Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 24 (2026), No. 1, pp. 71–86
DOI : 10.47974/JDSGT-2025-11003

An LMI-based framework for observer design in nonlinear systems with delayed measurements

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Abstract

In this paper, we address the problem of observer design for time-delay systems with delays in both the state and the input, under the global Lipschitz condition on the system nonlinearity, using a Linear Matrix Inequality (LMI) approach. Utilizing the recent results from Nguyen, Zemouche, and Trinh [28], this work further elaborates and expands upon them. The aim of this study is to present a novel design approach. Utilizing advanced mathematical techniques, the reformulation of the Lipschitz property enhances its practicality and reduces conservatism. Furthermore, the exponential stability of the augmented closed-loop system and the estimation error dynamics is established through a Halanay-inequality-based method. A numerical example, based on the dynamics of a directional antenna in a telecommunications system, is provided to illustrate the effectiveness of the proposed results.

Subject Classification: *34A30, 34A34.*

Keywords: *Observer design, Time-delay systems, Halanay-inequality.*

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1. Introduction

Time delay is a characteristic feature of many dynamical systems, including teleoperation systems, communication networks, and mechanical, electrical, and biological embedded systems, among others [2, 20]. However, state reconstruction for such systems has so far been achieved only in specific cases, and many open challenges remain, particularly with respect to the design of nonlinear observers for time-delay systems. Consequently, the design of observers for nonlinear systems with time delays has attracted increasing attention in recent years. Significant research efforts have been devoted to this problem, resulting in various observation strategies, such as asymptotic observers [13, 15] and exponential observers [5, 11]. State observation is a fundamental topic in the study of both linear and nonlinear systems. This paper focuses specifically on the observation of *linear* systems with known delays. The observation of delayed systems has been extensively investigated in the literature; see, for instance, the surveys in [30, 31]. However, in most existing approaches, the observer design assumes that the delay value is known *a priori*, that is, the delay must be either known or directly measurable. Furthermore, delay-free observers (i.e., observers without internal delay) [9, 11, 35] typically require access to the system output at both the current and past time instants. The general observer design framework typically emphasizes state reconstruction in most literature findings. Representative applications can be found in observer-based control strategies within robust control systems [22, 23], induction motor state estimation [26], observer design for chaotic systems [8, 32], and disturbance-decoupled observer synthesis [7]. However, most effective control processes involve systems with delays, making the design of state observers for time-delay systems a topic of significant importance. For various results on continuous time-delay systems, see [33, 34] and references therein. As highlighted in [11], the author has proposed observer designs for systems with delays in state variables. The presence of delays can be either beneficial or detrimental to the operation of a dynamical system. A feedback system that is stable without delay may become unstable for certain delay values [21] and [24]. Contrary to initial intuition, a delay can play a critical role in system stability. Mathematicians often refer to the now classic example presented in [19], which describes a system whose equilibrium remains stable despite the presence of a delay. In recent years, nonlinear delay systems have received significant attention, leading to numerous advancements. The recoil technique and the Lyapunov-Krasovskii method have been introduced to address a class of nonlinear, time-delayed, strict-feedback systems, as presented by [28]. Additionally, several researchers have explored fundamental concepts related to delay systems, which are essential for developing the main mathematical models of such systems, considering the different types of delays commonly encountered. One key aspect is the delay in acquiring information at a given time, which arises from material transport or information transmission. This delay can be constant, variable, localized, or distributed over time. Systems with delays are typically modeled using functional differential equations. The stabilization of nonlinear time-delay systems with unknown constant control coefficients has been investigated in [3].

The Linear Parameter-Varying (LPV) approach plays a crucial role in stabilizing non-linear systems by transforming them into linear systems with parameters that change over time or operational conditions. The application of linear control techniques is possible to ensure robust performance and stability because of this. A powerful framework for controller design can be achieved by combining LPV with Linear Matrix Inequalities (LMI). The synthesis of controllers that can adapt to varying conditions and maintain system stability is made possible by LMIs, which is a systematic method for handling constraints and uncertainties in LPV systems. The effectiveness and reliability of managing the complexities of nonlinear systems is ensured by this integration.

The addition of delay-dependent outputs to nonlinear time-delay systems is a significant addition and a significant generalization. It is based on judicious mathematical arrangements. This extension makes it possible to derive more general design conditions that can be easily reduced to those of the linear case. Furthermore, it is easy to implement, which makes it especially suitable for applications in real-world models. We now aim to compare our results with those of studies addressing similar issues. It is well established that the stabilization of nonlinear systems with time delay is generally more challenging than that of systems without delay. In [29], the authors analyze a class of nonlinear uncertain systems without delay. However, the system is a particular case of the system examined in our paper.

