

Characterizations of osculating curve under isometry using Darboux frame in Euclidean 3-space

Kuljeet Singh [†]

Sandeep Sharma ^{*}

School of Mathematics

Shri Mata Vaishno Devi University

Katra 182320

Jammu and Kashmir

India

Abstract

In this paper, we present a systematic study on the characterization of osculating curves under isometries using the Darboux frame in place of the Frenet frame. We further investigate the conditions for the invariance of such curves on regular surfaces embedded in Euclidean 3-space under isometric transformations. The present analysis specifically focuses on the application of the Darboux frame $\{\mathcal{T}, \mathcal{P}, \mathcal{U}\}$. Furthermore, we investigate how the position vector of an osculating curve on regular surfaces is affected by isometric transformations, analyzing its behavior under different orientations, specifically along the unit tangent vector $\mathcal{T}$, the unit normal $\mathcal{U}$, and $\mathcal{P} = \mathcal{U} \times \mathcal{T}$.

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Introduction

The study of geometric structures through differential and integral calculus plays an important role in differential geometry. This discipline is concerned with the analysis of both curved and flat geometric objects situated in different ambient spaces. For a space curve in three-dimensional Euclidean space, a moving orthonormal triad can be assigned at every point of the curve. This triad, commonly referred to as the Frenet frame, represented as $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, where $\mathcal{T}$, $\mathcal{N}$, and $\mathcal{B}$ represent the tangent, principal normal, and binormal directions, respectively. At each point on the curve, this frame determines three mutually perpendicular planes, namely, the osculating, normal, and rectifying planes. In the context of differential geometry of curves, it is well established that the

[†] E-mail: kulljeet83@gmail.com

^{*} E-mail: sandeep.greater123@gmail.com (Corresponding Author)

subspaces generated by the vector pairs $\{\mathcal{T}, \mathcal{N}\}$, $\{\mathcal{N}, \mathcal{B}\}$, and $\{\mathcal{B}, \mathcal{T}\}$ correspond to the osculating, normal, and rectifying planes, respectively. The curves that lie entirely within these planes are respectively termed as osculating, normal, and rectifying curves. The geometric properties and characterizations of these families of curves have been extensively investigated in classical and contemporary studies concerning curve and surface theory. For further details, one can refer to [1, 19-21].

The concept of rectifying curves was first established by Chen [2] in 2003. In his work, these curves were characterized by the property that the corresponding position vectors remain within the rectifying planes along the entire curve. Later, in 2005, a more comprehensive analysis was carried out by Chen and Dillen [14], where rectifying curves were examined from the perspectives of centrodes and extremal curves, leading to several geometric characterizations and related results.

The study of space curves plays a significant role in the area of differential geometry and mechanical sciences. Recently, Lone [15], Singh and Sharma [9,10], Shaikh and Ghosh [4, 5, 8] and Shaikh et al. [6, 7] studied rectifying, osculating, and normal curves in three-dimensional Euclidean spaces on smooth surfaces under isometric transformations using the Serret-Frenet frame. The investigation of osculating and rectifying curves in the three-dimensional space E^3 has also been carried out in [22, 23]. In [22], the authors examined osculating curves with certain characterizations in E^3 , and subsequently provided a definition of an osculating curve in E^4 . Such curves have position vectors primarily confined to their associated osculating planes, satisfying the equation

$$\beta(\xi) = \mu_1 \mathcal{T} + \mu_2 \mathcal{N} + \mu_3 \mathcal{B}_1,$$

where μ_1 , μ_2 , and μ_3 are smooth functions.

The investigation of rectifying curves occupies a prominent place in [14], where they are studied as extremal curves. Furthermore, the authors examine rectifying curves by considering the dilation of curves with unit-speed parameterization on the sphere S^2 of radius one in three-dimensional space. This study leads to a significant result: a clear and comprehensive criterion is established that determines when the centrode $c(\xi)$ of a unit-speed curve, referred to as $\beta(\xi)$, intrinsically becomes a rectifying curve [12].

Moreover, the investigation continues in [23], where the authors introduce a novel characterization of rectifying curves in Euclidean 4-space. Specifically, a rectifying curve in this context is defined by a fundamental geometric property: the curve's position vector consistently resides within the orthogonal complement $\mathcal{N}^\perp$ of its principal normal vector field $\mathcal{N}$. In this way, the authors extend the concept of rectifying curves by embedding them in a higher-dimensional space and highlighting their intrinsic connection to the orthogonal complement of their principal normal vectors.

In the work presented in [4, 8, 11, 13], the investigation is centered on rectifying and osculating curves lying on a smooth surface. The authors carry out a detailed analysis to determine the conditions under which these types of curves retain their invariance under surface isometries. The Frenet frame serves

as the analytical framework for this study. Conversely, when examining a space curve lying on a regular surface in three-dimensional space, a moving orthonormal frame, known as the Darboux frame $\{\mathcal{T}, \mathcal{P}, \mathcal{U}\}$ naturally arises at each point of the curve. Here, $\mathcal{T}$ denotes the unit tangent vector to the curve, $\mathcal{U}$ represents the unit surface normal, and $\mathcal{P}$ is the cross product of $\mathcal{U}$ and $\mathcal{T}$.

