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$\oplus$–irreducible elements of almost distributive lattices

Monikarchana Yathaluru [‡]
Department of Mathematics
Mohan Babu University
Tirupati 517102
Andhra Pradesh
India

Siva Mala Munnangi [†]
Rafi Noorbhasha [§]
Department of Mathematics
Siddhartha Academy of Higher Education
(Deemed to be University)
Vijayawada 520007
Andhra Pradesh
India

Vasu Babu Rajulapati *
Department of Mathematics
Shri Vishnu Engineering College for Women (A)
Bhimavaram 534202
Andhra Pradesh
India

[‡] *ORCID-ID:* **0000-0003-2146-4800**
[†] *ORCID-ID:* **0000-0003-2254-4650**
[§] *ORCID-ID:* **0000-0003-2027-0533**
* *ORCID-ID:* **0000-0002-0443-1276**

[‡] *E-mail:* **285archana@gmail.com**
[†] *E-mail:* **sivamala@vrsiddhartha.ac.in**
[§] *E-mail:* **rafimaths@gmail.com**
* *E-mail:* **vasukgr@svecw.edu.in (Corresponding Author)**

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Sulochana Nagam [@]
Department of Mathematics
Government College for Men (A)
Kadapa 516004
Andhra Pradesh
India

T. Satish [#]
Department of Engineering Mathematics and Humanities
SRKR Engineering College (A)
Bhimavaram 534204
Andhra Pradesh
India

Abstract

This study analyzes ADL frameworks by examining $\oplus$-divisibility alongside the distinct roles of $\oplus$-prime and $\oplus$-irreducible elements. A novel congruence relation is introduced via multiplier filters, leading to necessary and sufficient conditions under which the resulting quotient ADL assumes the structure of a Boolean algebra. Furthermore, the study provides a characterization of $\oplus-$prime and $\oplus-$irreducible elements through their intrinsic connection to multiplier filters. These theoretical developments have significant interdisciplinary potential, particularly in engineering fields such as digital circuit design, fault-tolerant computing, data encryption, and decision logic systems. The formalism and properties of $\oplus-$irreducible elements offer robust mathematical tools for enhancing the design and analysis of complex systems where logical structure, redundancy minimization, and computational integrity are paramount.

Subject Classification: *06D05, 06B10, 06B23.*

Keywords: *Almost distributive lattice (ADL), Dual annihilator, Congruence, $\oplus-$irreducible element.*

The graphical abstract, as shown in Figure 1, effectively bridges **pure mathematical theory with practical engineering relevance**, making it a valuable visual tool for researchers, engineers, and academics alike. It encapsulates the journey from algebraic structure to real-world application with clarity and precision.

[@] *ORCID-ID:* **0009-0009-4181-2017**

[@] *E-mail:* **sulochananagam@gmail.com**
[#] *E-mail:* **tsatishmaths@gmail.com**

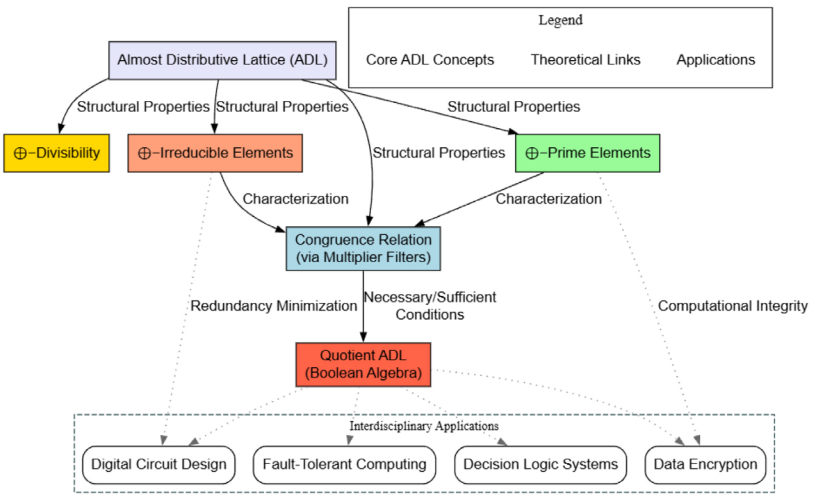


Figure 1

Graphical Abstract

1. Introduction

In 1981, Swamy and Rao [1] introduced the Almost Distributive Lattice to bridge the gap between these systems and standard distributive lattices. They proved that while an ADL itself might lack certain properties, the collection of its principal ideals behaves perfectly as a distributive lattice. Several researchers [2-4] studied the properties of annihilator ideals, micro −filters, and divisibility. D ivisibility and its related properties are introduced by [5] and [6]. Building on this, we present the ideas of ⊕−divisibility, ⊕−prime, and ⊕−irreducible elements within ADLs, and analyze their significant characteristics. We define a congruence relation θ on an ADL and derive equivalent conditions under which the quotient structure $\mathcal{R}/\theta$ becomes a Boolean algebra. Furthermore, we provide a characterization of ⊕−prime and ⊕−irreducible elements using prime and maximal filters. Finally, we establish that every ⊕−irreducible element in an ADL is also ⊕−prime.

The theory of ADLs initiated by Swamy and Rao [1] established a foundational link between ADLs and classical distributive lattices through the structure of principal ideals. Subsequent advancements by researchers [2, 3] have explored deeper algebraic properties, focusing on annihilator ideals, μ−filters, and general notions of divisibility within ADLs. Rafi, Bandaru, and Rao [5] and Kumar, Rao, and Babu [6], underscoring the need for a more generalized and robust framework, further expanded the concepts of divisibility and its algebraic variants.