This document focuses on the problem of designing an observer-based controller for a nonlinear system with delay and delayed outputs. By using an exponential stability criterion based on the Halanay inequality for the augmented system of the closed-loop system and the estimation error system, an observer-based controller can be designed for the delayed nonlinear system without invoking differentiability assumptions on the delay. The LPV approach is used in these papers, similar to those considered in [28], to address the inherent conservatism of classical Lipschitz conditions for a nonlinear system with delay. The main criterion added compared to the class considered in [28] is that the delayed output measurements, which were assumed to be available exclusively in a single form—either continuously in time or only as samples—can now be available in the current context in a mixed form. This means they can be continuous in time over certain intervals and only sampled over others. One of the main contributions of this work is the generalization of the Lipschitz property redefinition introduced by [36] for delay-free systems to systems with delays. This approach leads to less restrictive synthesis conditions than those reported in the literature [14, 26, 17]. By employing new mathematical tools, an LPV framework is directly derived for the general class of Lipschitz systems. No additional assumption on the nonlinearity is required. It is important to highlight that this new interpretation of the Lipschitz condition is the most comprehensive, as it fully accounts for all the properties of the nonlinearities in systems with delays.

The stability of nonlinear delay systems has been studied from various angles. Practical uniform stability under bounded perturbations—without observer or controller design—was examined in [6]. State-feedback strategies for systems with unknown, time-varying delays were developed in [27] using adaptive or compensation techniques. In contrast, our work tackles observer-based control with delayed output measurements, which poses additional estimation and stabilization challenges. Quasi-uniform stability for fractional-order and neural network systems with mixed delays was addressed in [25], but these studies concern different system classes and tools. Our approach, based on an LPV Lipschitz reformulation and Halanay-type inequalities, provides a tailored framework for observer-based control of integer-order nonlinear systems with state and output delays. Most existing observer-based control designs for nonlinear time-delay systems rely on a global Lipschitz condition expressed through a single scalar bound on the nonlinearity (see, e.g., [17, 36, 37]). Such a formulation often leads to conservative synthesis conditions, since it ignores the directional and componentwise structure of the nonlinear terms.

In contrast, the proposed approach relies on a novel LPV-based reformulation of the Lipschitz property, in which the nonlinearity is represented as a convex combination of bounded matrices. This representation preserves the intrinsic structure of the nonlinear function and enables a finer characterization of its influence on the system dynamics. As a result, the derived LMI conditions are less restrictive and typically admit feasible solutions in situations where classical Lipschitz-based designs fail. Moreover, unlike several existing methods, the proposed framework naturally accommodates delayed state and output measurements within a unified LMI formulation.

The paper is organized as follows: Section II introduces the new reformulation of the Lipschitz property and the system description. Section IV, which contains the main results, addresses the problem of designing an observer-based controller for a nonlinear system with delay. Within this framework, to leverage efficient and readily available numerical solvers, we transform the proposed problem into linear matrix inequalities (LMIs). The final section presents an illustrative example of a directional antenna in a telecommunications system.

2. System description and preliminary results

2.1 *System description*

To simplify analysis and gain insights into the system's behavior, a differential system with delay is often expressed as a linear path and a non-linear component. Since this paper deals with high-gain observers, we generalize the class of systems considered by [37] for free-delay systems. Consider the following the control nonlinear times-delay system:

$$\begin{cases} \dot{\varpi}(t) = A\varpi(t) + A_1\varpi(t-\tau) + f(\varpi(t), \varpi(t-\tau)) + Bu(t), \forall t \geq 0 \\ \varpi(t) = \varphi(t); \qquad \forall t \in [-\tau, 0] \\ \pi(t) = C\varpi(t) + D\varpi(t-\tau) \end{cases} \tag{1}$$

where $\varpi(t) \in \mathbb{R}^n$ is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the control input, $\pi(t) \in \mathbb{R}^p$ is the measured output, $f(\varpi(t),\ \varpi(t-\tau))$ is nonlinearities and $\tau > 0$ denotes the time delay. A, B, C, D: are constant matrices of appropriate dimensions and $\varphi(t)$ is the initial state. To finalize the system description, it is assumed that are considered.

Assumption 1: The nonlinearity $f(\varpi(t), \varpi(t-\tau))$ is smooth, Lipschitz, to ϖ and $\varpi(t-\tau)$, uniformly with respect to u and well-defined for all $\varpi(t) \in \mathbb{R}^n$ with $f(0,0) = 0$.

Assumption 2: For all $t \geq 0$, the delay τ is known and constant.

Remark 1: The class of nonlinear time delay systems examined in this paper is broader and more general than the Lipschitz nonlinear systems discussed in [12, 17].

