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Some existence and stability results for abstract evolution equations with perturbations

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Abstract

This study investigates the uniqueness and stability of solutions for infinite-dimensional systems with infinite delay, using almost-sectorial operators and specific conditions on the nonlinear terms. These findings are discussed in the context of recent progress on sectorial and almost-sectorial operators, expanding stability theory to include a wider range of time-varying operators and semilinear systems. The applicability of the theoretical findings is illustrated through several examples, including

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semilinear parabolic problems and abstract equations in Banach and Hölder spaces. These methods are demonstrated to be applicable to a wide range of infinite-dimensional systems with delay.

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1 Introduction

The study of Lyapunov stability for partial differential equations with delay has attracted significant attention in recent years due to its relevance in modeling complex dynamical systems [2]–[4],[17],[21],[25]. Various approaches have been proposed to investigate the asymptotic behavior of solutions for different classes of differential equations, including integral inequalities, functional analysis methods, and fixed-point techniques [5],[23],[17].

Partial differential equations with infinite delay present additional technical challenges, primarily due to the unboundedness of the delay. Several works have addressed these difficulties using fixed-point methods to establish existence and uniform asymptotic stability of solutions, [1],[7]. In particular, the choice of an appropriate state space plays a critical role in such analyses. Seminal contributions in this context were made by [26],[27],[32], with a comprehensive treatment provided in [24]. Recent developments have focused on abstract differential equations with time-dependent operators that may not satisfy classical sectoriality conditions. Classical results for sectorial operators and semilinear parabolic problems can be found in [22]. In [6],[31],[33], $\bar{\mathscr{A}}(\iota)$ is assumed to be sectorial with Hölder continuity in time. In contrast, the notion of almost sectorial operators has been introduced in [34], and further investigated in [29], [30], allowing functional calculus techniques to be applied to a broader class of operators. While these works provide significant insights, there remains a gap in explicitly comparing stability results for almost sectorial operators with classical sectorial frameworks, particularly for time-varying and perturbed systems with infinite delay. Our study addresses this gap by positioning the results relative to both the classical sectorial case and recent generalizations, emphasizing how the introduction of almost sectorial operators extends the scope of stability analyses. Furthermore, we build upon classical Lyapunov and exponential stability frameworks (see [9]–[16],[19],[20]) to provide new perspectives on robust stabilization for delayed systems. These advances motivate our study of abstract differential equations with almost sectorial operators.

This paper examines the well-posedness and stability of solutions for a class of abstract differential equations characterized by infinite delay, expressed as follows:

$$\frac{du(\iota)}{d\iota} + \bar{\mathscr{A}}(\iota)u(\iota) = \bar{\mathscr{B}}(\iota, u_\iota), \tag{1}$$

where $\bar{\mathscr{A}}(\iota)$ is almost sectorial and $\bar{\mathscr{B}}$ is Lipschitz continuous relative to the history segment u_ι, uniformly in ι. By explicitly positioning our results in relation to the recent literature, including both sectorial and almost-sectorial operator frameworks, we extend existing stability analyses to a broader class of time-varying operators and semilinear systems. We introduce a phase space $\mathfrak{B}$ of functions suitable for our analysis, and derive appropriate conditions for the existence and uniform asymptotic stability of trajectories using the contraction mapping principle. Finally, we illustrate our theoretical results with special cases and examples, highlighting the applicability of our approach to systems with infinite delay and perturbations.

The structure of the paper is as follows. Section 2 introduces the main results, Section 3 contains special cases and illustrative examples, and Section 4 concludes the paper.

2 Basic results

Consider:

$$\frac{d\Im(\iota)}{d\iota} + \bar{\mathscr{A}}(\iota)\Im(\iota) = \bar{\mathscr{B}}(\iota, \Im_\iota), \quad \iota > 0, \quad \Im_0(\iota) := \wp(\iota), \quad \iota \leq 0. \tag{2}$$

(H1) For each $\iota \in \mathbb{R}$, $\bar{\mathscr{A}}(\iota) : D(\bar{\mathscr{A}}(\iota)) \subset \mathscr{X} \longrightarrow \mathscr{X}$ is a closed sectorial operator on a Banach space $(\mathscr{X}, \|\cdot\|)$ associated a linear evolution process of growth $1-\mathfrak{a}$, $(0 < \mathfrak{a} < 1)$ and a family of fractional power spaces $D(\bar{\mathscr{A}}^\ell(\iota))$.

