s) & = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ \overline{\zeta}_2'(s) & = -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ \overline{\zeta}_3'(s) & = -\mathcal{K}_2(s)\overline{\zeta}_2(s) + \mathcal{K}_3(s)\overline{\zeta}_4(s), \\ \overline{\zeta}_4'(s) & = -\mathcal{K}_3(s)\overline{\zeta}_3(s), \end{cases}$$

where $\mathcal{K}_1(s) > 0$, $\mathcal{K}_2(s)$ and $\mathcal{K}_3(s)$ are curvatures. On the other hand, the following can be written:

$$\overline{r}'(s) = \varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s) + \varrho_3(s)\overline{\zeta}_3(s) + \varrho_4(s)\overline{\zeta}_4(s),$$

where $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) + \varrho_4^2(s) = 1$. It is seen that if $\varrho_1(s) = 1$, $\varrho_2(s) = 0$, $\varrho_3(s) = 0$ and $\varrho_4(s) = 0$, we have Frenet equations of a curve in Euclidean space E_4 (cf. [12, 17, 26]). Thanks to the studies [15] and [33], the fundamental theorem of invariants for versor field $(\mathcal{C}, \overline{\zeta})$ is given as follows (see also [21]):

Theorem 2.1: *If the functions $\mathcal{K}_1(s), \mathcal{K}_2(s), \mathcal{K}_3(s), \varrho_1(s), \varrho_2(s), \varrho_3(s), \varrho_4(s)$ with the condition $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) + \varrho_4^2(s) = 1$ are smooth functions for $s \in [a_1, a_2]$, then there exist a curve $\mathcal{C}: [a_1, a_2] \to E_4$ parameterized by arc-length s and a versor field $\overline{\zeta}(s)$. In that case, there exists a versor field $\overline{\zeta}(s)$ tangent to the curve $\mathcal{C}$ such that $\mathcal{K}_1(s), \mathcal{K}_2(s), \mathcal{K}_3(s)$ are curvatures and $\rho_i(s), i = 1, 2, 3, 4$ are the components of the versor field $\overline{\zeta}(s)$ in the Frenet-type frame associated to the $\overline{\zeta}(s)$.*

3. Osculating-Type Curves with Myller Configuration in Euclidean 4-Space

In this section, we introduce the osculating-type curves with Frenet-type frame in Myller configuration for Euclidean space E_4. We examine the osculating-type curves with Myller configuration in two parts: first kind osculating-type curves and second kind osculating-type curves. We give conditions for being osculating-type curves with Myller configuration and some geometric interpretations with respect to them. Then, we obtain some results for the cases of invariants $\varrho_1(s), \varrho_2(s), \varrho_3(s)$, and $\varrho_4(s)$ with the condition $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) + \varrho_4^2(s) = 1$. Also, we show that the osculating curves with classical Frenet frame in E_4 are one of the special cases of osculating-type curves with Frenet-type frame in Myller configuration for E_4.

3.1 *First kind osculating-type curves with Myller configuration*

Definition 3.1: $\overline{r}(s): I \to E_4$ is defined the first kind osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_4 if

$$\overline{r}(s) = \eta(s)\overline{\zeta}_1(s) + \omega(s)\overline{\zeta}_2(s) + \vartheta(s)\overline{\zeta}_3(s), \tag{1}$$

where $\eta(s), \omega(s)$, and $\vartheta(s)$ are smooth functions.

