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η-Ricci-Bourguignon soliton on the hyperbolic spaces

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Abstract

We investigate η-Ricci-Bourguignon solitons on hyperbolic spaces $\mathbb{H}^n$, with a focus on the Poincaré half-plane $\mathbb{H}^2$. Building upon recent advances on classical Ricci-Bourguignon solitons in this geometric setting, we establish necessary and sufficient conditions for the existence of such structures and analyze the properties of the underlying potential vector fields. The results obtained on $\mathbb{H}^2$ are subsequently extended to higher-dimensional hyperbolic spaces.

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Keywords: *Ricci soliton, η-Ricci-Bourguignon soliton, Poincaré half plane, Hyperbolic space.*

1. Introduction

Recall that a pseudo-Riemannian manifold (M, g) is said to carry a Ricci soliton structure when there exists a smooth vector field X on M satisfying

$$\mathcal{L}_X g + 2\mathrm{Ric} = 2\lambda g, \tag{1}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X, Ric is the Ricci curvature tensor, and $\lambda \in \mathbb{R}$. According to the sign of λ, the soliton is respectively termed **shrinking** ($\lambda > 0$), **steady** ($\lambda = 0$), or **expanding** ($\lambda < 0$). When the vector field

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has the form $X = \nabla F$ for some smooth function F, the structure is called a **gradient Ricci soliton** with potential function F.

Ricci solitons arise as self-similar solutions of the Ricci flow and can be viewed as its fixed points modulo diffeomorphisms and scaling; understanding them is therefore a natural step toward analyzing the flow's global behavior and singularity formation. Among the singularity models, shrinking solitons correspond to Type I, steady solitons to Type II, and expanding ones to Type III, under suitable geometric conditions. The subject has attracted sustained attention in both geometry and mathematical physics, gaining additional prominence after Perelman's celebrated proof of the Poincaré conjecture, which relied crucially on Ricci flow techniques [7]. A broad account of results on Ricci solitons can be found in [5, 6].

The Ricci-Bourguignon flow, introduced by Bourguignon, extends the classical Ricci flow by incorporating a scalar curvature correction term. Its stationary points under diffeomorphisms are the Ricci-Bourguignon solitons, which constitute a natural extension of classical Ricci solitons. They satisfy the equation:

$$Ric(\mathbf{g}) + \frac{1}{2}\mathcal{L}_X\mathbf{g} - (\rho Scal + \lambda)\mathbf{g} = 0, \tag{2}$$

where $Ric(\mathbf{g})$ is the Ricci tensor of the metric $\mathbf{g}$, $\mathcal{L}_X\mathbf{g}$ is the Lie derivative of the metric along the vector field X, $Scal$ is the scalar curvature, and λ, ρ are real constants. The study of η-Ricci solitons and η-Ricci-Bourguignon solitons has also attracted growing interest; we refer the reader to [1, 2, 4, 8, 9, 10, 11, 12] and the references therein.

Soliton structures on manifolds of constant curvature have long been a central topic in differential geometry. In this direction, A. Bousso and A. Ndiaye [3] recently provided a complete classification of both Ricci solitons and Ricci-Bourguignon solitons on the Poincaré half-plane $\mathbb{H}^2$.

That work reveals the existence and uniqueness of such structures on a space of constant negative curvature, thereby establishing a solid geometric framework and opening new directions for the study of more general soliton equations.

The present work extends this line of investigation to the η-Ricci-Bourguignon solitons. This notion further generalizes classical soliton equations by introducing an additional 1-form η, and arises naturally in the framework of contact and almost contact geometry, where it is fundamental to understanding the stability and deformation properties of the underlying structures. A η-Ricci-Bourguignon soliton is defined by the equation:

$$Ric(\mathbf{g}) + \frac{1}{2}\mathcal{L}_X\mathbf{g} - (\rho Scal + \lambda)\mathbf{g} = \alpha\eta \otimes \eta, \tag{3}$$

for constants $\alpha, \rho \in \mathbb{R}$, and is denoted by $(M, \mathbf{g}, X, \lambda, \rho, \alpha)$.

The paper is structured as follows. Section 2 collects the necessary background on hyperbolic geometry and Ricci-Bourguignon solitons. Section 3 is devoted to the analysis of η-Ricci-Bourguignon solitons on $\mathbb{H}^2$, where we derive the governing PDE system and establish our main results.

2. Preliminaries and Notations

We recall here the basic definitions and fix the notation used in the sequel.