Building upon this rich foundation, the present study introduces the concepts of ⊕−divisibility, ⊕−prime, and ⊕−irreducible elements in the

context of ADLs, which represent a novel extension of the classical structures. These elements are essential in understanding deeper congruence properties of ADLs, particularly in defining and analyzing a congruence relation θ that yields a quotient algebraic structure isomorphic to a Boolean algebra—a structure of paramount importance in digital logic, switching theory, and computational systems.

2. Need of the study

The characterization of $\oplus$−prime and $\oplus$−irreducible elements via prime and maximal filters offers a new lens for examining algebraic integrity and structural efficiency. Moreover, establishing that every $\oplus$−irreducible element is $\oplus$−prime not only unifies these concepts but also opens avenues for application in various engineering domains where binary decision-making, fault-tolerance, and logical consistency are vital. Thus, the study addresses a critical gap in the literature by extending the theoretical underpinnings of ADLs toward more applicable and engineering-relevant frameworks.

3. Preliminaries

The significant results of Rao [7] and Swamy and Rao [1] are gathered and given in the following lines.

Structure and Axioms: An ADL with zero is an algebra $(R, \vee, \wedge, 0)$ that mirrors a standard distributive lattice but lacks commutativity and right distributivity of join over meet. It is defined by left distributivity—$(u \vee v) \wedge s = (u \wedge s) \vee (v \wedge s)$ and $u \wedge (v \vee s) = (u \wedge v) \vee (u \wedge s)$—alongside absorption laws like $(u \vee v) \wedge u = u$ and the zero property $0 \wedge u = 0$.

Ideals, Filters, and Congruence: Structural subsets are organized into principal ideals $(u]$ and principal filters $[u)$. These maintain operational consistency, where $(u] \vee (v] = (u \vee v]$ and $(u] \cap (v] = (u \wedge v]$. Furthermore, congruence relations are equivalence relations that preserve the join and meet operations, allowing the creation of distinct congruence classes $[u]\theta$ for any element u in the lattice.

Annihilators and Density: For any element u, the dual annihilator $(u)^+$ comprises all elements m such that $u \vee m$ resides in a maximal set M. Elements that satisfy $(d)^+ = M$ are termed dual dense. The collection of these elements, E, forms an ideal within the ADL, provided the set is non-empty. Ultimately, these properties ensure that even without commutativity, the ADL maintains a rigorous internal logic suitable for algebraic analysis. Further, in Swamy's work, it is noted that an ADL denoted as R exhibits nearly all features of a distributive lattice [8-10], apart from the non-commutativity of $\vee$ and $\wedge$ and the right distributivity of $\vee$ over $\wedge$.

4. ⊕−divisibility of ADLs

We define ⊕divisibility in ADLs, proving that an element's multipliers constitute a filter. These filters collectively form a complete Boolean algebra. By identifying ⊕-prime and ⊕-irreducible elements, we link them to these multiplier-based filters.

Definition 3.1: An element μ $(\mathcal{R})$ is called an ⊕−divisor of an element $\pi \in \mathcal{R}$. This is denoted as $(\mu|\pi)_+$.

Lemma 3.2: *For each* $\mu, \pi, \chi \in R$*:*

(1) $(\mu|\mu)_+$;

(2) for every $\mu \leq \pi$, $(\mu|\pi)_+$;

(3) for every $(\mu)^+ = (\pi)^+$, $(\mu|\pi)_+$ and $(\pi|\mu)_+$;

(4) for every $(\mu|\pi)_+$ and $(\pi|\chi)_+$, $(\mu|\chi)_+$;

(5) for every $(\mu|\pi)_+$, $(\mu|(\pi \vee \nu))_+$ for all $\nu \in \mathcal{R}$;

(6) for every $(\mu|\pi)_+$, $((\mu \vee \nu)|(\pi \vee \nu))_+$ and $((\mu \wedge \nu)|(\pi \wedge \nu))_+$ for all $\nu \in \mathcal{R}$.

Proof:

(1) Since $(\mu)^+ = (\mu \vee \mu)^+$, we get $(\mu|\mu)_+$.

(2) Assume $\mu \leq \pi$. Then $\pi = \mu \vee \pi$ and hence $(\pi)^+ = (\mu \vee \pi)^+ = (\pi \vee \mu)^+$. Therefore $(\mu|\pi)_+$.

(3) Assume $(\mu)^+ = (\pi)^+$. Then $(\mu)^+ = (\pi \vee \pi)^+$ and $(\pi)^+ = (\mu \vee \mu)^+$. Therefore $(\pi|\mu)_+$ and $(\mu|\pi)_+$.

(4) Assume $(\mu|\pi)_+$ and $(\pi|\chi)_+$. Then $(\pi)^+ = (\mu \vee \omega)^+$ for some $\omega \in \mathcal{R}$ and $(\chi)^+ = (\pi \vee \nu)^+$, for some $\nu \in \mathcal{R}$. Now $(\chi)^{++} = (\pi \vee \nu)^{++} = (\pi)^{++} \cap (\nu)^{++} = (\mu \vee \omega)^{++} \cap (\nu)^{++} = (\mu)^{++} \cap (\omega)^{++} \cap (\nu)^{++} = (\mu \vee \omega \vee \nu)^{++}$. Hence $(\chi)^+ = (\mu \vee \omega \vee \nu)^+$. Therefore $(\mu|\chi)_+$.