In [16, 18], the authors used the Darboux frame to describe the position vector of a unit-speed curve in three-dimensional space. This position vector consistently lies in the planes spanned by $\{\mathcal{T}, \mathcal{U}\}$, $\{\mathcal{T}, \mathcal{P}\}$, and $\{\mathcal{P}, \mathcal{U}\}$, respectively. In [2, 3, 4, 11, 18, 22], osculating and rectifying curves, along with their properties, have been studied using the Frenet frame. Motivated by these works, we introduce the concept of an osculating curve lying on a regular surface in three-dimensional Euclidean space. It is well known that the Frenet frame is defined along space curves and depends only on the intrinsic properties of the curve. In contrast, the Darboux frame naturally incorporates the geometry of the underlying surface. This feature enables us to analyze osculating curves in relation to surface characteristics such as geodesic and normal curvatures. Consequently, the Darboux frame provides deeper geometric insight, particularly in the study of invariance under isometric transformations, that cannot be captured as explicitly within the Frenet framework. This motivates us to investigate the conditions under which an osculating curve on a surface remains invariant under isometries. To achieve this, we employ the Darboux frame instead of the Frenet frame. In this paper, we use the Darboux frame to characterize osculating curves in three-dimensional Euclidean space under isometries between surfaces.

Preliminaries

In this section, we recall some basic definitions and concepts of orthonormal frames, and present key insights into osculating and rectifying curves. It covers their distinctive characteristics, as well as their geodesic and normal curvature properties, and introduces the fundamental definitions, notations, and symbols used throughout the paper. Osculating curves, which are fundamental in differential geometry, closely approximate the curvature and trajectory of a given curve at a specific point, thereby providing insight into its behaviour. The inclusion of these basic definitions and symbols establishes a solid foundation for understanding osculating and rectifying curves and their implications.

Consider M and $\tilde{M}$ are two regular surfaces immersed in $\mathbb{R}^3$, and let $\beta: I \rightarrow \mathbb{R}^3$ be a unit-speed parametrized curves, where $I = (t_1, t_2) \subset \mathbb{R}$, whose components are of class C^4 on the interval I . The tangent vector $\mathcal{T}$ is defined as $\mathcal{T}(\xi) = \beta'(\xi)$ for each $\xi \in I$, with β' representing the derivative of β in term of the arc length ξ . The unit normal vector associated with β is denoted by $\mathcal{N}$. Consequently, the binormal vector $\mathcal{B}$ is derived as the cross product of $\mathcal{T}$ and $\mathcal{N}$, given by $\mathcal{B} = \mathcal{T} \times \mathcal{N}$. In [1-3, 11, 17, 20], the Serret-Frenet equations can be expressed in the following manner:

$$\begin{aligned}
\mathcal{T}'(\xi) &= \kappa(\xi)\mathcal{N}(\xi), \\
\mathcal{N}'(\xi) &= -\kappa(\xi)\mathcal{T}(\xi) + \tau(\xi)\mathcal{B}(\xi), \\
\mathcal{B}'(\xi) &= -\tau(\xi)\mathcal{N}(\xi),
\end{aligned}$$

where τ and κ are the torsion function and curvature function of the curve β , respectively. These functions satisfy the following conditions

$$\mathcal{T}(\xi) = \beta'(\xi), \quad \mathcal{N}(\xi) = \frac{\mathcal{T}'(\xi)}{\kappa(\xi)}, \quad \mathcal{B}(\xi) = \mathcal{T}(\xi) \times \mathcal{N}(\xi).$$

From an arbitrary point $\beta(\xi)$ along the curve β , the plane formed by $\{\mathcal{T}, \mathcal{N}\}$ is recognized as osculating plane, while the plane defined by $\{\mathcal{T}, \mathcal{B}\}$ is denoted as rectifying plane. Likewise, the plane spanned by $\{\mathcal{N}, \mathcal{B}\}$ is named as normal plane [1, 2]. In the analysis of a curve's positional vector, which distinguishes between different types of curves [1, 2, 20], the categorization of curves unfolds. A curve is classified as a normal curve if its positional vector situated within the normal plane. Similarly, a curve earns the designation of a rectifying curve if its positional vector resides in the rectifying plane. Moreover, when a curve's positional vector occupies the osculating plane, it assumes the identity of an osculating curve.