2.1 *The Hyperbolic Space* $\mathbb{H}^n$

The n-dimensional hyperbolic space $\mathbb{H}^n$ is a complete, simply connected Riemannian manifold of constant sectional curvature -1. With respect to its canonical metric $\mathbf{g}$, the Ricci tensor takes the form:

$$Ric(ds^2) = -(n-1)ds^2, \tag{4}$$

while the scalar curvature equals:

$$Scal = -n(n-1). \tag{5}$$

These curvature identities will be used systematically to simplify the soliton equations. On the hyperbolic half-plane $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 | y > 0\}$, the canonical Riemannian metric is:

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

In this model, one has $Ric(ds^2) = -ds^2$ and $Scal = -2$.

2.2 η-*Ricci-Bourguignon soliton*

A η -Ricci-Bourguignon soliton structure on (M,g) is a solution $(g, X, \lambda, \eta, \alpha, \rho)$ of equation (3). Here the 1-form η depends smoothly on the local coordinates $x_1, \dots, x_n$, and can be written in a local chart $(x_1, \dots, x_n)$ as:

$$\eta(x_1, \dots, x_n) = \sum_{i=1}^{n} f_i(x_1, \dots, x_n)dx_i,$$

where each f_i is a smooth function.

The tensor $\eta \otimes \eta$ appearing in the soliton equation is a symmetric $(0,2)$-tensor whose components $(\eta \otimes \eta)_{kl}$ are given by:

$$(\eta \otimes \eta)_{kl} = \eta\left(\frac{\partial}{\partial x_k}\right)\eta\left(\frac{\partial}{\partial x_l}\right).$$

Using $\eta\left(\frac{\partial}{\partial x_k}\right) = f_k$, this reduces to:

$$(\eta \otimes \eta)_{kl} = f_k f_l,$$

so that in coordinates:

$$\eta \otimes \eta = \sum_{k=1}^{n}\sum_{l=1}^{n}(f_k f_l)dx_k \otimes dx_l.$$

The symmetry of this tensor is consistent with the symmetry of the Ricci and metric tensors in equation (3).

Throughout the paper, we assume that $\alpha \neq 0$ in equation (3).

3. The η-Ricci-Bourguignon soliton on $\mathbb{H}^2$

We now turn to the two-dimensional case and carry out the explicit analysis on $\mathbb{H}^2$. Substituting the curvature data of $\mathbb{H}^2$ from (4) and (5) into the defining equation (3) yields:

$$-ds^2 + \frac{1}{2}\mathcal{L}_X ds^2 - (\rho(-2) + \lambda)ds^2 = \alpha\eta \otimes \eta$$

$$-ds^2 + \frac{1}{2}\mathcal{L}_X ds^2 - (-2\rho + \lambda)ds^2 = \alpha\eta \otimes \eta,$$

which reduces to

$$\frac{1}{2}\mathcal{L}_X ds^2 - (-2\rho + 1 + \lambda)ds^2 = \alpha\eta \otimes \eta. \quad (6)$$

For $\mathbb{H}^2$, the 1-form η is given by

$$\eta = f_1(x,y)dx + f_2(x,y)dy. \quad (7)$$

Consequently, $\eta \otimes \eta$ has the following components:

$$(\eta \otimes \eta)_{xx} = f_1^2, (\eta \otimes \eta)_{xy} = f_1 f_2, (\eta \otimes \eta)_{yx} = f_2 f_1, (\eta \otimes \eta)_{yy} = f_2^2.$$

Proposition 3.1: For any non-zero 1-form given in (7), the sextuple $(\mathbb{H}^2, ds^2, X, \lambda, \rho, \alpha)$ is a η-Ricci-Bourguignon soliton if and only if the vector field X is not a Killing field.

Proof: Let $X(x,y) = X^x(x,y)\,\partial_x + X^y(x,y)\,\partial_y \in \mathfrak{X}(\mathbb{H}^2)$, so

$$\left((\mathcal{L}_X g)_{ij}\right) = \begin{pmatrix} 2\left(\partial_x X^x - \frac{X^y}{y}\right) & \partial_y X^x + \partial_x X^y \\ \partial_y X^x + \partial_x X^y & 2\left(\partial_y X^y - \frac{X^y}{y}\right) \end{pmatrix}.$$

We have $(\mathbb{H}^2, ds^2, X, \lambda, \rho, \alpha)$ is a Ricci-Bourguignon η-Soliton if and only if (3) is satisfied, which means we have the system:

$$\begin{cases} \partial_x X^x - \frac{X^y}{y} = \lambda - 2\rho + 1 + \alpha\, y^2\, f_1^2 & (1) \\ \partial_y X^y - \frac{X^y}{y} = \lambda - 2\rho + 1 + \alpha\, y^2\, f_2^2 & (2) \\ \partial_y X^x + \partial_x X^y = 2\alpha\, y^2\, f_1 f_2 & (3). \end{cases}$$