(5) Assume $(\mu|\pi)_+$. Then $(\pi)^+ = (\mu \vee \omega)^+$, for some $\omega \in \mathcal{R}$. Let $\nu \in \mathcal{R}$. Now $(\pi \vee \nu)^{++} = (\pi)^{++} \cap (\nu)^{++} = (\mu \vee \omega)^{++} \cap \nu^{++} = (\mu \vee \omega \vee \nu)^{++}$. Therefore $(\pi \vee \nu)^+ = (\mu \vee \omega \vee \nu)^+$ and hence $(\mu|\pi \vee \nu)_+$.

(6) Assume $(\mu|\pi)_+$. Then $(\pi)^+ = (\mu \vee \omega)^+$, for some $\omega \in \mathcal{R}$. Let $\nu \in \mathcal{R}$. Now $(\pi \vee \nu)^{++} = (\pi)^{+++} \cap (\nu)^{++} = (\mu \vee \omega)^{++} \cap (\nu)^{++} = (\mu \vee \omega \vee \nu)^{++} = (\mu \vee \nu \vee \omega)^{++}$. Therefore $(\pi \vee \nu)^+ = (\mu \vee \nu \vee \omega)^+$ and hence $(\mu \vee \nu|\pi \vee \nu)_+$. Now $(\pi \wedge \nu)^+ = (\pi)^+ \cap (\nu)^+ = (\mu \vee \omega)^+ \cap (\nu)^+ = ((\mu \vee \omega) \wedge \nu)^+ = ((\mu \wedge \nu) \vee (\omega \wedge \nu))^+$. Therefore $((\mu \wedge \nu)|(\pi \wedge \nu))_+$.

Definition 3.3: Define $(\mu)^\perp = \{\nu \in \mathcal{R} \mid (\mu|\nu)_+\}$, for each $\mu \in \mathcal{R}$.

Lemma 3.4: *For any element* μ *of* $\mathcal{R}$*,* $(\mu)^\perp$ *- Filter of* $\mathcal{R}$.

Proof: As $(\mu|\mu)_+$, for each $\mu \in \mathcal{R}$, we obtain $(\mu)^\perp \varphi$. Consider $\omega, \nu \in (\mu)^\perp$. Then $(\mu|\omega)_+$ and $(\mu|\nu)_+$. It gives $(\omega)^+ = (\mu \vee \upsilon)^+$, $(\nu)^+ = (\mu \vee \vartheta)^+$, for some $\upsilon, \vartheta \in \mathcal{R}$. Now $(\omega \wedge \nu)^+ = (\omega)^+ \cap (\nu)^+ = (\mu \vee \upsilon)^+ \cap (\mu \vee \vartheta)^+ = ((\mu \vee \upsilon) \wedge (\mu \vee \vartheta))^+ = (\mu \vee$

$(\upsilon \wedge \vartheta))^{+}$. Therefore $(\mu|\omega \wedge \nu)_{+}$ and hence $\omega \wedge \nu \in (\mu)^{\perp}$. Let $\omega \in (\mu)^{\perp}$. Then $(\mu|\omega)_{+}$. That implies $(\omega)^{+} = (\mu \vee \upsilon)^{+}$, for some $\upsilon \in \mathcal{R}$. Let $\tau \in \mathcal{R}$. Now $(\tau \vee \omega)^{++} = (\tau)^{++} \cap (\omega)^{++} = (\tau)^{++} \cap (\mu \vee \upsilon)^{++} = (\tau \vee \mu \vee \upsilon)^{++} = (\mu \vee \tau \vee \upsilon)^{++}$. Therefore $(\tau \vee \omega)^{+} = (\mu \vee \tau \vee \upsilon)^{+}$ and hence $(\mu|(\tau \vee \omega))_{+}$. Which follows $\tau \vee \omega \in (\mu)^{\perp}$. Thus $(\mu)^{\perp}$ is a filter of $\mathcal{R}$.

Lemma 3.5: *Let $\mu, \pi, \chi \in \mathcal{R}$.*

(1) $\mu \leq \pi \Rightarrow (\pi)^{\perp} \subseteq (\mu)^{\perp}$;

(2) $(\mu)^{+} = (\pi)^{+} \Rightarrow (\mu)^{\perp} = (\pi)^{\perp}$;

(3) $(\mu)^{\perp} \cap (\pi)^{\perp} = (\mu \vee \pi)^{\perp}$;

(4) $(\mu \vee \pi)^{\perp} = (\pi \vee \mu)^{\perp}$ and $(\mu \wedge \pi)^{\perp} = (\pi \wedge \mu)^{\perp}$;

(5) $(\mu)^{\perp} = (\pi)^{\perp} \Rightarrow (\mu \vee \nu)^{\perp} = (\pi \vee \nu)^{\perp}$ and $(\mu \wedge \nu)^{\perp} = (\pi \wedge \nu)^{\perp}$, for all $\nu \in \mathcal{R}$.

Proof:

(1) As $(\mu|\mu)_{+}$, we may get $\mu \in (\mu)^{\perp}$.