First of all, we give some preliminary preparations before starting the relevant theorems. Differentiating the equation (1), we get:

$$\begin{aligned}
&\varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s) + \varrho_3(s)\overline{\zeta}_3(s) + \varrho_4(s)\overline{\zeta}_4(s)\\
&= \eta'(s)\overline{\zeta}_1(s) + \eta(s)\mathcal{K}_1(s)\overline{\zeta}_2(s) + \omega'(s)\overline{\zeta}_2(s) + \vartheta'(s)\overline{\zeta}_3(s)\\
&+\omega(s)\left(-\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s)\right) + \vartheta(s)\left(-\mathcal{K}_2(s)\overline{\zeta}_2(s) + \mathcal{K}_3(s)\overline{\zeta}_4(s)\right).
\end{aligned}$$

Then, we have:

$$\begin{cases}
\eta'(s) - \omega(s)\mathcal{K}_1(s) &= \varrho_1(s),\\
\omega'(s) + \eta(s)\mathcal{K}_1(s) - \vartheta(s)\mathcal{K}_2(s) &= \varrho_2(s),\\
\vartheta'(s) + \omega(s)\mathcal{K}_2(s) &= \varrho_3(s),\\
\vartheta(s)\mathcal{K}_3(s) &= \varrho_4(s).
\end{cases} \tag{2}$$

We should investigate the solutions of the equation (2) according to the cases $\varrho_4(s) = 0$ or $\varrho_4(s) \neq 0$.

Theorem 3.2: *Let $\overline{r}(s): I \to E_4$ is a curve with Frenet-type frame in Myller configuration for Euclidean space E_4 with $\rho_4(s) = 0$. Then, the curve $\overline{r}(s)$ is a first kind osculating-type curve if and only if $\mathcal{K}_3(s) = 0$ for every s.*

Proof: Assume that $\overline{r}(s)$ is a first kind osculating-type curve with Myller configuration $\rho_4(s) = 0$. By according to the equation (2), we have $\mathcal{K}_3(s) = 0$. Conversely, let us assume that $\mathcal{K}_3(s) = 0$. The position vector of $\overline{r}(s)$ is written by

$$\begin{aligned}
\overline{r}(s) = \ &\langle\overline{r}(s), \overline{\zeta}_1(s)\rangle\overline{\zeta}_1(s) + \langle\overline{r}(s), \overline{\zeta}_2(s)\rangle\overline{\zeta}_2(s) + \langle\overline{r}(s), \overline{\zeta}_3(s)\rangle\overline{\zeta}_3(s)\\
&+\langle\overline{r}(s), \overline{\zeta}_4(s)\rangle\overline{\zeta}_4(s).
\end{aligned}$$

Since $\mathcal{K}_3(s) = 0$, according to Frenet-type frame, we get $\overline{\zeta}_4(s)$ is a constant vector. Since $\rho_4(s) = 0$, we have $\langle\overline{r}(s), \overline{\zeta}_4(s)\rangle$=constant. We conclude that $\overline{r}(s)$ is a first kind osculating-type curve.

Theorem 3.3: *Let $\overline{r}(s): I \to E_4$ is a curve with Frenet-type frame in Myller configuration for Euclidean space E_4 with $\varrho_4(s) \neq 0$. Then, the curve $\overline{r}(s)$ is a first kind osculating-type curve if and only if*

$$\begin{aligned}
\overline{r}(s) = &\begin{pmatrix}
\frac{1}{\mathcal{K}_1(s)}\left(\frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)' + \frac{\varrho_4(s)\mathcal{K}_2(s)}{\mathcal{K}_1(s)\mathcal{K}_3(s)}\\
-\frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)' + \frac{\varrho_2(s)}{\mathcal{K}_1(s)}
\end{pmatrix}\overline{\zeta}_1(s)\\
&+\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)} - \frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)\overline{\zeta}_2(s) + \frac{\varrho_4(s)}{\mathcal{K}_3(s)}\overline{\zeta}_3(s).
\end{aligned} \tag{3}$$