Remark 2: Assumption 2 considers a constant and known delay in order to keep the analysis tractable and to highlight the main contribution of the paper, namely the LPV-based reformulation of the Lipschitz property and the observer-based control synthesis via LMIs. It is worth noting that extending the proposed framework to unknown but bounded delays, or to slowly time-varying delays satisfying

$$0 \leq \tau(t) \leq \bar{\tau}, \qquad \dot{\tau}(t) \leq \mu < 1,$$

would require a more elaborate stability analysis, typically involving delay-dependent Lyapunov--Krasovskii functionals or refined Halanay-type inequalities. Such extensions, although technically feasible, would significantly increase the complexity and conservatism of the resulting LMI conditions and are therefore left for future research.

Remark 3: This paper's main contributions are highlighted in Assumption A1. She shows that the investigated system (1) is fundamentally distinct from those in closely related works cited by [1] and [4]. Consequently, the outcomes obtained are less restrictive than the usual Lipschitz condition.

2.2 *Reformulation of the Lipschitz Property*

This section's main focus centers on a newly formulated Lipschitz property. This modification is essential for our advanced linear parameter varying (LPV) methodology. We begin by introducing some preliminary notion that will be crucial to the linear parameter varying approach developed for the Lipschitz condition of the system nonlinearity. Consider four vectors:

$$\eta = \begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \vdots \\ \varpi_n \end{bmatrix}, \sigma = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix}, \bar{\eta} = \begin{bmatrix} \bar{\varpi}_1 \\ \bar{\varpi}_2 \\ \vdots \\ \bar{\varpi}_n \end{bmatrix}, \bar{\sigma} = \begin{bmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \\ \vdots \\ \bar{\pi}_n \end{bmatrix} \in \mathbb{R}^n.$$

Let $n \geq 0$, for all $i = 1, \dots, n$, we define an auxiliary vector $\eta^{\bar{\eta}_i} \in \mathbb{R}^n$ corresponding to η and $\bar{\eta}$ as follows

$$\eta^{\bar{\eta}_0} = \eta, and\ \eta^{\bar{\eta}_i} = \begin{bmatrix} \bar{\varpi}_1 \\ \vdots \\ \bar{\varpi}_i \\ \varpi_{i+1} \\ \vdots \\ x_n \end{bmatrix} \text{for } i = 1, \dots, n.$$

$e_i = (0, \dots, 0, \overbrace{1}^{\text{ith}}, 0, \dots, 0) \in \mathbb{R}^n$ is a transposed vector of the canonical basis of $\mathbb{R}^n$.

The auxiliary vectors $\eta^{\bar{\eta}_i}$ and $\sigma^{\bar{\sigma}_i}$ allow the nonlinear increment to be decomposed component-wise rather than bounded globally. This incremental representation captures the individual contribution of each state component, which reduces conservatism and is well suited for the proposed LPV formulation.

Lemma 1: *Consider a function* $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*, there exists tow functions*

$$\psi_i, \varphi_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

For $i = 1, \dots, n$*, for all* $(\eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n, (\bar{\eta}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^n$ *so that:*

$$\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma}) = \sum_{i=1}^n \psi_i(\eta^{\bar{\eta}_{i-1}}, \sigma) e_i(\eta - \bar{\eta}) + \sum_{i=1}^n \varphi_i(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}}) e_i(\sigma - \bar{\sigma}) \tag{2}$$

Proof: We can writing $\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma})$ as :

$$\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma}) = \sum_{i=1}^n [\psi(\eta^{\bar{\eta}_{i-1}}, \sigma) - \psi(\eta^{\bar{\eta}_i}, \sigma)] + \sum_{i=1}^n [\psi(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}}) - \psi(\bar{\eta}, \sigma^{\bar{\sigma}_i})].$$

Now, defining two functions ψ_i, φ_i by:

$$\psi_i(\eta^{\bar{\eta}_{i-1}}, \sigma) = \begin{cases} 0 & if\ \varpi_i = \bar{\varpi}_i \\ \frac{\psi(\eta^{\bar{\eta}_{i-1}}, \sigma) - \psi(\eta^{\bar{\eta}_i}, \sigma)}{\varpi_i - \bar{\varpi}_i} & if\ \varpi_i \neq \bar{\varpi}_i \end{cases} \tag{3}$$

$$\varphi_i(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}}) = \begin{cases} 0 & if\ \pi_i = \bar{\pi}_i \\ \frac{\psi(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}}) - \psi(\bar{\eta}, \sigma^{\bar{\sigma}_i})}{\pi_i - \bar{\pi}_i} & if\ \pi_i \neq \bar{\pi}_i \end{cases} \tag{4}$$

we get:

$$\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma}) = \sum_{i=1}^n [\psi_i(\eta^{\bar{\eta}_{i-1}}, \sigma)](\varpi_i - \bar{\varpi}_i) + \sum_{i=1}^n [\varphi_i(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}})](\pi_i - \bar{\pi}_i)$$