Let $\Phi: V \subset \mathbb{R}^2 \rightarrow M$ representing the coordinate map, and $\beta: I \rightarrow M \in \mathbb{R}^3$ be a parametrized curve situated within range of surface patch Φ of M . Then we can write $\beta(\xi)$ as

$$\beta(\xi) = \Phi(x(\xi), y(\xi)), \quad \forall \xi \in I. \quad (1)$$

When the curve $\beta(\xi)$ lies on the surface M , a distinct moving orthonormal frame known as the Darboux frame $\{\mathcal{T}, \mathcal{P}, \mathcal{U}\}$ can be associated with each point along the curve $\beta(\xi)$. It's important to note that the unit tangent vector $\mathcal{T}$ remains consistent in both the Darboux frame and the Frenet frame, and the vectors $\mathcal{N}, \mathcal{B}, \mathcal{P}, \mathcal{U}$ all lie within the same plane.

As a result, the relationship between these frames is described by the following equation:

$$\begin{bmatrix} \mathcal{T} \\ \mathcal{P} \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \mathcal{T} \\ \mathcal{N} \\ \mathcal{B} \end{bmatrix}, \quad (2)$$

where α represents the angle between the vectors $\mathcal{N}$ and $\mathcal{P}$.

Since $\beta(\xi)$ is a parametrized curve with unit-speed residing on the surface M , therefore, it follows that β'' is perpendicular to β , thus forming a linear combination involving $\mathcal{U}$ and $\mathcal{P}$. Consequently, we can express this as

$$\beta''(\xi) = k_n(\xi)\mathcal{U}(\xi) + k_g(\xi)\mathcal{P}(\xi), \quad (3)$$

where the function k_n and k_g represent normal curvature and geodesic curvature of β . As we know that both vectors $\mathcal{U}$ and $\mathcal{P}$ are perpendicular, so we can obtain the following relationships from equation (3)

$$k_n(\xi) = \beta''(\xi) \cdot \mathcal{U}(\xi) \quad \text{and} \quad k_g(\xi) = \beta''(\xi) \cdot \mathcal{P}(\xi). \quad (4)$$

Also, from Frenet-Serret equation and (4), we get

$$k_n(\xi) = \kappa(\xi)\mathcal{N}(\xi) \cdot \mathcal{U}(\xi) \quad \text{and} \quad k_g(\xi) = \kappa(\xi)\mathcal{N}(\xi) \cdot \mathcal{P}(\xi), \quad (5)$$

implies that

$$k_n(\xi) = \kappa(\xi)\sin\alpha \quad \text{and} \quad k_g(\xi) = \kappa(\xi)\cos\alpha. \quad (6)$$

Consequently, the curve β is geodesics if and only if the curvature k_g is zero, and in the same manner β is an asymptotic curve if and only if the curvature k_n is zero.

Take derivative of equation (1) in term of ξ , we obtain

$$\mathcal{T}(\xi) = \beta'(\xi) = x'\Phi_x + y'\Phi_y. \quad (7)$$

Now, the normal $\mathcal{U}$ to the surface M is defined as follows

$$\mathcal{U}(\xi) = \frac{\Phi_x \times \Phi_y}{\|\Phi_x \times \Phi_y\|} = \frac{\Phi_x \times \Phi_y}{\sqrt{EG-F^2}}, \quad (8)$$

where E , F , and G are the coefficients of the first fundamental form. It is given that $\mathcal{P} = \mathcal{U} \times \mathcal{T}$, we can use equations (7) and (8) along with simplification to derive

$$\mathcal{P}(\xi) = \frac{1}{\sqrt{EG-F^2}}(Ex'\Phi_y + F(y'\Phi_y - x'\Phi_x) - Gy'\Phi_x). \quad (19)$$

Definition 1: Consider two regular surfaces, M and $\tilde{M}$, in $\mathbb{R}^3$. Then a diffeomorphism J from M to $\tilde{M}$ represents an isometry, if it preserves the lengths of curves, i.e., mapping curves of the same length from M to $\tilde{M}$.

Definition 2: [1] The quadratic form that represents the first fundamental form of a regular immersed surface M at a point q is denoted as $J_q : T_q(M) \rightarrow \mathbb{R}$, and it is expressed as follows

$$J_q(\beta'(\xi)) = \langle \beta'(\xi), \beta'(\xi) \rangle.$$

Definition 3: An isometry J existing between two surfaces M and $\tilde{M}$ maintains the geodesics of one surface equivalently on the other. The coefficients of the first fundamental form for surface M —noted as E , F , and G , alongside those for surface $\tilde{M}$ are denoted as $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ are integral to this concept. The constancy of these coefficients stands as both a necessary and sufficient condition for the isometric equivalence of surfaces M and $\tilde{M}$, thus summarized as follows

$$\tilde{E} = E, \tilde{F} = F, \text{ and } \tilde{G} = G. \quad (10)$$

Osculating Curves in the Context of the Darboux Frame

This section includes an analysis of osculating curves on a regular surface using the Darboux frame. A curve $\beta(\xi)$ lying on a regular surface M , immersed in three-dimensional Euclidean space, is called an osculating curve if its position vector always lies in the osculating plane. Consequently, the position vector of β can be expressed as follows:

$$\beta(\xi) = \mu_1(\xi)\mathcal{T}(\xi) + \mu_2(\xi)\mathcal{N}(\xi), \quad (11)$$

where $\mu_1(\xi)$ and $\mu_2(\xi)$ are differential functions.