Proof: Let us assume that $\overline{r}(s)$ is a first kind osculating-type curve with $\varrho_4(s) \neq 0$. Then, we have $\mathcal{K}_3(s) \neq 0$. From the equation (2), we have:

$$\begin{cases} \vartheta(s) &= \frac{\varrho_4(s)}{\mathcal{K}_3(s)}, \\ \omega(s) &= \frac{\varrho_3(s)}{\mathcal{K}_2(s)} - \frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)', \\ \eta(s) &= \frac{1}{\mathcal{K}_1(s)}\left(\frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)' + \frac{\varrho_4(s)\mathcal{K}_2(s)}{\mathcal{K}_1(s)\mathcal{K}_3(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)' + \frac{\rho_2(s)}{\mathcal{K}_1(s)}. \end{cases}$$

Therefore, if the equation (1) is considered, what is desired is shown. Conversely, suppose that (3) is provided when $\varrho_4(s) \neq 0$. Then, we get:

$$\begin{cases} \langle \overline{r}(s), \overline{\zeta}_2(s) \rangle &= \frac{\varrho_3(s)}{\mathcal{K}_2(s)} - \frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)', \\ \langle \overline{r}(s), \overline{\zeta}_3(s) \rangle &= \frac{\varrho_4(s)}{\mathcal{K}_3(s)}. \end{cases} \tag{4}$$

By differentiating of the last equation of the equation (4) and by using equation $\langle \overline{r}(s), \overline{\zeta}_2(s) \rangle$, we have:

$$\mathcal{K}_3(s)\langle \overline{r}(s), \overline{\zeta}_4(s) \rangle = 0.$$

Since $\mathcal{K}_3(s) \neq 0$, we get $\langle \overline{r}(s), \overline{\zeta}_4(s) \rangle = 0$. Therefore, $\overline{r}(s)$ is a first kind osculating-type curve.

Corollary 3.4: *Theorem 3.2 is a classification of the first kind of osculating curves in 4-dimensional Euclidean space when* $\varrho_1(s) = 1, \varrho_2(s) = \varrho_3(s) = \varrho_4(s) = 0$ *(cf. [17] and also [20]).*

Corollary 3.5: *Since* $\varrho_4(s) \neq 0$ *for Theorem 3.3, the results obtained are specific for osculating-type curves with Myller configurations.*

Theorem 3.6: *Let* $\overline{r}(s): I \to E_4$ *is a curve with Frenet-type frame in Myller configuration for Euclidean space* E_4 *with* $\varrho_4(s) \neq 0$. *Then,* $\overline{r}(s)$ *is a first kind osculating-type curve if and only if the curvatures* $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, $\mathcal{K}_3(s)$ *and the functions* $\varrho_1(s), \varrho_2(s), \varrho_3(s), \varrho_4(s)$ *satisfy the following relation:*

$$\begin{aligned} &\left(\frac{1}{\mathcal{K}_1(s)}\left(\frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)' + \frac{\varrho_4(s)\mathcal{K}_2(s)}{\mathcal{K}_1(s)\mathcal{K}_3(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)' + \frac{\rho_2(s)}{\mathcal{K}_1(s)}\right)' \\ &- \frac{\varrho_3(s)\mathcal{K}_1(s)}{\mathcal{K}_2(s)} + \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)' - \rho_1(s) = 0. \end{aligned} \tag{5}$$

Proof: Suppose that $\overline{r}(s)$ is a first kind osculating-type curve with $\varrho_4(s) \neq 0$. In the first equality of equation (2), by using $\eta(s)$ and $\omega(s)$, we have the equation (5). Conversely, assume that $\overline{r}(s)$ is a curve satisfying the relation (5). Then, we have:

$$\begin{aligned} &\frac{d}{ds}\left[\begin{array}{l} \overline{r}(s) - \left(\frac{1}{\mathcal{K}_1}\left(\frac{1}{\mathcal{K}_2}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)' + \frac{\varrho_4(s)\mathcal{K}_2(s)}{\mathcal{K}_1(s)\mathcal{K}_3(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)' + \frac{\varrho_2(s)}{\mathcal{K}_1(s)}\right)\overline{\zeta}_1(s) \\ - \left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)} - \frac{1}{\mathcal{K}_2(s)}\left(\frac{\varrho_4(s)}{\mathcal{K}_3(s)}\right)'\right)\overline{\zeta}_2(s) - \frac{\varrho_4(s)}{\mathcal{K}_3(s)}\overline{\zeta}_3(s) \end{array}\right] \\ &= 0. \end{aligned}$$

Therefore, $\overline{r}(s)$ is a first kind osculating-type curve.