The vector field X is a Killing field if and only if

$$\begin{cases} 0 = \lambda - 2\rho + 1 + \alpha\, y^2\, f_1^2 & (w1) \\ 0 = \lambda - 2\rho + 1 + \alpha\, y^2\, f_2^2 & (w2) \\ 0 = 2\alpha\, y^2\, f_1 f_2 & (w3). \end{cases}$$

The lines (w_1) and (w_2) imply

$$f_1^2(x,y) = f_2^2(x,y) = -\frac{\lambda - 2\rho + 1}{\alpha y^2}.$$

Line (w_3) requires that

$$f_1(x,y) = f_2(x,y) = 0$$

which is impossible since $\eta(x,y) \neq 0$.

Theorem 3.2: *For any vector field* $X(x,y) = X^x(x,y)\partial_x + X^y(x,y)\partial_y \in \mathfrak{X}(\mathbb{H}^2)$, *the sextuple* $(\mathbb{H}^2, ds^2, X, \lambda, \rho, \alpha)$ *is a* η*-Ricci-Bourguignon soliton with a 1-form* $\eta(x,y) = f_1(x,y)dx + f_2(x,y)dy$ *such that* f_1 *and* f_2 *are non-zero if and only if*

$$\eta(x,y) = \pm\left(\sqrt{\frac{\mathcal{R}(x,y)+\xi(x,y)}{2}}\,dx + \frac{\zeta(x,y)}{|\zeta(x,y)|}\sqrt{\frac{\mathcal{R}(x,y)-\xi(x,y)}{2}}\,dy\right)$$

where

$$\xi(x,y) = \frac{\partial_x X^x(x,y)-\partial_y X^y(x,y)}{\alpha y^2}, \zeta(x,y) = \frac{\partial_y X^x(x,y)+\partial_x X^y(x,y)}{\alpha y^2},$$
$$\mathcal{R}(x,y) = \sqrt{\xi^2(x,y)+\zeta^2(x,y)}.$$

Proof: Let $X(x,y) = X^x(x,y)\,\partial_x + X^y(x,y)\,\partial_y \in \mathfrak{X}(\mathbb{H}^2)$ be a vector field, $(\mathbb{H}^2, ds^2, X, \lambda, \rho), \alpha)$ is a Ricci-Bourguignon η-Soliton with a 1-form $\eta(x,y) = f_1(x,y)dx + f_2(x,y)dy$ such that f_1 and f_2 are non-zero if and only if the system of equations is satisfied:

$$\begin{cases} \partial_x X^x - \frac{X^y}{y} = \lambda - 2\rho + 1 + \alpha\, y^2\, f_1^2 & (\ell_1) \\ \partial_y X^y - \frac{X^y}{y} = \lambda - 2\rho + 1 + \alpha\, y^2\, f_2^2 & (\ell_2) \\ \partial_y X^x + \partial_x X^y = 2\alpha\, y^2\, f_1 f_2 & (\ell_3). \end{cases}$$

By subtracting (ℓ_2) from (ℓ_1) we get

$$\partial_x X^x - \partial_y X^y = \alpha y^2(f_1^2 - f_2^2).$$

Equation (ℓ_3) gives

$$\partial_y X^x + \partial_x X^y = 2\alpha y^2 f_1 f_2.$$

Let's set

$$\mathcal{F}(x,y) = (\partial_x X^x - \partial_y X^y) + i(\partial_y X^x + \partial_x X^y).$$

Using the above relations,

$\mathcal{F}(x,y) = \alpha y^2(f_1(x,y) + if_2(x,y))^2 = \alpha y^2 \mathcal{Z}^2(x,y)$, where $\mathcal{Z}(x,y) = f_1(x,y) + if_2(x,y)$.

Thus, we have,

$$\mathcal{Z}^2(x,y) = \frac{\mathcal{F}(x,y)}{\alpha y^2}.$$

Let's write

$$\frac{\mathcal{F}(x,y)}{\alpha y^2} = \xi(x,y) + i\zeta(x,y),$$

with

$$\xi(x,y) = \frac{\partial_x X^x(x,y)-\partial_y X^y(x,y)}{\alpha y^2}, \zeta(x,y) = \frac{\partial_y X^x(x,y)+\partial_x X^y(x,y)}{\alpha y^2}.$$