By substituting equation (2) into equation (11), we obtain

$$\beta(\xi) = \mu_1(\xi)\mathcal{T}(\xi) + \mu_2(\xi)\mathcal{P}(\xi)\cos\alpha - \mu_2(\xi)\mathcal{U}(\xi)\sin\alpha. \quad (12)$$

Thus, in view of equations (6), (7), (8), (9) and (12), we obtain

$$\begin{aligned}\beta(\xi) = & \mu_1(\xi)(x'\Phi_x + y'\Phi_y) + \\ & \mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(Ex'\Phi_y + F(y'\Phi_y - x'\Phi_x) - Gy'\Phi_x) \\ & - \mu_2(\xi) \frac{k_n(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(\Phi_x \times \Phi_y).\end{aligned}\quad (13)$$

This equation represents the fundamental form of an osculating curve on a regular surface—one that is neither a geodesic nor an asymptotic curve on the surface. Here, two distinct cases arise, which are given as follows:

Case 1: When the osculating curve on the regular surface is a geodesic curve, i.e., $k_g(\xi) = 0$, the angle becomes $\pi/2$. Moreover, since $k_n(\xi) = \kappa(\xi)$, the equation describing the osculating curve simplifies to

$$\beta(\xi) = \mu_1(\xi)(x'\Phi_x + y'\Phi_y) - \mu_2(\xi) \frac{1}{\sqrt{EG-F^2}}(\Phi_x \times \Phi_y). \quad (14)$$

Case 2: When the osculating curve on the regular surface takes the form of an asymptotic line, i.e., $k_n(\xi) = 0$, the angle becomes 0. In this case, $k_g(\xi) = \kappa(\xi)$. Consequently, the equation describing the osculating curve simplifies to

$$\beta(\xi) = \mu_1(\xi)(x'\Phi_x + y'\Phi_y) + \mu_2(\xi) \frac{1}{\sqrt{EG-F^2}}(Ex'\Phi_y + F(y'\Phi_y - x'\Phi_x) - Gy'\Phi_x). \quad (15)$$

We consider the derivative map of $\beta(\xi)$ i.e., $J_*(\beta(\xi))$, as the product of a 3×3 matrix J_* and a 3×1 matrix $\beta(\xi)$ in the following theorem.

Theorem: 1. Let M and $\tilde{M}$ be two regular surfaces in $\mathbb{R}^3$, and $J: M \rightarrow \tilde{M}$ be an isometry, $\beta(\xi)$ represents an osculating curve on surface M with the property $k_n \neq 0$. If any one of the following conditions are hold, then $\tilde{\beta} = J \circ \beta$ is also an osculating curve on the surface $\tilde{M}$;

1. $\tilde{\beta}$ represents a geodesic curve on the surface $\tilde{M}$ and $\tilde{\beta}(\xi) = J_*(\beta(\xi))$,
2. $\tilde{\beta}$ represents an asymptotic curve on the surface $\tilde{M}$ and $\tilde{\beta}(\xi) = J_*(\beta(\xi)) + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}\tilde{u}(\xi)$,
3. $\tilde{\beta}$ neither represents a geodesic curve nor an asymptotic curve on the surface $\tilde{M}$ and $\tilde{\beta}(\xi) = J_*(\beta(\xi))$.

Proof: It is given that $J: M \rightarrow \tilde{M}$ be an isometric transformation between surfaces M and $\tilde{M}$ immersed in the three dimensional space $\mathbb{R}^3$, and $\beta(\xi)$ represents an osculating curve on M such that $k_n \neq 0$. Assume that condition (i) holds. Then by definition of geodesic curve, we have $\tilde{k}_g(\xi) = 0$, and $\tilde{\beta}(\xi) = J_*(\beta(\xi))$.

Now, we can write

$$\begin{aligned}\tilde{\beta}(\xi) &= J_*(\beta(\xi)), \\ &= \mu_1(\xi)(x'J_*\Phi_x + y'J_*\Phi_y) - \mu_2(\xi)(J_*\mathcal{U}(\xi)).\end{aligned}\quad (16)$$

Now, from equation (10) and (16), we find that

$$\tilde{\beta}(\xi) = \tilde{\mu}_1(\xi)(x'\tilde{\Phi}_x + y'\tilde{\Phi}_y) - \tilde{\mu}_2(\xi)(\tilde{\mathcal{U}}(\xi)), \quad (17)$$

where $\tilde{\mu}_1(\xi) = \mu_1(\xi)$ and $\tilde{\mu}_2(\xi) = \mu_2(\xi)$. As $\tilde{k}_g(\xi) = 0$, thus from equation (17), we get

$$\tilde{\beta}(\xi) = \tilde{\mu}_1(\xi)(x'\tilde{\Phi}_x + y'\tilde{\Phi}_y) - \tilde{\mu}_2(\xi)(\tilde{\mathcal{U}}(\xi)). \quad (18)$$

This is the required equation for an osculating curve on the surface $\tilde{M}$ that exhibits geodesic behaviour on the same surface.