3.2 *Second kind osculating-type curves*

Definition 3.7: $\overline{r}(s): I \to E_4$ is defined second kind osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_4 if

$$\overline{r}(s) = \eta(s)\overline{\zeta}_1(s) + \omega(s)\overline{\zeta}_2(s) + \vartheta(s)\overline{\zeta}_4(s), \tag{6}$$

where $\eta(s), \omega(s)$, and $\vartheta(s)$ are smooth functions.

Theorem 3.8: *Let $\overline{r}(s): I \to E_4$ is a curve with Frenet-type frame in Myller cnfiguration for Euclidean space E_4 with non-zero curvatures $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, and $\mathcal{K}_3(s)$. Then, $\overline{r}(s)$ is a second kind osculating-type curve if and only if*

$$\begin{aligned}\overline{r}(s) = &\left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right)\right)' - \left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)\overline{\zeta}_1(s) \\ &+ \left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right) + \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)\overline{\zeta}_2(s) + \left(\int \varrho_4(s)ds\right)\overline{\zeta}_4(s).\end{aligned} \tag{7}$$

Proof: Suppose that $\overline{r}(s)$ is a second kind osculating-type curve. Then, we have the equation (6). By differentiating the equation (6), we get:

$$\begin{aligned}&\varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s) + \varrho_3(s)\overline{\zeta}_3(s) + \varrho_4(s)\overline{\zeta}_4(s) \\ &= \eta'(s)\overline{\zeta}_1(s) + \eta(s)\mathcal{K}_1(s)\overline{\zeta}_2(s) + \omega(s)\left(-\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s)\right) \\ &\quad + \vartheta'(s)\overline{\zeta}_4(s) + \vartheta(s)\left(-\mathcal{K}_3(s)\overline{\zeta}_3(s)\right) + \omega'(s)\overline{\zeta}_2(s).\end{aligned}$$

Then, we have:

$$\begin{cases} \eta'(s) - \omega(s)\mathcal{K}_1(s) &= \varrho_1(s), \\ \omega'(s) + \eta(s)\mathcal{K}_1(s) &= \varrho_2(s), \\ \omega(s)\mathcal{K}_2(s) - \vartheta(s)\mathcal{K}_3(s) &= \varrho_3(s), \\ \vartheta'(s) &= \varrho_4(s). \end{cases} \tag{8}$$

From the equation (8), we obtain:

$$\begin{cases} \vartheta(s) &= \int \varrho_4(s)ds, \\ \omega(s) &= \frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right) + \frac{\varrho_3(s)}{\mathcal{K}_2(s)}, \\ \eta(s) &= \frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right)\right)' - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'. \end{cases}$$

Therefore, if the equation (6) is considered, what is desired is shown. Conversely, suppose that the equation (7) is provided. Then, we have:

$$\left\langle \overline{r}(s), \overline{\zeta}_4(s) \right\rangle = \int \varrho_4(s)ds. \tag{9}$$

By differentiating the equation (9), we can write:

$$\mathcal{K}_3(s)\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = 0.$$

Since $\mathcal{K}_3(s) \neq 0$, we get $\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = 0$. Hence, $\overline{r}(s)$ is a second kind osculating-type curve.