Then

$$\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma}) = \sum_{i=1}^n [\psi_i(\eta^{\bar{\eta}_{i-1}}, \sigma)] e_i(\eta - \bar{\eta}) + \sum_{i=1}^n [\varphi_i(\bar{\eta}, \sigma^{\bar{\sigma}_{i-1}})] e_i(\sigma - \bar{\sigma})$$

Lemma 2: *Let the function* $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$*. Then, for all* $\eta,\ \sigma,\ \bar{\eta},\ \bar{\sigma} \in \mathbb{R}^n$*, the two following items are equivalent:*

1. *There exist positives constants* γ_1, γ_2, *such that :*

$$\|\psi(\eta,\sigma)-\psi(\bar{\eta},\bar{\sigma})\| \le \gamma_1\|\eta-\bar{\eta}\|+\gamma_2\|\sigma-\bar{\sigma}\| \tag{5}$$

2. *There exist two functions* $\psi_{ij}, \varphi_{ij}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *for all* $i,j=1,\dots,n$ *and constants* $\gamma_{1\psi_{ij}}, \gamma_{2\psi_{ij}}, \gamma_{1\varphi_{ij}}, \gamma_{2\varphi_{ij}}$ *such that*

$$\begin{aligned}\psi(\eta,\sigma)-\psi(\bar{\eta},\bar{\sigma}) &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \psi_{ij}(\eta^{\bar{\eta}_{i-1}},\sigma)e_i^T e_j(\eta-\bar{\eta}) \\ &+ \textstyle\sum_{i=1}^n \sum_{j=1}^n \varphi_{ij}(\bar{\eta},\sigma^{\bar{\sigma}_{i-1}})e_i^T e_j(\sigma-\bar{\sigma})\end{aligned} \tag{6}$$

and

$$\gamma_{1\psi_{ij}} \le \psi_{ij} \le \gamma_{2\psi_{ij}}, \quad \gamma_{1\varphi_{ij}} \le \varphi_{ij} \le \gamma_{2\varphi_{ij}}. \tag{7}$$

Proof: The necessity part of the Lemma. For all $\eta,\sigma \in \mathbb{R}^n$, we have:

$$\psi(\eta,\sigma) = (\psi_1(\eta,\sigma),\dots,\psi_n(\eta,\sigma)) = \textstyle\sum_{i=1}^n e_i^T\psi_i(\eta,\sigma)$$

$$\begin{aligned}\|\psi(\eta,\sigma)-\psi(\bar{\eta},\bar{\sigma})\|^2 &= \textstyle\sum_{i=1}^n |\psi_i(\eta,\sigma)-\psi_i(\bar{\eta},\bar{\sigma})|^2 \\ &\le (\gamma_1\|\eta-\bar{\eta}\|+\gamma_2\|\sigma-\bar{\sigma}\|)^2\end{aligned} \tag{8}$$

By inequality (8) we have:

$$|\psi_i(\eta,\sigma)-\psi_i(\bar{\eta},\bar{\sigma})| \le \gamma_1\|\eta-\bar{\eta}\|+\gamma_2\|\sigma-\bar{\sigma}\|.$$

Which implies that, ψ_i is Lipschitz, for all $i=1,\dots,n$. By using Lemma 1, we have

$$\psi_i(\eta,\sigma)-\psi_i(\bar{\eta},\bar{\sigma}) = \textstyle\sum_{j=1}^n \psi_{ij}(\eta^{\bar{\eta}_{i-1}},\sigma)e_j(\eta-\bar{\eta}) + \sum_{j=1}^n \varphi_{ij}(\bar{\eta},\sigma^{\bar{\sigma}_{i-1}})e_j(\sigma-\bar{\sigma})$$

Where ψ_{ij} and φ_{ij} are given by (3) and (4) respectively. By replacing ψ by ψ_i, there exist $\gamma_{1\psi_{ij}}$ and $\gamma_{2\psi_{ij}}$ such that

$$|\psi_i(\eta,\sigma)-\psi_i(\bar{\eta},\bar{\sigma})| \le \gamma_{1\psi_{ij}}\|\eta-\bar{\eta}\|+\gamma_{2\psi_{ij}}\|\sigma-\bar{\sigma}\|$$

with

$$\gamma_{1\psi_{ij}} \le \gamma_1 \text{ and } \gamma_{2\psi_{ij}} \le \gamma_2$$