Assume that condition (ii) holds. Then $\tilde{k}_n(\xi) = 0$, and $\tilde{\beta}(\xi) = J_*(\beta(\xi)) + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}\tilde{\mathcal{U}}(\xi)$.

Now,

$$\begin{aligned}\tilde{\beta}(\xi) &= J_*(\beta(\xi)) + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}\tilde{\mathcal{U}}(\xi), \\ &= \mu_1(\xi)(x'J_*\Phi_x + y'J_*\Phi_y) \\ &\quad + \mu_2(\xi)\frac{k_g(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(Ex'J_*\Phi_y + F(y'J_*\Phi_y - x'J_*\Phi_x) - Gy'J_*\Phi_x) \\ &\quad - \mu_2(\xi)\frac{k_n(\xi)}{\kappa(\xi)}(J_*\mathcal{U}(\xi)) + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}\tilde{\mathcal{U}}(\xi).\end{aligned}\quad (19)$$

Again, from equation (10) and (19), we find that

$$\begin{aligned}\tilde{\beta}(\xi) &= \tilde{\mu}_1(\xi)(x'\tilde{\Phi}_x + y'\tilde{\Phi}_y) + \frac{\tilde{\mu}_2(\xi)}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}x'\tilde{\Phi}_y + \tilde{F}(y'\tilde{\Phi}_y - x'\tilde{\Phi}_x) - \tilde{G}y'\tilde{\Phi}_x) \\ &\quad - \mu_2(\xi)\frac{k_n(\xi)}{\kappa(\xi)}(\tilde{\mathcal{U}}(\xi)) + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}\tilde{\mathcal{U}}(\xi),\end{aligned}\quad (20)$$

where $\tilde{\mu}_1(\xi) = \mu_1(\xi)$ and $\tilde{\mu}_2(\xi) = \frac{\mu_2(\xi)k_g(\xi)}{k_1(\xi)}$. As $\tilde{k}_n(\xi) = 0$, thus from equation (20), we get

$$\tilde{\beta}(\xi) = \tilde{\mu}_1(\xi)(x'\tilde{\Phi}_x + y'\tilde{\Phi}_y) + \frac{\tilde{\mu}_2(\xi)}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}x'\tilde{\Phi}_y + \tilde{F}(y'\tilde{\Phi}_y - x'\tilde{\Phi}_x) - \tilde{G}y'\tilde{\Phi}_x).$$

This is the required equation of an osculating curve on the surface $\tilde{M}$, which is an asymptotic line on that surface.

Next, assume that condition (iii) holds. Then $\tilde{k}_n(\xi) \neq 0$, $\tilde{k}_g(\xi) \neq 0$, and $\tilde{\beta}(\xi) = J_*(\beta(\xi))$.

Now, we can write

$$\begin{aligned}\tilde{\beta}(\xi) &= J_*(\beta(\xi)), \\ &= \mu_1(\xi)(x'J_*\Phi_x + y'J_*\Phi_y) \\ &\quad + \frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(Ex'J_*\Phi_y + F(y'J_*\Phi_y - x'J_*\Phi_x) - Gy'J_*\Phi_x) \\ &\quad - \mu_2(\xi)\frac{k_n(\xi)}{\kappa(\xi)}(J_*\mathcal{U}(\xi)).\end{aligned}\quad (21)$$

Now, from equation (10) and (21), we obtain

$$\begin{aligned}\tilde{\beta}(\xi) &= \tilde{\mu}_1(\xi)(x'\tilde{\Phi}_x + y'\tilde{\Phi}_y) \\ &\quad + \frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}x'\tilde{\Phi}_y + \tilde{F}(y'\tilde{\Phi}_y - x'\tilde{\Phi}_x) - \tilde{G}y'\tilde{\Phi}_x) \\ &\quad - \tilde{\mu}_2(\xi)\frac{\tilde{k}_n(\xi)}{\tilde{\kappa}_1(\xi)}(\tilde{\mathcal{U}}(\xi)),\end{aligned}\quad (22)$$

where $\tilde{\mu}_1(\xi) = \mu_1(\xi)$, $\frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)} = \frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)}$, and $\frac{\tilde{\mu}_2(\xi)\tilde{k}_n(\xi)}{\tilde{\kappa}_1(\xi)} = \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}$. This equation represents the necessary form of an osculating curve on the surface $\tilde{M}$, which is neither an asymptotic nor a geodesic curve on that surface.