(H2) There exists $C_{\bar{\mathscr{A}}} > 0$ and $\varepsilon \in (0, 1]$, such that

$$\|[\bar{\mathscr{A}}(\iota) - \bar{\mathscr{A}}(\tau)]\bar{\mathscr{A}}^{-1}(\bar{s})\| \leq C_{\bar{\mathscr{A}}}(\iota - \hbar)^\varepsilon, \quad \forall \iota, \hbar, \bar{s} \in \mathbb{R}.$$

$\bar{\mathscr{A}}(\iota)$ satisfies uniform Hölder continuity.

(H3) The nonlinear term $\bar{\mathscr{B}} : \mathbb{R}^+ \times D(\bar{\mathscr{A}}^\ell(\iota)) \longrightarrow \mathscr{X}$ satisfies:

$$\|\bar{\mathscr{B}}(\iota, \wp) - \bar{\mathscr{B}}(\iota, \psi)\| \leq C(\iota)\|\wp - \psi\|_{\mathfrak{B}}, \quad \forall \wp, \psi \in \mathfrak{B}, \quad \iota \geq 0,$$

where $\bar{C}(\iota) = \aleph(\iota)\triangle(\iota)$ with $\triangle(\cdot)$ is a nonnegative measurable function belonging to $L^{\mathfrak{p}}_{loc}(\mathbb{R}^+)$ and

$$\sup_{\iota \geq 0} \int_0^\iota (\iota - \varpi)^{(-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}})q}[\aleph(\iota)]^{\mathfrak{q}} d\varpi < \infty$$

for $\frac{1}{\mathfrak{q}} + \frac{1}{\mathfrak{p}} := 1.$

Here, $\wp$ belongs to the phase space $\mathfrak{B}$, consisting of functions from $(\infty, 0]$ into $D(\bar{\mathscr{A}}^{\ell}(\iota))$, which will be specified later. For any function $\Im : (\infty, \Bbbk] \to D(\bar{\mathscr{A}}^{\ell}(\iota))$, $\Bbbk > 0$, and for each $\iota \in [0, \Bbbk]$, the history segment $\Im_{\iota}$ is defined as the element of $\mathfrak{B}$ given by $\Im_{\iota}(\jmath) = \Im(\iota + \jmath)$, $\forall \jmath \in (\infty, 0]$.

Let $\mathscr{X}$ be a real Banach space with norm $\|\cdot\|$. The phase space $\mathfrak{B}$ is defined as the space of functions mapping $(\infty, 0]$ to $\mathscr{X}$, endowed with a semi-norm $\|\cdot\|_{\mathfrak{B}}$, and meeting the following basic rules:

(A1) Suppose that $x : (\infty, \sigma + \bar{a}] \to \mathscr{X}$, where $\bar{a} > 0$, is a continuous function on $[\sigma, \sigma + a)$ and its initial history x_{σ} lies in $\mathfrak{B}$. Then, for all $\iota \in [\sigma, \sigma + \bar{a})$, the following statements hold true:

1. $x_{\iota} \in \mathfrak{B}$,
2. $\|x(\iota)\|_X \leq \Lambda \|x_{\iota}\|_{\mathfrak{B}}$,
3. $\|x_{\iota}\|_{\mathfrak{B}} \leq \Pi(\iota - \bar{\sigma}) \sup_{\bar{\sigma} \leq \bar{s} \leq \iota} \|x(\bar{s})\| + \|x_{\bar{\sigma}}\|_{\mathfrak{B}} M(\iota - \bar{\sigma})$,

in which Λ is a fixed constant, Π is continuous on $[0, \infty)$, M is locally bounded on $[0, \infty)$, and neither function depends on x.

(A2) For any function x satisfying **(A1)**, the map $\iota \mapsto x_{\iota}$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

Let x satisfy **(A1)**. Then $\iota \mapsto x_{\iota}$ defines a continuous mapping on $[\sigma, \sigma + a)$ into $\mathfrak{B}$.