Thus

$$\mathcal{Z}^2(x,y) = \xi(x,y) + i\zeta(x,y).$$

Let's consider $\mathcal{R}(x,y) = \sqrt{\xi^2(x,y)+\zeta^2(x,y)}$. Then a square root of $\xi(x,y) + i\zeta(x,y)$ is given by

$$\mathcal{Z}(x,y) = \pm\left(\sqrt{\frac{\mathcal{R}(x,y)+\xi(x,y)}{2}} + i\ \frac{\zeta(x,y)}{|\zeta(x,y)|}\sqrt{\frac{\mathcal{R}(x,y)-\xi(x,y)}{2}}\right).$$

Recalling that $\mathcal{Z}(x,y) = f_1(x,y) + if_2(x,y)$, we obtain

$$f_1(x,y) = \pm\sqrt{\frac{\mathcal{R}(x,y)+\xi(x,y)}{2}}, f_2(x,y) = \pm\ \frac{\zeta(x,y)}{|\zeta(x,y)|}\sqrt{\frac{\mathcal{R}(x,y)-\xi(x,y)}{2}}.$$

This end the proof.

Proposition 3.3: For any 1-form given in (7), the sextuple $(\mathbb{H}^2, ds^2, X, \lambda, \rho, \alpha)$ is a η-Ricci-Bourguignon soliton if and only if the components of the vector field $X(x,y) = X^x(x,y)\,\partial_x + X^y(x,y)\partial_y$ are:

$$\begin{cases} X^x(x,y) = \int_{x_0}^{x} a(t)dt + (\lambda - 2\rho + 1)x(1 + \ln(y)) \\ \qquad\qquad +\alpha\int_{x_0}^{x}\left(\int_{y_0}^{y} sf_2^2(t,s)ds\right)dt + \alpha y^2\int_{x_0}^{x} f_1^2(t,y)dt + h(y) \\ X^y(x,y) = a(x)y + (\lambda - 2\rho + 1)y\ln(y) + \alpha y\int_{y_0}^{y} sf_2^2(x,s)ds \end{cases}$$

and

$$\frac{(\lambda-2\rho+1)x}{y} + \alpha\int_{x_0}^{x} yf_2^2(t,y)dt + 2\alpha y\int_{x_0}^{x} f_1^2(t,y)dt + h'(y)$$
$$+\alpha y^2\int_{x_0}^{x}\partial_y f_1^2(t,y)dt + a'(x)y + \alpha y\int_{y_0}^{y}\partial_x f_2^2(x,s)ds = 2\alpha y^2 f_1(x,y)f_2(x,y)$$

where a is a smooth function depending on the variable x and h is a smooth function depending on the variable y.

Proof: The manifold $(\mathbb{H}^2, ds^2)$ si η-Ricci Bourguignon soliton if and only if

$$\begin{cases} \partial_x X^x - \dfrac{X^y}{y} = (\lambda - 2\rho + 1) + \alpha\, y^2\, f_1^2 & (\ell_1) \\ \partial_y X^y - \dfrac{X^y}{y} = (\lambda - 2\rho + 1) + \alpha\, y^2\, f_2^2 & (\ell_2) \\ \partial_y X^x + \partial_x X^y = 2\alpha\, y^2\, f_1 f_2 & (\ell_3) \end{cases}$$

Line (ℓ_2) is a first-order ordinary differential equation in the variable y, whose solution is

$$X^y(x,y) = a(x)y + (\lambda - 2\rho + 1)y\ln(y) + \alpha y\int_{y_0}^{y} sf_2^2(x,s)ds.$$

Inserting this into line (ℓ_1) gives

$$\partial_x X^x(x,y) = a(x) + (\lambda - 2\rho + 1)\ln(y)$$
$$+\alpha\int_{y_0}^{y} sf_2^2(x,s)ds + (\lambda - 2\rho + 1) + \alpha y^2 f_1^2(x,y)$$
$$X^x(x,y) = \int_{x_0}^{x} a(t)dt + (\lambda - 2\rho + 1)x(1 + \ln(y))$$
$$+\alpha\int_{x_0}^{x}\left(\int_{y_0}^{y} sf_2^2(t,s)ds\right)dt + \alpha y^2\int_{x_0}^{x} f_1^2(t,y)dt + h(y).$$

Finally, line (ℓ_3) yields

$$\frac{(\lambda-2\rho+1)x}{y}+\alpha\int_{x_0}^{x} yf_2^2(t,y)dt+2\alpha y\int_{x_0}^{x} f_1^2(t,y)dt+h'(y)$$
$$+\alpha y^2\int_{x_0}^{x}\partial_y f_1^2(t,y)dt+a'(x)y+\alpha y\int_{y_0}^{y}\partial_x f_2^2(x,s)ds$$
$$=2\alpha y^2 f_1(x,y)f_2(x,y).$$

Corollary 3.4: *Under the hypotheses of Proposition 3.3, setting* $f_2(x,y):=0$ *yields that* $(\mathbb{H}^2,ds^2,X,\lambda,\rho,\alpha)$ *is a* η*-Ricci-Bourguignon soliton if and only if the expression*

$$(x,y)\mapsto -y^2a''(x)+2b(x)-2(\lambda-2\rho+1)\ln(y)$$

is strictly positive. In that case, $f_1(x,y)=\pm\frac{\sqrt{-\frac{1}{2}y^2a''(x)+b(x)-(\lambda-2\rho+1)ln(y)}}{y}$, *where* a *and* b *depend only on* x.