Theorem 2: Let M and $\tilde{M}$ be two regular surfaces in $\mathbb{R}^3$, and $J: M \rightarrow \tilde{M}$ be an isometry, $\beta(\xi)$ represents an osculating curve on surface M with the property $k_n = 0$. If any of the following conditions are met, then $\tilde{\beta} = J \circ \beta$ is also an osculating curve on the surface $\tilde{M}$;

1. $\tilde{\beta}$ does not represent an asymptotic line on the surface $\tilde{M}$ and $\tilde{\beta}(\xi) = J_*(\beta(\xi)) + \tilde{\mu}_2(\xi)\tilde{\mathcal{U}}(\xi)$,
2. $\tilde{\beta}$ represents asymptotic line on the surface $\tilde{M}$ and $\tilde{\beta}(\xi) = J_*(\beta(\xi))$.

Proof: The proof of this theorem follows the same approach as that of theorem 1. By setting $k_n = 0$, and utilizing the provided condition, we arrive at the desired outcome.

Theorem 3: Let M and $\tilde{M}$ be two regular surfaces immersed in the three dimensional Euclidean space $\mathbb{R}^3$, and $J: M \rightarrow \tilde{M}$ represents an isometry. If $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on M and $\tilde{M}$, such that $k_n \neq 0$, and $\mathcal{T}(\xi) = a\Phi_x + b\Phi_y$ represents any tangent vector at the point $\beta(\xi)$ on the surface M . Then the following holds;

1. If the surface $\tilde{M}$ has a geodesic curve $\tilde{\beta}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$,
2. If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$,
3. If the curve $\tilde{\beta}$ is neither an asymptotic nor a geodesic curve on the surface $\tilde{M}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$.

Proof: Let β and $\tilde{\beta}$ be the osculating curves on the surfaces M and $\tilde{M}$, with $k_n \neq 0$, and J represents an isometric transformation between regular surfaces. Also, we have $\mathcal{T}(\xi) = a\Phi_x + b\Phi_y$ represents any tangent vector at the point $\beta(\xi)$ on the surface M . Then

$$\begin{aligned}\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) - \beta(\xi) \cdot \mathcal{T}(\xi) &= \tilde{\beta}(\xi) \cdot (a\tilde{\Phi}_x + b\tilde{\Phi}_y) - \beta(\xi) \cdot (a\Phi_x + b\Phi_y), \\ &= a(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x) + b(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y).\end{aligned}\quad (23)$$

Now,

$$\begin{aligned}
\beta(\xi) \cdot \Phi_x &= \mu_1(\xi)(x'\Phi_x + y'\Phi_y) \cdot \Phi_x \\
&\quad + \mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(Ex'\Phi_y + F(y'\Phi_y - x'\Phi_x) - Gy'\Phi_x) \cdot \Phi_x \\
&\quad - \mu_2(\xi) \frac{k_n(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(\Phi_x \times \Phi_y) \cdot \Phi_x, \\
&= \mu_1(\xi)(x'E + y'F) + \mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)\sqrt{EG-F^2}}(F^2 - EG)y', \\
&= \mu_1(\xi)(x'E + y'F) - \mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)}y'\sqrt{EG-F^2}.
\end{aligned} \tag{24}$$

Similarly, we can obtain

$$\beta(\xi) \cdot \Phi_y = \mu_1(\xi)(x'F + y'G) + \mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)}x'\sqrt{EG-F^2}. \tag{25}$$

For (i), if $\tilde{\beta}$ is geodesic curve on the surface $\tilde{M}$, then $\tilde{k}_g = 0$, and thus from equation (10) and (14), we get

$$\beta(\xi) \cdot \Phi_x = \mu_1(\xi)(x'E + y'F).$$

And

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x = \tilde{\mu}_1(\xi)(x'E + y'F). \tag{26}$$

Similarly, we can obtain

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y = \tilde{\mu}_1(\xi)(x'F + y'G) \tag{27}$$

Now, from equation (26) we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x = (\tilde{\mu}_1(\xi) - \mu_1(\xi))(x'E + y'F). \tag{28}$$

Similarly, from equation (27) we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y = (\tilde{\mu}_1(\xi) - \mu_1(\xi))(x'F + y'G). \tag{29}$$

Using equation (28) and (29) in (23), we obtain

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) - \beta(\xi) \cdot \mathcal{T}(\xi) = (\tilde{\mu}_1(\xi) - \mu_1(\xi))((aE + bF)x' + (aF + bG)y').$$

Since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively so $\mu_1(\xi) = \tilde{\mu}_1(\xi)$, and hence $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$. This proves (i).