(2) Let $\mu \in (\pi)^{\perp}$. Then $(\pi|\mu)_{+}$. Let $\nu \in (\mu)^{\perp}$. Then $(\mu|\nu)_{+}$. As $(\pi|\mu)_{+}$, weget $(\pi|\nu)_{+}$ and hence $\nu \in (\pi)^{\perp}$. Therefore $(\mu)^{\perp} \subseteq (\pi)^{\perp}$.

(3) Assume $\mu \leq \pi$. Then $\pi = \mu \vee \pi$. Let $\nu \in (\pi)^{\perp}$. Then $(\pi|\nu)_{+}$. That implies $(\nu)^{+} = (\pi \vee \omega)^{+}$, for some $\omega \in \mathcal{R}$. Therefore $(\nu)^{+} = (\mu \vee \pi \vee \omega)^{+}$ and hence $(\mu|\nu)_{+}$. Which follows $\nu \in (\mu)^{\perp}$. Thus $(\pi)^{\perp} \subseteq (\mu)^{\perp}$.

(4) Assume $(\mu)^{+} = (\pi)^{+}$. Let $\nu \in (\mu)^{\perp}$. Then $(\mu|\nu)_{+}$. That implies $(\nu)^{+} = (\mu \vee \omega)^{+}$, for some $\omega \in \mathcal{R}$. Now $(\nu)^{+} = (\mu \vee \omega)^{+++} = ((\mu)^{++} \cap (\omega)^{++})^{+} = ((\pi)^{++} \cap (\omega)^{++})^{+} = (\pi \vee \omega)^{+++} = (\pi \vee \omega)^{+}$. Therefore $(\pi|\nu)_{+}$ and hence $\nu \in (\pi)^{\perp}$. Thus $(\mu)^{\perp} \subseteq (\pi)^{\perp}$. Similarly, we get $(\pi)^{\perp} \subseteq (\mu)^{\perp}$. Hence $(\mu)^{\perp} = (\pi)^{\perp}$.

(5) Let

$\nu \in (\mu)^{\perp} \cap (\pi)^{\perp}$. Then $(\mu|\nu)_{+}$ and $(\pi|\nu)_{+}$. That implies $(\nu)^{+} = (\mu \vee \upsilon)^{+}$, for some $\upsilon \in \mathcal{R}$, $(\nu)^{+} = (\pi \vee \vartheta)^{+}$, for some $\vartheta \in \mathcal{R}$. Now $(\mu \vee \pi \vee \upsilon \vee \vartheta)^{++} = (\mu \vee \upsilon)^{++} \cap (\pi \vee \vartheta)^{++} = (\nu)^{++} \cap (\nu)^{++} = (\nu)^{++}$. It follows $((\mu \vee \pi)|\nu)_{+}$. That implies $\nu \in (\mu \vee \pi)^{\perp}$.

Statements (6) and (7) are evident.

Theorem 3.6: *For each $\mu \in \mathcal{R}$, $(\mu)^{+} = \mathcal{M}$ iff $(\mu)^{\perp} = \mathcal{R}$.*

Proof: Let's assume $(\mu)^{+} = \mathcal{M}$. In this case, we have $(\mu)^{+} = (0)^{+}$, which implies that $(\mu)^{\perp} = (0)^{\perp}$. Given that $\mu = \mu \vee 0$ holds for any $\mu \in \mathcal{R}$,it follows that $(\mu)^{+} = (\mu \vee 0)^{+}$, meaning $(0|\mu)_{+}$ for all $\mu \in \mathcal{R}$. Therefore, every $\mu \in \mathcal{R}$belongs to $(0)^{\perp}$, which implies $(0)^{\perp} = \mathcal{R}$. Since $(\mu)^{\perp} = (0)^{\perp}$, it follows that $(\mu)^{\perp} = \mathcal{R}$.

Now, for the reverse direction, assume that $(\mu)^{\perp} = \mathcal{R}$, which means $0 \in (\mu)^{\perp}$. This gives $(\mu|0)_{+}$, implying that $(0)^{+} = (\mu \vee \nu)^{+}$ for some $\nu \in \mathcal{R}$.

As a result, $(\mu)^+ = (\mu \vee 0)^+ = (\mu \vee 0)^{+++} = ((\mu)^{++} \cap (0)^{++})^+ = ((\mu)^{++} \cap (\mu \vee \nu)^{++})^+ = (\mu \vee \mu \vee \nu)^{+++} = (\mu \vee \nu)^+ = (0)^+ = \mathcal{M}$. Since $(0)^+ = \mathcal{M}$,we conclude that $(\mu)^+ = \mathcal{M}$.

Figure 2. Theoretical and practical framework of Almost Distributive Lattices (ADLs). Figure 2a presents a meet table illustrating the internal structure of an ADL, highlighting the partial ordering and interactions among elements, which reflect non-distributive behavior. Figure 2b complements this by mapping the conceptual flow of the study—starting from ADL structural properties such as ⊕−divisibility, ⊕−prime, and ⊕−irreducible elements, leading through a congruence relation defined by multiplier filters, and culminating in a quotient ADL that forms a Boolean algebra. This theoretical progression is directly connected to real-world applications in digital circuit design, fault-tolerant computing, logic systems, and data encryption.

The collection $\mathfrak{F}^{\perp}(\mathcal{R})$ contains all filters expressed in the form $(\mu)^{\perp}$ for μ in $\mathcal{R}$.