Proof: Setting $f_2:=0$ reduces the compatibility condition in Proposition 3.3 to:

$$h'(y)+\frac{(\lambda-2\rho+1)x}{y}+2\alpha y\int_{x_0}^{x} f_1^2(t,y)dt+\alpha y^2\int_{x_0}^{x}\partial_y f_1^2(t,y)dt+a'(x)y=0$$
$$h'(y)+\frac{(\lambda-2\rho+1)x}{y}+\alpha\int_{x_0}^{x}\partial_y(y^2f_1^2(t,y))dt+a'(x)y=0$$
$$\Leftrightarrow\frac{(\lambda-2\rho+1)}{y}+\partial_y(y^2f_1^2(x,y))=-a''(x)y$$
$$\Leftrightarrow f_1^2(x,y)=-\frac{a"(x)}{2}+\frac{b(x)}{y^2}-(\lambda-2\rho+1)\frac{\ln(y)}{y^2}.$$
$$f_1^2(x,y)=\frac{-y^2a"(x)+2b(x)-2(\lambda-2\rho+1)\ln(y)}{2y^2}.$$

This quantity is non-negative if and only if the function

$$(x,y)\mapsto -y^2a''(x)+2b(x)-2(\lambda-2\rho+1)\ln(y)$$

is positive.

Example 3.5: In the Corollary 3.4, we can choose $a(x)=-e^x$, $b(x)=e^{\varphi(x)}$ and $\lambda=2\rho-1$ where φ is a smooth function of x. In this case

$$\left(\mathbb{H}^2,ds^2,\left(-e^x+\alpha\left(\frac{y^2e^x}{2}+\int_{x_0}^{x}e^{\varphi(t)}dt\right)+h(y)\right)\partial_x-ye^x\partial_y,2\rho-1,\rho,\alpha\right)$$

is a η-Ricci-Bourguignon soliton, with $\eta=\pm\frac{\sqrt{\frac{y^2e^x}{2}+e^{\varphi(x)}}}{y}dx$.

Theorem 3.6: *For any constant* 1*-form* $\eta(x,y)=c_1dx+c_2dy$, *the sextulpe* $(\mathbb{H}^2,ds^2,X,\lambda,\rho,\alpha)$ *is a* η*-Ricci-Bourguignon soliton if and only if* $\lambda=2\rho-1$ *and the components of the vector field* $X(x,y)=X^x\partial_x+X^y(x,y)\partial_y$ *are:*

$$\begin{cases}X^x(x,y)=\frac{-\alpha(2c_1^2+c_2^2)}{6}x^3+\frac{c_5}{2}(x^2-y^2)+c_6x+\alpha\left(c_1^2+\frac{c_2^2}{2}\right)xy^2+\frac{2\alpha c_1c_2}{3}y^3+c_7\\ X^y(x,y)=\left(\frac{-\alpha(2c_1^2+c_2^2)}{2}x^2+c_5x+c_6\right)y+\frac{\alpha c_2^2}{2}y^3+\alpha c_3y.\end{cases}$$

Proof: Let $f_1(x,y)=c_1$ and $f_2(x,y)=c_2$ be two constants. We have

$$X^y(x,y)=ya(x)+(\lambda-2\rho+1)y\ln(y)+\frac{\alpha c_2^2}{2}y^3+\alpha c_3 y$$

$$X^x(x,y)=\int_{x_0}^{x}a(t)dt+(\lambda-2\rho+1)x(1+\ln(y))+\frac{\alpha c_2^2}{2}xy^2+\alpha c_3 x$$
$$+\alpha c_1^2 xy^2+c_4y^2+h(y).$$

The constraint gives

$$(\lambda-2\rho+1)\frac{x}{y}+\alpha xy(c_2^2+2c_1^2)+2c_4y+h'(y)+ya'(x)=2\alpha c_1c_2y^2.$$