Theorem 3.9: *Let $\overline{r}(s): I \to E_4$ is a curve with Frenet-type frame in Myller configuration for Euclidean space E_4 with non-zero curvatures $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$ and $\mathcal{K}_3(s)$. Then, $\overline{r}(s)$ is a second kind osculating-type curve if and only if the curvatures $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, $\mathcal{K}_3(s)$ and the functions $\varrho_1(s), \varrho_2(s), \varrho_3(s), \varrho_4(s)$ satisfy the following relation*

$$\begin{aligned}&\left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)}\right)' - \left(\frac{1}{\mathcal{K}_1(s)}\left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right)\right)'\right)' - \left(\frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)' \\ &- \frac{\mathcal{K}_1(s)\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right) - \frac{\varrho_3(s)\mathcal{K}_1(s)}{\mathcal{K}_2(s)} - \varrho_1(s) = 0.\end{aligned} \tag{10}$$

Proof: Assume that $\overline{r}(s)$ is a second kind osculating-type curve. In the first equation of the equation (8), by using $\eta(s)$ and $\omega(s)$, we have the equation (10). Conversely, assume that $\overline{r}(s)$ is a curve satisfying the relation given in the equation (10). Then, we have:

$$\frac{d}{ds}\left[\begin{aligned}&\overline{r}(s) - \left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right)\right)' - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)\overline{\zeta}_1(s) \\ &- \left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\left(\int \varrho_4(s)ds\right) + \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)\overline{\zeta}_2(s) - \left(\int \varrho_4(s)ds\right)\overline{\zeta}_4(s)\end{aligned}\right]$$
$$= 0.$$

Hence, $\overline{r}(s)$ is a second kind osculating-type curve.

Special Case 3.10: Let $\overline{r}(s): I \to E_4$ be a second kind osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_4 with non-zero curvatures $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, and $\mathcal{K}_3(s)$. If $\rho_1(s) = 1$ and $\varrho_2(s) = \varrho_3(s) = \varrho_4(s) = 0$, by according to the equation (10), we get:

$$\left(\frac{1}{\mathcal{K}_1(s)}\left(\frac{\mathcal{K}_3(s)}{\mathcal{K}_2(s)}\right)'\right)' + \frac{\mathcal{K}_1(s)\mathcal{K}_3(s)}{\mathcal{K}_2(s)} = -\frac{1}{c}, \tag{11}$$

where c is a constant. Equation (11) is a characterization of second-type osculating curve in Euclidean space with Frenet frame [17].

Now, we give a numerical example in order to support this notion. By inspiring the study [9], we construct the following example for the Frenet-type frame in Myller configuration for E_4.

Example 3.11: Consider the following elements of the Frenet-type frame

$$\begin{cases}\overline{\zeta}_1(s) &= \left(\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos(s), -\frac{1}{\sqrt{2}}\sin(s)\right), \\ \overline{\zeta}_2(s) &= \left(-\frac{1}{\sqrt{3}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{3}}\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{2}{\sqrt{6}}\sin(s), -\frac{2}{\sqrt{6}}\cos(s)\right), \\ \overline{\zeta}_3(s) &= \left(\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\sin(s)\right), \\ \overline{\zeta}_4(s) &= \left(-\frac{2}{\sqrt{6}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{2}{\sqrt{6}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{3}}\sin(s), \frac{1}{\sqrt{3}}\cos(s)\right).\end{cases}$$

Let us choose $\varrho_1(s) = \frac{1}{\sqrt{2}}, \varrho_2(s) = \frac{1}{\sqrt{2}}, \varrho_3(s) = 0, \varrho_4(s) = 0$. Then, we obtain the following curve:

$$\bar{r}(s): I \to E_4$$

$$s \mapsto \bar{r}(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{s}{\sqrt{2}}\right), \\ \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{2}\sin(s) + \frac{1}{\sqrt{3}}\cos(s), \\ \frac{1}{2}\cos(s) - \frac{1}{\sqrt{3}}\sin(s) \end{pmatrix}$$