Example: 3.7: Consider $\mathcal{R} = \{0, \upsilon, \vartheta, \sigma, d, e, \rho, \kappa\}$ and define $\vee$, $\wedge$ on $\mathcal{R}$ as follows:

a)

$\wedge$	0	υ	θ	σ	d	e	ρ	κ
0	0	0	0	0	0	0	0	0
υ	0	υ	θ	σ	d	e	ρ	κ
θ	0	υ	θ	σ	d	e	ρ	κ
σ	0	σ	σ	σ	0	0	σ	0
d	0	d	e	0	d	e	κ	κ
e	0	d	e	0	d	e	κ	κ
ρ	0	ρ	ρ	σ	κ	κ	ρ	κ
κ	0	κ	κ	0	κ	κ	κ	κ

b)

$\vee$	0	υ	θ	σ	d	e	ρ	κ
0	0	υ	θ	σ	d	e	ρ	κ
υ	υ	υ	υ	υ	υ	υ	υ	υ
θ	θ	θ	θ	θ	θ	θ	θ	θ
σ	σ	υ	θ	σ	υ	θ	ρ	ρ
d	d	υ	υ	υ	d	d	υ	d
e	e	θ	θ	θ	e	e	θ	e
ρ	ρ	υ	θ	ρ	υ	θ	ρ	ρ
κ	κ	υ	θ	ρ	d	e	ρ	κ

Figure 2. Theoretical and Practical Framework of Almost Distributive Lattices (ADLs): (a) Meet Operation Table of a Finite Lattice with Distinguished Elements b) Join Operation Table ($\vee$) of Elements in an Almost Distributive Lattice (ADL).

Then $(\mathcal{R}, \vee, \wedge)$ is an ADL. $\mathcal{H} = \{\upsilon, \vartheta, \rho\}$ is a filter of $\mathcal{R}$. Also, we have that $(\sigma)^{\perp} = \{\upsilon, \vartheta, \sigma, \rho\}$ and $(d)^{\perp} = \{\upsilon, \vartheta, d, e, \rho, \kappa\}$. Therefore $(\sigma)^{\perp} \vee (d)^{\perp} = \{\upsilon, \vartheta, \rho\}$ and $(\sigma \wedge d)^{\perp} = (0)^{\perp} = \mathcal{R}$. Hence $(\sigma)^{\perp} \vee (d)^{\perp} \neq (\sigma \wedge d)^{\perp}$.

Theorem 3.8: *$\mathfrak{F}^{\perp}(\mathcal{R})$ could form a complete-distributive-lattice.*

Proof: Let $(\mu)^{\perp}, (\pi)^{\perp} \in \mathfrak{F}^{\perp}(\mathcal{R})$. Consider $(\mu)^{\perp} \cap (\pi)^{\perp} = (\mu \vee \pi)^{\perp}$ and $(\mu)^{\perp} \sqcup (\pi)^{\perp} = (\mu \wedge \pi)^{\perp}$. It is evident that $(\mu)^{\perp} \cap (\pi)^{\perp} = (\mu \vee \pi)^{\perp}$. It gives $(\mu \vee \pi)^{\perp}$ is the glb of $(\mu)^{\perp}$ and $(\pi)^{\perp}$ in $\mathfrak{F}^{\perp}(R)$. Its obviously that $(\mu)^{\perp} \subseteq (\mu \wedge \pi)^{\perp}$and $(\pi)^{\perp} \subseteq (\mu \wedge \pi)^{\perp}$. Which leads $(\mu \wedge \pi)^{\perp}$ is an upper bound of $(\mu)^{\perp}, (\pi)^{\perp}$. Let $(\chi)^{\perp}$ be any upper bound of $(\mu)^{\perp}, (\pi)^{\perp}$. Then $(\mu)^{\perp} \subseteq (\chi)^{\perp}$ and $(\pi)^{\perp} \subseteq (\chi)^{\perp}$. It gives $\mu, \pi \in (\chi)^{\perp}$ and hence $\mu \wedge \pi \in (\chi)^{\perp}$. Thus $(\mu \wedge \pi)^{\perp} \subseteq (\chi)^{\perp}$. $(\mu \wedge \pi)^{\perp}$ is the lub of $(\mu)^{\perp}$ and $(\pi)^{\perp}$. It leads $(\mu)^{\perp} \sqcup (\pi)^{\perp} = (\mu \wedge \pi)^{\perp}$. Therefore, $\mathfrak{F}^{\perp}(\mathcal{R})$ is a lattice.

Definition 3.9: Let us define a relation θ on $\mathcal{R}$ such that for any elements $\mu, \pi \in \mathcal{R}$, the pair (μ, π) belongs to θ if $(\mu)^{\perp} = (\pi)^{\perp}$. It is straightforwardto confirm that θ qualifies as congruence on $\mathcal{R}$.

For any congruence θ on $\mathcal{R}$, it can be established easily that the set $\mathcal{R}/\theta = \{[\mu]_{\theta} \mid \mu \in \mathcal{R}\}$ forms a quotient lattice. This is achieved by defining the operations as follows:

We have $[\mu]_{\theta} \wedge [\pi]_{\theta} = [\mu \wedge \pi]_{\theta}$ and $[\mu]_{\theta} \vee [\pi]_{\theta} = [\mu \vee \pi]_{\theta}$.

Next, consider the function f: $\mathcal{R} \rightarrow \mathcal{R}/\theta$ defined as $f(\mu) = [\mu]_{\theta}$ for every $\mu \in \mathcal{R}$. It is clear that f acts as a natural surjective homomorphism.