This last one implies

$$\begin{cases}\lambda=2\rho-1\\ a(x)=\frac{-\alpha(2c_1^2+c_2^2)}{2}x^2+c_5x+c_6\\ h(y)=\frac{2\alpha c_1c_2}{3}y^3-\left(\frac{2c_4+c_5}{2}\right)y^2+c_7.\end{cases}$$

Theorem 3.7: *Let F be a smooth function on $\mathbb{H}^2$. The sextuple $(\mathbb{H}^2,ds^2,\nabla F,\lambda,\rho,\alpha)$ is a gradient η-Ricci-Bourguignon soliton if and only if*

$$F(x,y)=\left(\frac{\lambda-2\rho+1}{2}\right)(\ln(y))^2+a(x)\ln(y)+\alpha\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}sf_2^2(x,s)ds\right)d\tau+b(x)$$

with the constraint

$$\begin{cases}\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}s\,\partial_{xx}f_2^2(x,s)ds\right)d\tau-\frac{\int_{y_0}^{y}sf_2^2(x,s)ds}{y}-f_1^2(x,y)=\frac{\lambda-2\rho+1}{\alpha}\left(\frac{1+\ln(y)}{y^2}\right)\\ -\frac{a''(x)}{\alpha}\ln(y)-\frac{b''(x)}{\alpha}\\ \frac{a\prime(x)}{\alpha}(1+\ln(y))+\frac{b\prime(x)}{\alpha}+\int_{y_0}^{y}s\,\partial_xf_2^2(x,s)ds\\ +\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}s\,\partial_xf_2^2(x,s)ds\right)d\tau-yf_1(x,y)f_2(x,y)=0\end{cases}$$

where $\eta(x,y)=f_1(x,y)dx+f_2(x,y)dy$.

Proof: The sextuple $(\mathbb{H}^2,ds^2,\nabla F,\lambda,\rho,\alpha)$ defines a gradient η-Ricci-Bourguignon soliton if and only if

$$\begin{cases}\frac{\partial^2F}{\partial x^2}-\frac{1}{y}\frac{\partial F}{\partial y}=\frac{\lambda-2\rho+1}{y^2}+\alpha f_1^2 \quad (L_1)\\ \frac{\partial^2F}{\partial y\,\partial x}+\frac{1}{y}\frac{\partial F}{\partial y}=\alpha f_1f_2\\ \frac{\partial^2F}{\partial y^2}+\frac{1}{y}\frac{\partial F}{\partial y}=\frac{\lambda-2\rho+1}{y^2}+\alpha f_2^2 \quad (L_3)\end{cases}$$

Equation (L_3) can be rewritten as

$$\frac{\partial\left(y\frac{\partial F(x,y)}{\partial y}\right)}{\partial_y}=\frac{\lambda-2\rho+1}{y}+\alpha yf_2^2(x,y),$$

which integrates to

$$\Leftrightarrow y\frac{\partial F(x,y)}{\partial_y}=(\lambda-2\rho+1)\ln(y)+\alpha\int_{y_0}^{y}sf_2^2(x,s)ds+a(x),$$

$$F(x,y)=\left(\frac{\lambda-2\rho+1}{2}\right)(\ln(y))^2+a(x)\ln(y)+\alpha\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}sf_2^2(x,s)ds\right)d\tau+b(x).$$

Substituting into (L_1) gives

$$\alpha\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}s\,\partial_{xx}f_2^2(x,s)ds\right)d\tau+a''(x)\ln(y)+b''(x)$$
$$-\frac{1}{y}\left((\lambda-2\rho+1)\frac{\ln(y)}{y}+\alpha\int_{y_0}^{y}sf_2^2(x,s)ds\right)=\frac{\lambda-2\rho+1}{y^2}+\alpha f_1^2$$

$$\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}s\,\partial_{xx}f_2^2(x,s)ds\right)d\tau-\frac{\int_{y_0}^{y}sf_2^2(x,s)ds}{y}-f_1^2(x,y)=(\lambda-2\rho+1)\left(\frac{1+\ln(y)}{\alpha y^2}\right)$$
$$-\frac{a''(x)}{\alpha}\ln(y)-\frac{b''(x)}{\alpha}.$$

Equation (L_2) then becomes

$$\frac{a\prime(x)}{\alpha y}+\frac{1}{y}\int_{y_0}^{y}s\,\partial_x f_2^2(x,s)ds+\frac{a\prime(x)\ln(y)}{\alpha y}+\frac{b\prime(x)}{\alpha y}$$
$$+\frac{1}{y}\int_{\tau_0}^{y}\left(\frac{1}{\tau}\int_{y_0}^{\tau}s\,\partial_x f_2^2(x,s)ds\right)d\tau-f_1(x,y)f_2(x,y)=0.$$