Since ψ_{ij} and φ_{ij} are Lipschitz, we have

$$|\psi_i(\eta^{\bar{\eta}_{i-1}},\sigma)-\psi_i(\eta^{\bar{\eta}_i},\sigma)| \le \gamma_{1\psi_{ij}}\|\eta^{\bar{\eta}_{i-1}}-\eta^{\bar{\eta}_i}\| = \gamma_{1\psi_{ij}}|\varpi_i-\bar{\varpi}_i|$$

$$|\varphi_j(\bar{\eta},\sigma^{\bar{\sigma}_{i-1}})-\varphi_j(\bar{\eta},\sigma^{\bar{\sigma}_i})| \le \gamma_{2\varphi_{ij}}\|\sigma^{\bar{\sigma}_{i-1}}-\sigma^{\bar{\sigma}_i}\| = \gamma_{2\varphi_{ij}}|\pi_i-\bar{\pi}_i|$$

with $\gamma_{1\psi_{ij}} \le \psi_{ij} \le \gamma_{2\psi_{ij}}$ and $\gamma_{1\varphi_{ij}} \le \varphi_{ij} \le \gamma_{2\varphi_{ij}}$.

Let us prove the sufficiency part. Assume that for all $i,j=1,\dots,n$, there exist two functions ψ_{ij}, $\varphi_{ij}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and constants: $\gamma_{1\psi_{ij}}$, $\gamma_{2\psi_{ij}}$, $\gamma_{1\varphi_{ij}}$, $\gamma_{2\varphi_{ij}}$, so that , hold. Then, for all η, $\sigma \in \mathbb{R}^n$, we have:

$$\begin{aligned}\|\psi_i(\eta,\sigma)-\psi_i(\bar{\eta},\bar{\sigma})\| &\le \textstyle\sum_{i=1}^n \sum_{j=1}^n |\psi_{ij}|\|\eta-\bar{\eta}\| + \sum_{i=1}^n \sum_{j=1}^n |\varphi_{ij}|\|\sigma-\bar{\sigma}\| \\ &\le \textstyle(\sum_{i=1}^n \sum_{j=1}^n \lambda_{ij})\|\eta-\bar{\eta}\| + (\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij})\|\sigma-\bar{\sigma}\|\end{aligned}$$

where

$$\lambda_{ij} = \max(|\gamma_{1\psi_{ij}}|, |\gamma_{2\psi_{ij}}|)$$

$$\alpha_{ij} = \max(|\gamma_{1\varphi_{ij}}|, |\gamma_{2\varphi_{ij}}|)$$

Remark 4: Lemma 2 provides a Lipschitz condition that is less conservative and more refined than the traditional global Lipschitz assumption on the nonlinear functions. The reformulation given in (6)-(7) enables a more precise characterization of the nonlinearity and fully exploits the structural properties of the system's nonlinearity under consideration.

The following lemma is a generalization of the Lemma 1 [28] and Lemma 7 [36]. For the corresponding the proof is omitted.

Lemma 3: *Consider a nonlinear function $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. If ψ is γ_1, γ_2 Lipschitz, ie verify equation (5). Then there exist two matrix Q_1, Q_2 such that*

$$\psi(\eta, \sigma) - \psi(\bar{\eta}, \bar{\sigma}) = Q_1(\eta - \bar{\eta}) + Q_2(\sigma - \bar{\sigma})$$

$Q_1, Q_2 \in \mathbb{M}(n, \mathbb{R})$ *is a varying in a bounded convex set H_ψ, H_φ whose the vertices set is :*

$$V_{H_\psi} = \{\theta = (\theta_{ij}) \in \mathbb{R}^{n\times n}, \theta_{ij} \in \{\gamma_{1\psi_{ij}}, \gamma_{2\psi_{ij}}\}\}$$
$$V_{H_\varphi} = \{\varepsilon = (\varepsilon_{ij}) \in \mathbb{R}^{n\times n}, \varepsilon_{ij} = \{\gamma_{1\varphi_{ij}}, \gamma_{2\varphi_{ij}}\}\}$$

Proof: The result is obtained as a direct corollary of Lemma 2. Specifically, applying Lemma 2 allows the estimation error dynamics and the closed-loop system to be rewritten in an LPV form. Since the steps involved are standard and follow exactly the same reasoning as in [28], the proof is straightforward and is therefore omitted.

3. Main result

This section is dedicated to the design of an observer-based controller, aiming to estimate the internal states of the system.