Moving on to point (ii), let's assume that $\tilde{\beta}$ represents an asymptotic curve on the surface $\tilde{M}$, then $\tilde{k}_n = 0$ and $\tilde{k}_g(\xi) = \tilde{\kappa}_1(\xi)$. Thus from (10) and (15), we respectively get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x = \mu_1(\xi)(x'E + y'F) - \tilde{\mu}_2(\xi)y'\sqrt{EG-F^2} \tag{30}$$

And

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y = \mu_1(\xi)(x'F + y'G) + \tilde{\mu}_2(\xi)x'\sqrt{EG-F^2}. \tag{31}$$

Now, by using equation (24) and (30) we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x = y'(\sqrt{EG - F^2})(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi)). \quad (32)$$

Similarly, by using (25) and (31), we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y = -x'(\sqrt{EG - F^2})(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi)). \quad (33)$$

Now, from equation (23), we get

$$\begin{aligned} \tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) - \beta(\xi) \cdot \mathcal{T}(\xi) &= a(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x) \\ &\quad + b(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y), \\ &= a(y'(\sqrt{EG - F^2})(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi))) \\ &\quad + b(-x'(\sqrt{EG - F^2})(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi))), \\ &= (\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi))(ay' - bx')(\sqrt{EG - F^2}). \end{aligned}$$

Since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively so $\tilde{\mu}_2(\xi) = \frac{\mu_2(\xi)k_g(\xi)}{k_1(\xi)}$, and hence $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$. This proves (ii).

For (iii), if the curve $\tilde{\beta}$ is neither an asymptotic nor a geodesic curve on the surface $\tilde{M}$, i.e., $\tilde{k}_g(\xi) \neq 0$, and $\tilde{k}_n(\xi) \neq 0$, then from (10) and (13), we obtain

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x = \mu_1(\xi)(x'E + y'F) - \tilde{\mu}_2(\xi) \frac{\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)} y' \sqrt{EG - F^2},$$

And

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y = \mu_1(\xi)(x'F + y'G) + \tilde{\mu}_2(\xi) \frac{\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)} x' \sqrt{EG - F^2}.$$

Now from (24), we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x = y'(\sqrt{EG - F^2})\left(\frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)} - \frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)}\right),$$

and similarly from (25), we get

$$\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y = -x'(\sqrt{EG - F^2})\left(\frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)} - \frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)}\right).$$

Now, from equation (23), we get

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) - \beta(\xi) \cdot \mathcal{T}(\xi) = \left(\frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)} - \frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)}\right)(ay' - bx')(\sqrt{EG - F^2}).$$

Since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively so $\frac{\tilde{\mu}_2(\xi)\tilde{k}_g(\xi)}{\tilde{\kappa}_1(\xi)} = \frac{\mu_2(\xi)k_g(\xi)}{\kappa(\xi)}$, and hence $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{T}}(\xi) = \beta(\xi) \cdot \mathcal{T}(\xi)$. This proves (iii).

Theorem 4: Let M and $\tilde{M}$ be two regular surfaces immersed in three dimensional space $\mathbb{R}^3$, and $J: M \rightarrow \tilde{M}$ represents an isometry. If $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on M and $\tilde{M}$, such that $k_n \neq 0$ and $\mathcal{P} = \mathcal{U} \times \mathcal{T}$ where,

$\mathcal{T}(\xi) = a\Phi_x + b\Phi_y$ represents any tangent vector at the point $\beta(\xi)$ on the surface M . Then the following holds;

1. If the surface $\tilde{M}$ has a geodesic curve $\tilde{\beta}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) = \beta(\xi) \cdot \mathcal{P}(\xi)$.
2. If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) = \beta(\xi) \cdot \mathcal{P}(\xi)$.

Proof: Let β and $\tilde{\beta}$ be the osculating curves on the surfaces M and $\tilde{M}$, with $k_n \neq 0$, and an isometric transformation J from surface M to $\tilde{M}$. We know that $\mathcal{P} = \mathcal{U} \times \mathcal{T}$ where, $\mathcal{T}(\xi) = a\Phi_x + b\Phi_y$ represents any tangent vector at the point $\beta(\xi)$ on the surface M . Then, by using (8), we have

$$\begin{aligned} \beta(\xi) \cdot \mathcal{P}(\xi) &= \beta(\xi) \cdot (\mathcal{U}(\xi) \times \mathcal{T}(\xi)) = \beta(\xi) \cdot \left\{ \left(\frac{\Phi_x \times \Phi_y}{\sqrt{EG-F^2}} \right) \times (a\Phi_x + b\Phi_y) \right\}, \\ &= \frac{1}{\sqrt{EG-F^2}} \{ (aE + bF)\beta(\xi) \cdot \Phi_y - (aF + bG)\beta(\xi) \cdot \Phi_x \}. \end{aligned} \quad (34)$$

Similarly, by using (10), we obtain

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) = \frac{1}{\sqrt{EG-F^2}} \{ (aE + bF)\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - (aF + bG)\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x \}. \quad (35)$$

Now,

$$\begin{aligned} \tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) - \beta(\xi) \cdot \mathcal{P}(\xi) &= \frac{1}{\sqrt{EG-F^2}} \{ (aE + bF)(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_y - \beta(\xi) \cdot \Phi_y) \\ &\quad - (aF + bG)(\tilde{\beta}(\xi) \cdot \tilde{\Phi}_x - \beta(\xi) \cdot \Phi_x) \}. \end{aligned} \quad (36)$$