Lemma 3.10: *$\mathcal{E}$ is an ideal of an ADL $\mathcal{R}$.*

Proof: Clearly, $0 \in \mathcal{E}$ and hence $\mathcal{E} \neq \varphi$. Let $\upsilon, \vartheta \in \mathcal{E}$. Then $(\upsilon)^{+} = \mathcal{M} = (\vartheta)^{+}$. Now $(\upsilon \vee \vartheta)^{+} = (\upsilon \vee \vartheta)^{+++} = ((\upsilon)^{++} \cap (\vartheta)^{++})^{+} = (\mathcal{M}^{+} \cap \mathcal{M}^{+})^{+} = \mathcal{R}^{+} = \mathcal{M}$. That implies $\upsilon \vee \vartheta \in \mathcal{E}$. Let $\upsilon \in \mathcal{E}$. Then $(\upsilon)^{+} = \mathcal{M}$. Let $\tau \in R$. Clearly $\mathcal{M} \subseteq (\upsilon \wedge \tau)^{+} \subseteq (\upsilon)^{+} = \mathcal{M}$ and hence $(\upsilon \wedge \tau)^{+} = \mathcal{M}$.

Therefore, $\upsilon \wedge \tau \in \mathcal{E}$. Hence, E is an ideal of $\mathcal{R}$.

Theorem 3.11: *In $\mathcal{R}/\theta$, $\mathcal{E}$ -zero congruence and $\mathcal{M}$ is the unit congruence.*

Proof: We get $(\mu)^{+} = \mathcal{M} = (\pi)^{+}$, for every $\mu, \pi \in \mathcal{E}$. That implies $(\mu)^{\perp} = (\pi)^{\perp}$ and hence $(\mu, \pi) \in \theta$. Thus $\mathcal{E}$ is a congruence class of $\mathcal{R}/\theta$. Let $\upsilon \in \mathcal{R}$. For each $\vartheta \in \mathcal{E}$, we have $\vartheta \wedge \upsilon \in \mathcal{E}$, for all $\upsilon \in \mathcal{R}$. Since $\vartheta, \vartheta \wedge \upsilon \in \mathcal{E}$, we get that $(\vartheta)^{+} = \mathcal{M} = (\vartheta \wedge \upsilon)^{+}$ and hence $(\vartheta)^{\perp} = (\vartheta \wedge \upsilon)^{\perp}$. Therefore $(\vartheta, \vartheta \wedge \upsilon) \in \theta$, which gives $\vartheta \in [\vartheta \wedge \upsilon]_{\theta} \subseteq [\upsilon]_{\theta}$, for all $\upsilon \in \mathcal{R}$. Therefore $\mathcal{E} \subseteq [\upsilon]_{\theta}$, for every $\upsilon \in \mathcal{R}$. It concludes that $\mathcal{E}$ is the zero congruence of $\mathcal{R}/\theta$. For each $\mu, \pi \in \mathcal{M}$, we have $(\mu)^{+} = \mathcal{R} = (\pi)^{+}$, and it leads $(\mu)^{\perp} = (\pi)^{\perp}$. Thus $\mathcal{M}$ is a congruence class of $\mathcal{R}/\theta$. For $m \in \mathcal{M}$, we have $m \vee \mu = m$, for each $\mu \in \mathcal{R}$. Therefore, $[\mu]_{\theta} \subseteq (\mu \vee m)_{\theta} = (m)_{\theta}$, which gives $[\mu]_{\theta} \subseteq [m]_{\theta} = \mathcal{M}$,for all $\mu \in \mathcal{R}$.

Theorem 3.12: *The equivalent conditions are as follows in an ADL* $\mathcal{R}$:

Proof: (1) ⇒ (2): Consider (1). Now define κ: $\mathcal{R}|\theta \to \mathfrak{F}^{\perp}(\mathcal{R})$ by $\kappa([\upsilon]_\theta) = (\upsilon)^{\perp}$, for every $[\upsilon]_\theta \in \mathcal{R}/\theta$. Let $\upsilon, \vartheta \in \mathcal{R}$ with $[\upsilon]_\theta = [\vartheta]_\theta$. Then $(\upsilon, \vartheta) \in \theta$, which implies $(\upsilon)^{\perp} = (\vartheta)^{\perp}$, and thus $\kappa([\upsilon]_\theta) = \kappa([\vartheta]_\theta)$. Therefore, κ is well defined. Now, let $\upsilon, \vartheta \in \mathcal{R}$ be such that $(\upsilon)^{\perp} = (\vartheta)^{\perp}$. This implies $(\upsilon, \vartheta) \in \theta$, leading to $[\upsilon]_\theta = [\vartheta]_\theta$, which shows that κ is injective. Consider an element $\upsilon \in \mathcal{R}$. Since $[\upsilon]_\theta \in \mathcal{R}/\theta$ and f is surjective, there is a $\vartheta \in \mathcal{R}$ satisfies $f(\vartheta) = [\upsilon]_\theta$. Consequently, $\kappa([\upsilon]_\theta) = (\upsilon)^{\perp}$, demonstrating that κ is surjective. It is also evident that κ is a homomorphism, which implies that κ is an isomorphism. Thus, $\mathfrak{F}^{\perp}(\mathcal{R})$ is a Boolean algebra.