(A3) $\mathfrak{B}$ is a complete space.

Remark 1. *Axiom **(A1)**(2) is equivalent to*

$$\|\wp(0)\| \leq \Lambda \|\wp\|_{\mathfrak{B}}, \quad \forall \wp \in \mathfrak{B}.$$

Remark 2. *If $\mathscr{X}_{\ell} = D(\bar{\mathscr{A}}(\iota)^{\ell})$ denotes the fractional domain of a family of operators $\bar{\mathscr{A}}(\iota)$, we explicitly assume that the spaces $\mathscr{X}_{\ell}$ are compatible for all $\iota \geq 0$. That is, there exists a common Banach space structure on $\mathscr{X}_{\ell}$ independent of ι, so that all estimates involving $\|\cdot\|_{\ell}$ are meaningful uniformly in ι (see [8, Theorem 2.1, p. 45] and [30, Sec. 3.2, p. 78] for details).*

Here are some examples of functional spaces that satisfy Axioms **(A1)**, **(A2)**, and **(A3)**.

Example 1. *Let $\varphi : (\infty, 0] \to (0, \infty)$, be a continuous function. Define*

$$C^0_\varphi = \Big\{ \wp \in \mathcal{C}(\infty, 0]; \mathscr{X}) \ \Big| \ \lim_{\jmath \to \infty} \frac{\|\wp(\jmath)\|}{\varphi(\jmath)} = 0 \Big\},$$

equipped with the norm

$$\|\wp\|_\varphi = \sup_{\jmath \leq 0} \frac{\|\wp(\jmath)\|}{\varphi(\jmath)}.$$

*It was shown in [24] that if φ is non-increasing, then $(C^0_\varphi, \|\cdot\|_\varphi)$ fulfills **(A1) – (A3)**.*

Consider the case $\varphi(\jmath) = e^{-\lambda \jmath}$, $\lambda > 0$, which yields the example below.

Example 2. *Define*

$$C^0_\nu = \Big\{ \wp \in C((\infty, 0]; \mathscr{X}) \ \Big| \ \lim_{\jmath \to \infty} e^{\nu \jmath} \wp(\jmath) = 0 \Big\}, \quad \nu > 0,$$

with the norm

$$\|\wp\|_\nu = \sup_{\theta \leq 0} e^{\nu\theta} \|\wp(\theta)\|, \quad \wp \in C^0_\nu.$$

For this space, as shown in [24], we can choose $\Lambda = 1$, $\Pi(\iota) = 1$, and $M(\iota) = e^{-\nu\iota}$.

Remark 3. *Within C^0_ν, Axiom **(A1)**(3) can be written as*

$$\|x_\iota\|_\nu \leq \max\{\Pi(\iota) \sup_{0 \leq s \leq \iota} \|x(s)\|, \ M(\iota)\|x_0\|_\nu\},$$

where $\Pi(\iota) = 1$, together with $M(\iota) = \exp(-\nu\iota)$. In fact, $\forall \iota \geq 0$,

$$\|x_\iota\|_\nu \leq \max\{ \sup_{0 \leq \eta \leq \iota} \|x(\eta)\|, \ \exp(-\nu\iota)\|x_0\|_\nu\}.$$

Definition 1. *Let $\bar{\mathscr{A}}(\iota)$ be a closed linear operator acting on a Banach space $\mathscr{X}$. We say that $\bar{\mathscr{A}}(\iota)$ is* almost sectorial *if there exists an angle $\jmath \in \left(0, \frac{\pi}{2}\right)$ so that*

$$\Sigma_\jmath := \{\xi \in \mathbb{C} : |\arg \xi| \leq \pi - \jmath\} \cup \{0\} \subset \rho(-\bar{\mathscr{A}}(\iota)),$$

and for $0 < \mathfrak{a} < 1$, the resolvent satisfies

$$\|(\xi + \bar{\mathscr{A}}(\iota))^{-1}\| \leq \frac{a}{|\xi|^{\mathfrak{a}}}, \quad \forall \xi \in \Sigma_\jmath \setminus \{0\}. \tag{3}$$

As a result, the following estimate also holds:

$$\|\bar{\mathscr{A}}(\iota)(\xi + \bar{\mathscr{A}}(\iota))^{-1}\| \leq 1 + a|\xi|^{1-\mathfrak{a}}, \quad \forall \xi \in \Sigma_\jmath. \tag{4}$$

From the preceding estimates, one cannot infer that $\bar{\mathscr{A}}(\iota)$ generates a strongly continuous semigroup.