3.1 Design observer

The observer design methodology is fundamentally based on assessing the system's state boundaries and constraining the nonlinearities of the observer based on the estimated ranges of variation in the true states. The observer of system is defined by the following form :

$$\begin{cases} \dot{\widehat{\varpi}}(t) = A\widehat{\varpi}(t) + A_1\widehat{\varpi}_\tau + f(\widehat{\varpi}(t), \widehat{\varpi}(t-\tau)) + Bu(t) + L(\pi - C\widehat{\varpi} - D\widehat{\varpi}_\tau) \\ \hat{\pi}(t) = C\widehat{\varpi} + D\widehat{\varpi}_\tau \end{cases} \tag{9}$$

where $\widehat{\varpi}_\tau = \widehat{\varpi}(t-\tau)$, $\widehat{\varpi}(t) \in \mathbb{R}^n$ is the state estimation of $\varpi(t)$, the gain matrix $L \in \mathbb{R}^{n\times p}$ is to be determined. The state estimation error: $e(t) = \varpi(t) - \widehat{\varpi}(t)$. The dynamic equation of the observation error $e(t)$ is given by:

$$\begin{aligned} \dot{e}(t) &= (A - LC)e + (A_1 - LD)e_\tau + (f(\varpi(t), \varpi(t-\tau)) - f(\widehat{\varpi}(t), \widehat{\varpi}(t-\tau))) \\ &= (A - LC)e(t) + (A_1 - LD)e_\tau(t) + \bar{f}, \end{aligned} \tag{10}$$

where $\bar{f} = f(\varpi(t), \varpi(t-\tau)) - f(\hat{\varpi}(t), \hat{\varpi}(t-\tau))$.

By Lemma 3, there exist matrices $G > 0,\ H > 0$ such as:

$$f(\varpi(t), \varpi(t-\tau)) - f(\hat{\varpi}(t), \hat{\varpi}(t-\tau)) = Ge(t) + He(t-\tau)$$

$$A - LC = A_L, \text{ and } A_1 - LD = A_{1D}$$

then

$$\dot{e}(t) = (A_L + G)e + (A_{1D} + H)e_\tau \tag{11}$$

Remark 5: The observer structure (9) involves the real-time computation of the nonlinear function $f(\hat{\varpi}(t), \hat{\varpi}(t-\tau))$. In many practical applications, this function is explicitly known and already used for modeling or simulation purposes, making its online evaluation feasible. For systems with complex nonlinearities, approximate implementations may be employed, such as polynomial approximations, lookup tables, or data-driven surrogate models. In addition, thanks to the LPV-based Lipschitz reformulation adopted in this work, the nonlinear term can be bounded or represented through its LPV equivalent, which may be used to reduce computational burden while preserving stability guarantees.

3.2 *Observer-Based controller structure*

This section focuses on the design of an observer-based controller. We demonstrate that a controller augmented with a linear observer can effectively drive the states of a nonlinear time-delay system to the origin, independently of the delay magnitude. The proposed approach leverages advanced observer design techniques to compensate for the delay, thereby improving both the stability and overall performance of the closed-loop system.

Consider the following observer-based controller for system (1):

$$u(t) = -K\hat{\varpi}(t) \tag{12}$$

Then, the closed-loop system given by:

$$\begin{aligned} \dot{\varpi}(t) &= A\varpi(t) + A_\tau\varpi(t-\tau) + f(\varpi(t), \varpi(t-\tau)) - BK(\varpi - e) \\ &= A_K\varpi(t) + BKe + A_1\varpi_\tau + f(\varpi(t), \varpi(t-\tau)). \end{aligned}$$

Where $A_K = (A - BK)$, by Lemma 3, we have $f(\varpi(t), \varpi(t-\tau)) = M\varpi + N\varpi_\tau$, then

$$\dot{\varpi}(t) = (A_K + M)\varpi + (A_1 + N)\varpi_\tau + BKe \tag{13}$$

The augmented system of the closed-loop system and the estimation error is:

$$\begin{bmatrix} \dot{\varpi}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A_K + M & BK \\ 0 & A_L + G \end{bmatrix} \begin{bmatrix} \varpi(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} A_1 + N & 0 \\ 0 & A_{1D} + H \end{bmatrix} \begin{bmatrix} \varpi_\tau \\ e_\tau \end{bmatrix} \tag{14}$$

Proposition 1: [28] Consider the augmented system (14). If there exist matrices $P > 0$, $S > 0$ and positive scalars $\alpha > \beta$ such that the following inequality condition holds

$$\dot{V}(t) + \alpha V(t) - \beta\varpi^T{}\tau S\varpi_\tau \leq 0 \tag{15}$$

where

$$V(t) = g^T(t)\vartheta g(t) \tag{16}$$

$$g(t) = \begin{bmatrix} \varpi(t) \\ e(t) \end{bmatrix} \text{ and } \vartheta = \begin{bmatrix} S & 0 \\ 0 & P \end{bmatrix}.$$

then the system is exponentially stable.

By applying Proposition 1, the observer-based controller defined in (11)-(12) can be designed to ensure the exponential stabilization of both the closed-loop system (13) and the state estimation error dynamics (10). To this end, we use condition (15) as the starting point for deriving a set of Linear Matrix Inequality (LMI) conditions that serve as sufficient criteria for the exponential stability of system (14). Once established, these LMI conditions allow for the explicit computation of the gain matrices F and L, which are instrumental in shaping the dynamics of the observer and controller, thereby enhancing the overall performance and robustness of the closed-loop system.