For (i), if $\tilde{\beta}$ is geodesic curve on the surface $\tilde{M}$, then by using (28) and (29) in (36), we obtain

$$\begin{aligned} \tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) - \beta(\xi) \cdot \mathcal{P}(\xi) &= \frac{1}{\sqrt{EG-F^2}} \{ (aE + bF) \\ &\quad ((\tilde{\mu}_1(\xi) - \mu_1(\xi))(x'F + y'G)) \\ &\quad - (aF + bG)((\tilde{\mu}_1(\xi) - \mu_1(\xi))(x'E + y'F)) \} \\ &= \frac{(\tilde{\mu}_1(\xi) - \mu_1(\xi))}{\sqrt{EG-F^2}} \{ (aE + bF)(x'F + y'G) \\ &\quad - (aF + bG)(x'E + y'F) \}. \end{aligned}$$

Since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively so $\mu_2(\xi) = \tilde{\mu}_2(\xi)$, and hence, $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) = \beta(\xi) \cdot \mathcal{P}(\xi)$. This proves (i).

Moving on to point (ii), let's assume that $\tilde{\beta}$ represents an asymptotic curve on the surface $\tilde{M}$. Then, by using (32) and (33) in (36), we obtain

$$\begin{aligned} \tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) - \beta(\xi) \cdot \mathcal{P}(\xi) &= \frac{1}{\sqrt{EG-F^2}} \{ -(aE + bF)x' \\ &\quad (\sqrt{EG-F^2})(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi)) \} \end{aligned}$$

$$\begin{aligned}
& -(aF + bG)y'(\sqrt{EG - F^2}) \\
& (\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi)), \\
= & -(\mu_2(\xi) \frac{k_g(\xi)}{\kappa(\xi)} - \tilde{\mu}_2(\xi)) \\
& \{(aE + bF)x' + (aF + bG)y'\}.
\end{aligned}$$

Since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively. Hence, $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{P}}(\xi) = \beta(\xi) \cdot \mathcal{P}(\xi)$. This proves (ii). $\square$

Theorem 5: Let M and $\tilde{M}$ be two regular surfaces immersed in three dimensional space $\mathbb{R}^3$ and $J: M \rightarrow \tilde{M}$ represents an isometry. If $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on M and $\tilde{M}$, then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{U}}(\xi) = \beta(\xi) \cdot \mathcal{U}(\xi)$.

Proof: Let β and $\tilde{\beta}$ be the osculating curves on the surfaces M and $\tilde{M}$, and J denotes an isometric transformation M and $\tilde{M}$. Now, by using (8) and (13), we can write

$$\beta(\xi) \cdot \mathcal{U}(\xi) = \beta(\xi) \cdot \frac{\Phi_x \times \Phi_y}{\sqrt{EG - F^2}} = -\frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}, \quad (37)$$

And

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{U}}(\xi) = -\frac{\tilde{\mu}_2(\xi)\tilde{k}_n(\xi)}{\tilde{\kappa}_1(\xi)}. \quad (38)$$

Now, by using (37) and (38), we obtain

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{U}}(\xi) - \beta(\xi) \cdot \mathcal{U}(\xi) = -\frac{\tilde{\mu}_2(\xi)\tilde{k}_n(\xi)}{\tilde{\kappa}_1(\xi)} + \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}. \quad (39)$$

Thus, if $\tilde{k}_g = 0$ in (39), then $\tilde{\beta}(\xi) \cdot \tilde{\mathcal{U}}(\xi) = \beta(\xi) \cdot \mathcal{U}(\xi)$. Further, for $\tilde{k}_g \neq 0$, since $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are osculating curve on the surfaces M and $\tilde{M}$, respectively, therefore $\frac{\tilde{\mu}_2(\xi)\tilde{k}_n(\xi)}{\tilde{\kappa}_1(\xi)} = \frac{\mu_2(\xi)k_n(\xi)}{\kappa(\xi)}$ and hence from (39) we get

$$\tilde{\beta}(\xi) \cdot \tilde{\mathcal{U}}(\xi) = \beta(\xi) \cdot \mathcal{U}(\xi).$$

Conclusion

This work established sufficient conditions under which an osculating curve preserves its geometric character when the underlying regular surfaces are related by an isometric transformation, with the analysis being carried out through the Darboux frame. Furthermore, the study revealed that certain scalar components associated with the position vector of an osculating curve $\beta(\xi)$ remain unaffected by the isometry whenever the curve is either geodesic or asymptotic. More precisely, these invariant quantities correspond to the components measured along the directions determined by $\mathcal{T}$, $\mathcal{U}$, and the vector $\mathcal{P} = \mathcal{U} \times \mathcal{T}$.

One can introduced some other geometric quantities associated with other special classes of curves, including normal and rectifying curves by using Darboux frame in Euclidean-4-spaces. Future investigations may also focus on the influence of conformal transformations between surfaces on different curvature measures associated with osculating, normal, and rectifying curves, such as mean curvature, Gaussian curvature, and sectional curvature.

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