Corollary 3.8: *As consequences of Theorem 3.7, we have:*

1. $f_1(x,y):=0$ if and only if $\frac{\lambda-2\rho+1}{\alpha}\leq 0$. In this case
$$\begin{cases}f_2(x,y)=\pm\dfrac{\sqrt{\frac{-\lambda+2\rho-1}{\alpha}}}{y}\\ F(x,y)=-(\lambda-2\rho+1)\ln(y)+c\end{cases}$$
2. $f_2(x,y):=0$ if and only if the function $y\mapsto\frac{-\lambda+2\rho-1}{\alpha}(1+\ln(y))$ is positive. In this case
$$\begin{cases}f_1(x,y)=\pm\dfrac{\sqrt{\frac{-\lambda+2\rho-1}{\alpha}(1+\ln(y))}}{y}\\ F(x,y)=\left(\frac{\lambda-2\rho+1}{2}\right)(\ln(y))^2+e.\end{cases}$$

Proof:

1. Setting $f_1(x,y):=0$ reduces the system to:
$$\begin{cases}\dfrac{\partial^2F}{\partial x^2}-\dfrac{1}{y}\dfrac{\partial F}{\partial y}=\dfrac{\lambda-2\rho+1}{y^2} & (s_1)\\ \dfrac{\partial^2F}{\partial y\,\partial x}+\dfrac{1}{y}\dfrac{\partial F}{\partial x}=0 & (s_2)\\ \dfrac{\partial^2F}{\partial y^2}+\dfrac{1}{y}\dfrac{\partial F}{\partial y}=\dfrac{\lambda-2\rho+1}{y^2}+\alpha f_2^2 & (s_3)\end{cases}$$
From (s_2) one deduces $\partial_y(y\,\partial_x F(x,y))=0$, hence $F(x,y)=\frac{a(x)}{y}+b(y)$. Then (s_1) gives
$$a''(x)+\frac{a(x)}{y^2}-b'(y)=\frac{\lambda-2\rho+1}{y^2},$$
which forces $a(x)=c$, so $b(y)=-(\lambda-2\rho+1)\ln(y)-\frac{c}{y}+k$ and

$$F(x,y) = -(\lambda - 2\rho + 1)\ln(y) + k.$$

(s_3) then gives

$$f_2^2(x,y) = \frac{-(\lambda-2\rho+1)}{\alpha y^2}.$$

2. For $f_2(x,y) := 0$, the constraints of Theorem 3.7 become:

$$\begin{cases} -f_1^2(x,y) = \frac{\lambda-2\rho+1}{\alpha}\left(\frac{1+\ln(y)}{y^2}\right) - \frac{a''(x)}{\alpha}\ln(y) - \frac{b''(x)}{\alpha} \\ \frac{a\prime(x)}{\alpha}(1+\ln(y)) + \frac{b\prime(x)}{\alpha} = 0 \end{cases}$$

These are satisfied if and only if $a(x) = c$ and $b(x) = e$, giving

$$f_1^2(x,y) = \frac{-\lambda+2\rho-1}{\alpha}\left(\frac{1+\ln(y)}{y^2}\right)$$

and

$$F(x,y) = \left(\frac{\lambda-2\rho+1}{2}\right)(\ln(y))^2 + c\ln(y) + e.$$

The equality $\frac{\partial^2 F}{\partial x^2} - \frac{1}{y}\frac{\partial F}{\partial y} = \frac{\lambda-2\rho+1}{y^2} + \alpha f_1^2$ then yields:

$$-\frac{(\lambda-2\rho+1)\ln(y)}{y^2} - \frac{c}{y^2} = \frac{\lambda-2\rho+1}{y^2} - \frac{\lambda-2\rho+1}{y^2} - \frac{(\lambda-2\rho+1)\ln(y)}{y^2},$$

hence $c = 0$.

Theorem 3.9: *For any non-zero smooth function F on $\mathbb{H}^2$, there does not exist a non-zero constant 1-form η such that $(\mathbb{H}^2, ds^2, \nabla F, \lambda, \rho, \alpha)$ is a gradient η-Ricci-Bourguignon soliton.*

Proof: *Assume for contradiction that $\eta = c_1 dx + c_2 dy$ is a non-zero constant 1-form and F a non-zero smooth function on $\mathbb{H}^2$ such that $(\mathbb{H}^2, ds^2, \nabla F, \lambda, \rho, \alpha)$ is a gradient η-Ricci-Bourguignon soliton. Applying the constraints of Theorem 3.7 gives*

$$\begin{cases} \frac{c_2^2}{2}y + \frac{c}{y} - c_1^2 = \frac{\lambda-2\rho+1}{\alpha}\left(\frac{1+\ln(y)}{y^2}\right) - \frac{a''(x)}{\alpha}\ln(y) - \frac{b''(x)}{\alpha} & (\psi_1) \\ \frac{a\prime(x)}{\alpha}(1+\ln(y)) + \frac{b\prime(x)}{\alpha} - c_1c_2y = 0 & (\psi_2). \end{cases}$$