Also, we can see that

$$\bar{r}(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{s}{\sqrt{2}}\right), \\ \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{2}\sin(s) + \frac{1}{\sqrt{3}}\cos(s), \\ \frac{1}{2}\cos(s) - \frac{1}{\sqrt{3}}\sin(s) \end{pmatrix}$$

$$= \eta(s)\bar{\zeta}_1(s) + \omega(s)\bar{\zeta}_2(s) + \vartheta(s)\bar{\zeta}_4(s),$$

where $\eta(s) = \frac{\sqrt{2}}{\sqrt{3}}, \omega(s) = -\frac{\sqrt{6}}{3}$ and $\vartheta(s) = -\frac{\sqrt{3}}{6}$. We see that the curve $\bar{r}(s)$ is a second kind osculating-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4.

4. Conclusion

In this paper, we investigated the generalized osculating-type curves with Frenet-type frame in Myller configuration Euclidean space E_4. This study enhanced the theory of curves with Myller configuration related to the osculating-type curves and also presented the relationships among the special cases and general form with respect to the osculating-type curves. We introduced some characterizations of the first kind and second kind osculating-type curves. Also, we obtained some correlations between curvatures and invariants of generalized osculating-type curves. Because of the fact that the geometry of versor fields along a curve with Myller configuration in E_4 is a generalization of the usual theory of curves in E_4, one can say that the osculating curves with Frenet frame in E_4 are one of the natural special cases for generalized osculating-type curves. Additionally, we present new theorems that are not valid for osculating curves in E_4, as well. Then, we give a numerical example with respect to the osculating-type curves with Frenet-type frame in Myller configuration for E_4. We absolutely believe that this study contributes to the literature and leads future studies related to the theory of curves of Myller configuration.

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References

[1] M. Akyiğit and Ö. G. Yıldız, "On the framed normal curves in Euclidean 4-space," *Fundamental Journal of Mathematics and Applications,* vol. 4, no. 4, pp. 258–263 (2021), doi: 10.33401/fujma.992917.

[2] Ö. Bektaş, N. Gürses, and S. Yüce, "Quaternionic osculating curves in Euclidean and semi-Euclidean space," *Journal of Dynamical Systems and Geometric Theories,* vol. 14, no. 1, pp. 65–84 (2016), doi: 10.1080/1726037X.2016.1177935.

[3] B.-Y. Chen, "When does the position vector of a space curve always lie in its rectifying plane?," *The American Mathematical Monthly*, vol. 110, no. 2, pp. 147–152 (2003), doi: 10.1080/00029890.2003.11919949.

[4] B.-Y. Chen and F. Dillen, "Rectifying curves as centrodes and extremal curves," *Bulletin of the Institute of Mathematics*, Academia Sinica, vol. 33, no. 2, pp. 77–90 (2005).

[5] O. Constantinescu, "Myller configurations in Finsler spaces," *Differential Geometry–Dynamical Systems,* vol. 7, pp. 69–76 (2005).

[6] M. Craioveanu, Topologia si geometria varietăților Banach (in Romanian), Ph.D. dissertation, Iași, Romania (1968).

[7] B. Doğan Yazıcı and M. Tosun, "Quasi-type frame and quasi-type osculating curves in Myller configuration," *Filomat,* vol. 38, no. 8, pp. 2813–2820 (2024), doi: 10.2298/FIL2408813D.

[8] B. Doğan Yazıcı, S. Özkaldı Karakuş, and M. Tosun, "Characterizations of framed curves in four-dimensional Euclidean space," *Universal Journal of Mathematics and Applications,* vol. 4, no. 4, pp. 125–131 (2021), doi: 10.32323/ujma.1008148.

[9] A. Duyan, "Associated Curves According to Bishop Frame in 4-Dimensional Euclidean Space," M.S. thesis, Fırat University, Elazığ, Türkiye (2020).

[10] Gh. Gheorghiev, "On Dini nets," Academia Republicii Populare Române. Buletin Științific, Seria Matematică, Fizică și Chimie, vol. 2, pp. 17–20 (1950) (in Romanian).