Remark 6: According to Lemma 3, the nonlinear terms of the system and the observation error dynamics can be expressed in an LPV form as

$$f(\varpi, \varpi_\tau) = M\varpi + N\varpi_\tau, \qquad f(\varpi, \varpi_\tau) - f(\hat{\varpi}, \hat{\varpi}_\tau) = Ge + He_\tau,$$

where the matrices M, N, G, and H belong to bounded convex sets induced by the Lipschitz constants of the nonlinear function. These sets are described as convex hulls of finitely many vertices corresponding to the extreme values of the Lipschitz bounds.

The proposed LMI-based stability conditions are enforced for all admissible matrices within these convex sets. As a result, stability is guaranteed uniformly over the entire uncertainty domain, and the boundedness of M, N, G, and H is naturally captured through their polytopic structure.

Theorem 1: *Assume that there exist positive definite matrices P and S with appropriate dimensions satisfying the LMIs presented below for given parameters α and β. Then, the augmented closed-loop system (14) corresponding to the observer-based controller (13)-(11) is exponentially stable.*

$$G = \begin{bmatrix} G_{11} & SBK & SN + SA_1 & 0 \\ & G_{22} & 0 & PH + PA_{1D} \\ & * & -\beta S & 0 \\ & * & * & 0 \end{bmatrix} < 0. \tag{17}$$

where

$$\begin{cases} G_{11} = S(A_K + M) + (S(A_K + M))^T + \alpha S \\ G_{22} = P(A_L + G) + (P(A_L + G))^T + \alpha P \end{cases}$$

Proof: Consider the following quadratic function (16). The evolution of V along the solutions given in is described by its derivative

$$\dot{V}(t) = \varpi^T (A_K + M)^T S\varpi + \varpi^T S(A_K + M)x + e^T K^T B^T S\varpi + \varpi^T SBKe$$
$$+\varpi_\tau (A_1 + N)^T S\varpi + \varpi^T S(A_1 + N)\varpi_\tau + e^T (A_L^T P + PA_L + PG + G^T P)e$$
$$+e_\tau^T (A_{1D}^T P + H^T P)e + e^T (PA_{1D} + PH)e_\tau$$

By Proposition , there exist positive scalars α, β such that the following inequality condition holds:

$$\dot{V}(t) + \alpha V(t) - \beta \varpi_\tau^T S\varpi_\tau = \varpi^T [(A_K + M)^T S + S(A_K + M) + \alpha S]\varpi$$
$$+e^T K^T B^T S\varpi + \varpi^T SBKe + \varpi_\tau^T (A_1 + N)^T S\varpi + \varpi^T S(A_1 + N)\varpi_\tau$$
$$+e^T [A_L^T P + PA_L + PG + G^T P + \alpha P]e + e_\tau^T (A_{1D}^T P + H^T P)e$$
$$+e^T (PA_{1D} + PH)e_\tau - \beta \varpi_\tau^T S\varpi_\tau = \psi^T(t) G\psi(t).$$

where $\alpha > \beta > 0$,

$$\psi(t) = \begin{bmatrix} \varpi \\ e \\ \varpi_\tau \\ e_\tau \end{bmatrix} \text{ and } G = \begin{bmatrix} G_{11} & SBK & SN + SA_1 & 0 \\ & G_{22} & 0 & PH + PA_{1D} \\ & * & -\beta S & 0 \\ & * & * & 0 \end{bmatrix} < 0.$$

where

$$\begin{cases} G_{11} = S(A_K + M) + (S(A_K + M))^T + \alpha S \\ G_{22} = P(A_L + G) + (P(A_L + G))^T + \alpha P \end{cases}$$

Then, by proposition the system is exponentially stable if the condition (17) is verified.

Remark 7: The observer is designed for delays belonging to a known bounded interval. As long as the actual delay τ remains within this bound, LMI (17) guarantees exponential convergence of the estimation error. Variations of τ mainly influence transient performance without affecting stability. If the delay exceeds the prescribed bound, stability can no longer be ensured.

Remark 8: Our method's improved reformulation reduces conservativeness and allows for better handling of nonlinearities, unlike classical approaches based on the global Lipschitz condition [12] and [17].

Remark 9: The precise knowledge of the delay is generally required by existing observers for delayed systems [1, 28]. Our approach provides stability even when the delay is unknown but limited, which is a significant improvement.

Remark 10: LMIs are used in our design to provide stability conditions that can easily be verified, unlike the more complex analytical methods proposed in [20].