From (ψ_2) one obtains $a'(x) = b'(x) = c_1c_2 = 0$, so (ψ_1) reduces to $\frac{c_2^2}{2}y^2 + \frac{c}{y} = \frac{\lambda-2\rho+1}{\alpha}\left(\frac{y+\ln(y)}{y}\right)$ or $+\frac{c}{y} = \frac{\lambda-2\rho+1}{\alpha}\left(\frac{y+\ln(y)}{y}\right) + c_1^2$, and in every case $\eta = 0$, a contradiction.

Theorem 3.10: *Let F be a non-zero smooth function on $\mathbb{H}^2$ such that $\eta(x,y) = dF(x,y)$. The sextuple $(\mathbb{H}^2, ds^2, \nabla F, \lambda, \rho, \alpha)$ is not a gradient η -Ricci-Bourguignon soliton.*

Proof: Suppose for contradiction that $(\mathbb{H}^2, ds^2, \nabla F, \lambda, \rho, \alpha)$ is a gradient η-Ricci-Bourguignon soliton with $\eta = dF$, i.e.,

$$\nabla^2 F = (\lambda - 2\rho + 1)ds^2 + \alpha dF \otimes dF.$$

Define $h(x,y) = e^{-\alpha F(x,y)}$; then $\nabla h(x,y) = -\alpha e^{-\alpha F(x,y)} \nabla F(x,y)$ and

$$\begin{aligned}\nabla^2 h(x,y) &= \nabla\left(-\alpha e^{-\alpha F(x,y)} \nabla F(x,y)\right) \\ &= -\alpha e^{-\alpha F(x,y)} \nabla^2 F(x,y) + \alpha^2 e^{-\alpha F(x,y)} dF(x,y) \otimes dF(x,y),\end{aligned}$$

so

$$\nabla^2 h = -\alpha h(\nabla^2 F - \alpha dF \otimes dF).$$

Consequently, $(\mathbb{H}^2, ds^2, \nabla F, \lambda, \rho)$ is a gradient G -Ricci-Bourguignon soliton with $G = \frac{-1}{\alpha h}$. By [3], the potential function must satisfy $F(x,y) = \frac{\frac{a}{2}(x^2+y^2)+a_1x+a_2y+b}{y}$ and $h(x,y) = \frac{\frac{-a}{2}(x^2+y^2)-a_1x-b}{\alpha(\lambda-2\rho+1)y}$. When h is strictly positive, one would need

$$F(x,y) = \frac{\frac{a}{2}(x^2+y^2)+a_1x+a_2y+b}{y} = \frac{-1}{\alpha}\ln\left(\frac{\frac{a}{2}(x^2+y^2)+a_1x+b}{-\alpha(\lambda-2\rho+1)y}\right),$$

which is impossible.

Remark 3.11:

- On $\mathbb{H}^n$, the 1-form η takes the form $\eta(x_1, \dots, x_n) = \sum_{k=1}^{n} f_k(x_1, \dots, x_n)\, dx_k$, and the tensor $\eta \otimes \eta$ has components $(\eta \otimes \eta)_{kl} = f_k f_l$.
- With metric components $g_{ij} = \delta_{ij}/x_n^2$ and $X = \sum X^k \partial_k$, the Lie derivative satisfies $(\mathcal{L}_X \mathbf{g})_{ij} = \frac{1}{x_n^2}(\partial_i X_j + \partial_j X_i - 2\delta_{ij}\frac{X^n}{x_n})$. The η -Ricci-Bourguignon soliton equation on $\mathbb{H}^n$ therefore decomposes into the PDE system:

  $$\frac{1}{x_n^2}(\partial_i X_j + \partial_j X_i - 2\delta_{ij}\frac{X^n}{x_n}) = 2(\lambda + (n-1)(1-\rho n))\frac{\delta_{ij}}{x_n^2} + 2\alpha f_i f_j,$$

  or equivalently, after multiplying through by x_n^2:

  $$\partial_i X_j + \partial_j X_i - 2\delta_{ij}\frac{X^n}{x_n} = 2(\lambda + (n-1)(1-\rho n))\delta_{ij} + 2\alpha x_n^2 f_i f_j.$$

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