[11] Gh. Gheorghiev, "Câteva probleme geometrice legate de un câmp de vectori unitari," Buletinul Științific al Academiei Republicii Populare Române, vol. VI (1955) (in Romanian).

[12] H. Gluck, "Higher curvatures of curves in Euclidean space," *The American Mathematical Monthly,* vol. 73, no. 7, pp. 699–704 (1966), doi: 10.2307/2313974.

[13] F. Gökçelik, Z. Bozkurt, İ. Gök, F. N. Ekmekçi, and Y. Yaylı, "Parallel transport frame in 4-dimensional Euclidean space," *Caspian Journal of Mathematical Sciences,* vol. 3, no. 1, pp. 91–103 (2014).

[14] A. Haimovici, Grupuri de transformări: Introducere elementară. București, Romania: Editura Didactică și Pedagogică (1963).

[15] B. Heroiu, "Versor fields along a curve in a four-dimensional Lorentz space," *Journal of Advanced Mathematical Studies*, vol. 4, no. 1, pp. 49–57 (2011).

[16] K. İlarslan, "Spacelike normal curves in Minkowski space $\mathbb{E}_1^3$," *Turkish Journal of Mathematics,* vol. 29, no. 2, pp. 53–63 (2005).

[17] K. İlarslan and E. Nešović, "Some characterizations of osculating curves in the Euclidean spaces," *Demonstratio Mathematica*, vol. 41, no. 4, pp. 931–940 (2008), doi: 10.1515/dema-2008-0421.

[18] K. İlarslan and E. Nešović, "Some characterizations of rectifying curves in Euclidean 4-spaces E^4," *Turkish Journal of Mathematics*, vol. 32, no. 1, pp. 21–30 (2008).

[19] K. İlarslan and E. Nešović, "Spacelike and timelike normal curves in Minkowski space-time," Publications de l'Institut Mathématique, vol. 85, no. 99, pp. 111–118 (2009), doi: 10.2298/PIM0999111I.

[20] K. İlarslan and E. Nešović, "The first kind and the second kind osculating curves in Minkowski space-time," *Comptes Rendus de l'Académie Bulgare des Sciences,* vol. 62, no. 6, pp. 677–686 (2009).

[21] Z. İşbilir and M. Tosun, "A new insight on rectifying-type curves in Euclidean 4-space," *International Electronic Journal of Geometry*, vol. 16, no. 2, pp. 644–652 (2023), doi: 10.36890/iejg.1291893.

[22] Z. İşbilir and M. Tosun, "On generalized osculating-type curves in Myller configuration," *Analele Universității Ovidius Constanța – Seria Matematică,* vol. 32, no. 2, 13 pages (2024), doi: 10.2478/auom-2024-0020.

[23] Z. İşbilir and M. Tosun, "Generalized Smarandache curves with Frenet-type frame," *Honam Mathematical Journal*, vol. 46, no. 2, pp. 181–197 (2024), doi: 10.5831/HMJ.2024.46.2.181.

[24] Z. İşbilir, B. Doğan Yazıcı, and M. Tosun, "On special curves in Lie groups with Myller configuration," *Mathematical Methods in the Applied Sciences,* vol. 47, no. 14, pp. 11693–11708 (2024), doi: 10.1002/mma.10149.

[25] Ö. Keskin and Y. Yaylı, "Rectifying-type curves and rotation minimizing frame R^n," arXiv preprint (2019). doi: 10.48550/arXiv.1905.04540.

[26] W. Kühnel, Differential Geometry: Curves–Surfaces–Manifolds. Braunschweig/Wiesbaden, Germany: Vieweg Verlag (1999).

[27] T. Levi-Civita, "Rendiconti del Circolo di Palermo," vol. 42, p. 173 (1917).