4. Numerical Example

In this section, we study the dynamics of a directional antenna in a telecommunications system. To maximize signal reception in a telecommunications system, a directional antenna must change its orientation. The antenna's position can be adjusted with a time delay τ due to processing and transmission delays. The control system's stability and performance could be negatively impacted by this delay. We model a nonlinear time-delay system that represents the antenna dynamics in the form (1), with the following parameter matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1.5 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 1 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0], D = [0.5 \quad 0.2]$$

$\varpi_1(t)$ represents the antenna orientation angle, while $\varpi_2(t)$ corresponds to its angular velocity. The term $u(t)$ denotes the control input applied to the antenna motor. Finally, D introduces a delayed effect in the measurement output.

We are considering a genuine nonlinear model for wind perturbations and sudden directional changes

$$f(\varpi(t), \varpi(t-\tau)) = \begin{bmatrix} 0 \\ \sin(\varpi_1(t)) + 0.3, \ \varpi_2(t-\tau) \end{bmatrix} \tag{19}$$

with a constant delay $\tau = 0.5$ seconds. The vertices sets are:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0.3 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.2345 & 0.5678 \\ 0.5678 & 2.3456 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.9876 & 0.1234 \\ 0.1234 & 1.8765 \end{bmatrix}$$

$\alpha = 3.4567$ and $\beta = 1.2345$ This Figure 1 and Figure 2 illustrate the evolution of $\varpi_1(t)$, $\varpi_2(t)$ the antenna orientation angle and the angular velocity respectively, along with their estimation obtained using a moving average. The actual trajectory of $\varpi_1(t)$ and $\varpi_2(t)$ is represented by the blue curve, and the estimated value is represented by the green dash curve. The estimation generally follows the actual trajectory, but it appears slightly smoothed due to the moving average effect.

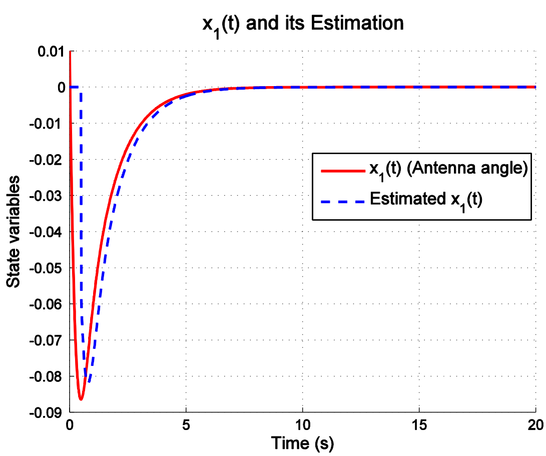

Figure 1
The evolution of ϖ_1 and its estimate $\hat{\varpi}_1$ for $\tau = 0.5s$

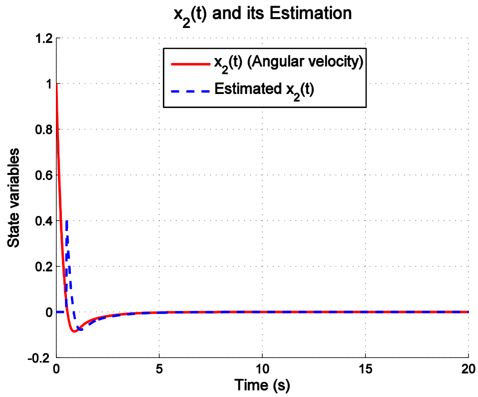

Figure 2
The evolution of ϖ_2 and its estimate $\hat{\varpi}_2$ for $\tau = 0.5s$

Table 1
Comparison between existing methods

Method	state delays	output delays	unknown delays
Nguyen, Zemouche, and Trinh [28]	Yes	No	No
Dong and Yang [10]	Yes	No	Partial
Proposed Method	Yes	Yes	Bounded

As illustrated in Table 1, the proposed method expands upon existing methodologies by managing both state and output lags, while also considering bounded uncertain delays.

Remark 11: The numerical example shows how our method can be applied to directional antenna control, and it has potential for communication systems with different delays, as opposed to the more theoretical approaches in [18].

5. Conclusion

We have investigated the challenge of creating an observer-based controller for a nonlinear system that includes delayed data and output delays. An exponential stability criterion was established by leveraging the Halanay inequality and subsequently applied to design an observer-based controller for the nonlinear delayed system without requiring differentiability assumptions on the delay. Using the LPV (Linear Parameter-Varying) approach, less conservative conditions compared to the classical Lipschitz conditions were derived to manage the system nonlinearities. Lastly, an investigation was carried out to demonstrate the efficacy of the suggested hypothesis. As a future perspective, this work can be extended to systems involving conformable derivatives, following the approach proposed in [16].

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Received November, 2025