(2) ⇒ (3): Assume (2). We have $(\upsilon)^{\perp} \in \mathfrak{F}^{\perp}(\mathcal{R})$, for each $\upsilon \in \mathcal{R}$. By (2), There is $(\vartheta)^{\perp} \in \mathfrak{F}^{\perp}(\mathcal{R})$ satisfies $(\upsilon \vee \vartheta)^{\perp} = (\upsilon)^{\perp} \cap (\vartheta)^{\perp} = (0)^{\perp}$ and $(\upsilon \wedge \vartheta)^{\perp} = (\upsilon)^{\perp} \sqcup (\vartheta)^{\perp} = \mathcal{R}$. Hence $\upsilon \vee \vartheta \in \mathcal{M}$ and $(\upsilon \wedge \vartheta)^{+} = \mathcal{M}$. Therefore $\upsilon \vee \vartheta \in \mathcal{M}$ and $\upsilon \wedge \vartheta \in \mathcal{E}$.

(2) ⇒ (1): Assume (3). Let $[\upsilon]_\theta \in \mathcal{R}/\theta$, for each $\upsilon \in \mathcal{R}$. Then $\upsilon \wedge \vartheta \in \mathcal{E}$, $\upsilon \vee \vartheta \in \mathcal{M}$, for some $\vartheta \in \mathcal{M}$. That implies $[\upsilon]_\theta \wedge [\vartheta]_\theta = [\upsilon \wedge \vartheta]_\theta = \mathcal{E}$ and $[\upsilon]_\theta \vee [\vartheta]_\theta = [\upsilon \vee \vartheta]_\theta = \mathcal{M}$. Hence $\mathcal{R}/\theta$ is a Boolean algebra.

Definition 3.13: An element μ of $\mathcal{R}$ which is not in $\mathcal{M}$ is referred to as $\oplus$−prime if for any $\pi, \chi \in \mathcal{R}$, $(\mu|\pi \vee \chi)_+$ implies $(\mu|\pi)_+$ or $(\mu|\chi)_+$.

Theorem 3.14: *For any* $x \notin \mathcal{E}$, *μ is* $\oplus$*−prime if and only if* $(\mu)^{\perp}$ *is a prime filter of* $\mathcal{R}$.

Proof: Let $x \notin \mathcal{E}$. Suppose μ is $\oplus$−prime. Let $\upsilon, \vartheta \in \mathcal{R}$ such that $\upsilon \vee \vartheta \in (\mu)^{\perp}$. Then $(\mu|\upsilon \vee \vartheta)_+$. Based on our assumption, this leads to $(\mu|\upsilon)_+$ or $(\mu|\vartheta)_+$. Consequently, this means that $\upsilon \in (\mu)^{\perp}$ or $\vartheta \in (\mu)^{\perp}$. Thus, $(\mu)^{\perp}$ is prime.

Conversely, let us assume $(\mu)^{\perp}$ is prime. Let $\upsilon, \vartheta \in \mathcal{R}$ such that $(\mu|\upsilon \vee \vartheta)_+$. This implies $\upsilon \vee \vartheta \in (\mu)^{\perp}$. From our assumption, it follows that $\upsilon \in (\mu)^{\perp}$ or $\vartheta \in (\mu)^{\perp}$. Hence, we conclude that $(\mu|\upsilon)_+$ or $(\mu|\vartheta)_+$. Therefore, μ is $\oplus$- prime.

Definition 3.15: An element μ of $\mathcal{R}$ is called as $\oplus$−irreducible if it is not dual dense and satisfies the following condition:

For any elements π and χ in $\mathcal{R}$, if $(\mu)^{+} = (\pi \vee \chi)^{+}$, then it must be true that either $(\pi)^{+} = \mathcal{M}$ or $(\chi)^{+} = \mathcal{M}$.

Lemma 3.16: *Each element in* $\mathcal{E}$ *is* $\oplus$*−irreducible.*

Proof: Let $\mu \in \mathcal{E}$. Suppose $\upsilon, \vartheta \in \mathcal{R}$ with $(\mu)^{+} = (\upsilon \vee \vartheta)^{+}$. Then $(\mu)^{++} = (\upsilon)^{++} \cap (\vartheta)^{++}$. That implies $\mathcal{M}^{+} = (\upsilon)^{++} \cap (\vartheta)^{++}$ and hence $\mathcal{R} = (\upsilon)^{++} \cap (\vartheta)^{++}$. Therefore $\mathcal{R} \subseteq (\upsilon)^{++}$ and $\mathcal{R} \subseteq (\vartheta)^{++}$. That implies $(\upsilon)^{+++} = \mathcal{M}$ and $(\vartheta)^{+++} = \mathcal{M}$, which gives $(\upsilon)^{+} = \mathcal{M}$ and $(\vartheta)^{+} = \mathcal{M}$. Thus, μ is $\oplus$−irreducible.