[28] T. Levi-Civita, Lezioni di Calcolo Differenziale Assoluto. Bologna, Italy: Zanichelli (1925).

[29] G. Macsim, A. Mihai, and A. Olteanu, "On rectifying-type curves in a Myller configuration," *Bulletin of the Korean Mathematical Society*, vol. 56, no. 2, pp. 383–390 (2019), doi: 10.4134/BKMS.b180279.

[30] G. Macsim, A. Mihai, and A. Olteanu, "Curves in a Myller configuration," in Proc. Int. Conf. Applied and Pure Mathematics (ICAPM), Iași, Romania, Institute of Mathematics, Romanian Academy (2017).

[31] G. Macsim, A. Mihai, and A. Olteanu, "Special curves in a Myller configuration," in Proc. 16th Workshop on Mathematics, Computer Science and Technical Education, vol. 2 (2019).

[32] N. Mihăileanu, "Configurații Myller în spații Minkowski," Analele Universității București, Matematică–Mecanică, vol. 16, pp. 117–119 (1967).

[33] R. Miron, The Geometry of Myller Configurations: Applications to Theory of Surfaces and Nonholonomic Manifolds. Bucharest, Romania: Romanian Academy (2010).

[34] R. Miron, "Geometria unor configurații Myller," Analele Științifice ale Universității, vol. 6, no. 3 (1960).

[35] R. Miron, "Myller configurations and Vrănceanu nonholonomic manifolds," *Scientific Studies and Research,* vol. 21, no. 1 (2011).

[36] R. Miron, "Asupra sferei neolonome și planului neolonom," Analele Științifice ale Universității 'Al. I. Cuza' din Iași, Seria Nouă, Secțiunea I, vol. 1, nos. 1–2, pp. 43–53 (1955).

[37] R. Miron, "Configurații Myller M(C, ζi_1, Tn−1) în spații Riemann Vn. Aplicații la studiul hipersuprafețelor din Vn," Studii și Cercetări Științifice Matematice Iași, vol. 12, no. 1 (1962).

[38] R. Miron, "Despre reducerea la forma canonică a grupului intrinsec al unui spațiu neolonom," *Buletin Științific, Secția de Științe Matematice și Fizice,* vol. 8, no. 3, pp. 631–645 (1956).

[39] R. Miron, "Despre torsiunea totală a unei suprafețe," *Gazeta Matematică, Seria A*, vol. 8 (1955).

[40] R. Miron, “Configurații Myller M(C, ζi_1, Tm) în spații Riemann Vn. Aplicații la studiul varietăților Vm din Vn,” Studii și Cercetări Științifice Matematice Iași, vol. 13, no. 1 (1962).

[41] R. Miron, “Les configurations de Myller M(C, ζi_1, Tn−1) dans les espaces de Riemann Vn (I),” Tensor, vol. 12, no. 3, Japan.

[42] R. Miron, “Les configurations de Myller M(C, ζi_1, Tm) dans les espaces de Riemann Vn,” Tensor, vol. 5, no. 1 (1964).

[43] R. Miron and I. Pop, Topologie Algebrică: Omologie, Omotopie, Spații de Acoperire. București, Romania: Editura Academiei (1974).

[44] R. Miron and D. Brânzei, Backgrounds of Arithmetic and Geometry, Series on Pure Mathematics, vol. 23. Singapore: World Scientific Publishing (1995).

[45] R. Miron, Geometria Configurațiilor Myller. București, Romania: Editura Tehnică (1966).

[46] M. N. Şavlı, “4-Boyutlu Öklid Uzayında Çatlandırılmış Oskülatör ve Rektifiyan Eğriler,” M.S. thesis, Bilecik Şeyh Edebali University, Bilecik, Türkiye (2022).

[47] I. Vaisman, Symplectic Geometry and Secondary Characteristic Classes, Progress in Mathematics, vol. 72. Basel, Switzerland: Birkhäuser Verlag (1994).

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