Theorem 3.17: *Consider an element $\mu \in \mathcal{R}$ for which $(\mu)^+ \neq \mathcal{M}$. The subsequent conditions are equivalent:*

Proof: (1) ⇒ (2): Assume (1). Consider there is $\pi \in \mathcal{R}$ satisfies $(\mu)^\perp \subseteq (\pi)^\perp \neq \mathcal{R}$. Then $\mu \in (\pi)^\perp$. Which gives $(\pi|\mu)_+$. It leads that there is $\chi \in \mathcal{R}$ satisfies $(\mu)^+ = (\pi \vee \chi)^+$. By (1), $(\pi)^+ = \mathcal{M}$ or $(\chi)^+ = \mathcal{M}$. Since $(\pi)^\perp \neq \mathcal{M}$. we get $(\pi)^+ \neq \mathcal{M}$ and hence $(\chi)^+ = \mathcal{M}$. That implies $(\chi)^+ = (0)^+$ and hence $(\mu)^+ = (\pi)^+$. Therefore $(\mu)^\perp = (\pi)^\perp$. Thus $(\mu)^\perp$ is maximal among all filters of the form $(\pi)^\perp$. Assume $(\mu)^+ = (\mu \vee \pi)^+$, for some $\pi \in \mathcal{M}$. By(1), we get $(\mu)^+ = \mathcal{M}$ or $(\pi)^+ = \mathcal{M}$. Since $(\mu)^+ \neq \mathcal{M}$, we get $(\pi)^+ = \mathcal{M}$.

(1) ⇒ (1): Assume (2). Let $\mu \in \mathcal{R}$ with $(\mu)^+ \neq \mathcal{M}$. Suppose $(\mu)^+ = (\pi \vee \chi)^+$, for some $\pi, \chi \in \mathcal{R}$. It gives $(\chi|\mu)_+$. which implies $\mu \in (\chi)^\perp$ and hence $(\mu)^\perp \subseteq (\chi)^\perp$. By (2)(i), we obtain $(\mu)^\perp = (\chi)^\perp$ or $(\chi)^\perp = \mathcal{R}$. Assume $(\mu)^\perp = (\chi)^\perp$. we get $\chi \in (\mu)^\perp$. It implies $(\mu|\chi)_+$. Which leads $(\chi)^+ = (\mu \vee \upsilon)^+$, for some $\upsilon \in \mathcal{R}$. Hence $(\chi \vee \pi)^+ = (\mu \vee \upsilon \vee \pi)^+$. By 2(ii), we get $(\upsilon \vee \pi)^+ = \mathcal{M}$. Since $(\pi)^+ \subseteq (\upsilon \vee \pi)^+$, we get $(\pi)^+ = \mathcal{M}$. Suppose $(\chi)^\perp = \mathcal{R}$. It gives $0 \in (\chi)^\perp$ and hence $(\chi/0)_+$. So $(0)^+ = (\chi \vee \upsilon)^+$, for some $\upsilon \in \mathcal{R}$. Which gives $\mathcal{M} = (\chi \vee \upsilon)^+$. This implies $(\chi)^+ \subseteq (\chi \vee \upsilon)^+ = \mathcal{M}$, and it leads $(\chi)^+ = \mathcal{M}$. Thus μ is $\oplus$−irreducible.

Theorem 3.18: *In an ADL, each $\oplus$irreducible element is $\oplus$prime.*

Proof: Let μ be any $\oplus$irreducible element in $\mathcal{R}$. Consider $\upsilon, \vartheta \in \mathcal{R}$ such that $(\mu|\upsilon \vee \vartheta)_+$. Assume that $\upsilon \notin (\mu)^\perp$ and $\vartheta \notin (\mu)^\perp$. It follows that $(\mu)^\perp \subsetneq (\mu)^\perp \sqcup (\upsilon)^\perp \subseteq (\mu \wedge \upsilon)^\perp$ and $(\mu)^\perp \subsetneq (\mu)^\perp \sqcup (\vartheta)^\perp \subseteq (\mu \wedge \vartheta)^\perp$. By the maximality of $(\mu)^\perp$, we obtain $(\mu \wedge \upsilon)^\perp = \mathcal{R}$ and $(\mu \wedge \vartheta)^\perp = \mathcal{R}$. This implies $\mathcal{R} = (\mu \wedge \upsilon)^\perp \cap (\mu \wedge \vartheta)^\perp$. Hence, $\mathcal{R} = ((\mu)^\perp \sqcup (\upsilon)^\perp) \cap ((\mu)^\perp \sqcup (\vartheta)^\perp) = (\mu)^\perp \sqcup ((\upsilon)^\perp \cap (\vartheta)^\perp) = (\mu)^\perp \sqcup (\upsilon \wedge \vartheta)^\perp$. Since $\upsilon \wedge \vartheta \in (\mu)^\perp$, we deduce $(\upsilon \vee \vartheta)^\perp \subseteq (\mu)^\perp$, leading to $(\mu)^\perp = \mathcal{R}$. This results in a contradiction. Therefore, either $\upsilon \in (\mu)^\perp$ or $\vartheta \in (\mu)^\perp$. Consequently, we conclude that $(\mu|\upsilon)_+$ or $(\mu|\vartheta)_+$. Thus, μ is $\oplus$−prime.

5. Conclusion

This study investigates the structure of Almost Distributive Lattices (ADLs) by introducing and analyzing the concepts of $\oplus$−divisibility, $\oplus$−prime, and $\oplus$−irreducible elements. A new congruence relation using multiplier filters is defined, and conditions are established for the resulting quotient ADL. The link between $\oplus$−prime/$\oplus$−irreducible elements and multiplier filters is clearly characterized. These findings not only contribute to the theoretical development of lattice theory but also open up possibilities for practical applications in engineering, especially in areas where logic and reliability are critical.

Future research can explore the implementation of $\oplus$−irreducible elements in designing efficient digital circuits, secure communication systems, and fault-tolerant architectures. The concept of multiplier filters can also be extended to

other algebraic structures for broader applications. Moreover, the integration of ADL-based models into software tools and computational frameworks could provide practical advantages in system modeling and decision-making processes.

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