Definition 2 ([28]). *Set* $\mathfrak{a} < 1$. *An operator family* $\{\Gamma(\iota), \iota \geq 0\}$ *on* $\mathscr{X}$ *is called a* semigroup of growth $1 - \mathfrak{a}$ *if it satisfies:*

1. $\Gamma(0) = I$.
2. $\Gamma(\iota + \varsigma) = \Gamma(\iota)T(\varsigma), \forall \iota, \varsigma \geq 0$,
3. *If* $\Gamma(\iota)\Im = 0, \forall \iota > 0$, *then* $\Im = 0$,
4. $\|\iota^{1-\mathfrak{a}}\Gamma(\iota)\|$ *is bounded as* $\iota \to 0^+$,
5. $\mathscr{X}_0 = \bigcup_{\iota > 0} \Gamma(\iota)[\mathscr{X}]$ *is dense in* $\mathscr{X}$.

Definition 3. *Set* $\mathfrak{a} < 1$. *A family* $\{U(\iota, \jmath) : \iota > \jmath, \jmath \in \mathbb{R}\}$ *is called a* linear evolution process of growth $1 - \mathfrak{a}$ *if it satisfies:*

1. $U(\iota, \jmath)U(\jmath, \tau) = U(\iota, \tau), \quad \forall \iota > \jmath > \tau$,
2. $\|(\iota - \jmath)^{1-\mathfrak{a}} U(\iota, \jmath)\| \leq M, \quad \forall \iota > \jmath$,
3. *The map* $(\iota, \jmath) \mapsto U(\iota, \jmath)v_0$ *is continuous for* $\iota > \jmath$ *and* $\forall v_0 \in \mathscr{X}$.

We now recall some definitions concerning stability.

Definition 4. *The solution of system (2) is defined as follows:*

- *It is said to be uniformly stable if, for every* $\jmath > 0$, *there exists* $\hbar > 0$ *so that*
$$\|\Im(\iota, \wp)\| < \jmath \quad \text{if } \|\wp\| < \hbar.$$
- *It is said to be uniformly asymptotically stable if it is uniformly stable and*
$$\lim_{\iota \to \infty} \|\Im(\iota, \wp)\| := 0.$$

Denote by $\mathfrak{h}$ the boundary of the sector Σ_θ, whose orientation is taken in the direction of increasing imaginary part. Define

$$\Gamma_{\bar{\mathscr{A}}(\tau)}(\iota) = \frac{1}{2\pi i} \int_{\hbar} e^{\xi \iota} (\xi + \bar{\mathscr{A}}(\xi))^{-1} \, d\xi,$$

which is the semigroup of growth order $1 - \mathfrak{a}$ associated with $\bar{\mathscr{A}}(\xi)$.

The linear evolution process of growth $\mathfrak{a}$, $\{U(\iota, \xi) : \iota > \xi, \xi \in \mathbb{R}\}$, is the solution of the linear homogeneous problem

$$\frac{d\Im(\iota)}{d\iota} + \bar{\mathscr{A}}(\iota)\Im(\iota) = 0, \quad \Im(\xi) = \Im_0 \in \mathscr{X}, \tag{5}$$

given by

$$U(\iota,\xi)=\Gamma_{\bar{\mathscr{A}}(\tau)}(\iota-\xi)+\int_{\xi}^{\iota}U(\iota,\zeta)\,[\bar{\mathscr{A}}(\xi)-\bar{\mathscr{A}}(\zeta)]\,\Gamma_{\bar{\mathscr{A}}(\xi)}(\zeta-\xi)\,d\zeta.$$

Proposition 1. *([8]) The operator family* $\{\Gamma_{\bar{\mathscr{A}}(\xi)}(\iota),\ \iota>0\}\subset L(\mathscr{X})$ *satisfies, for some constant* $\Omega>0$,

$$\|\Gamma_{\bar{\mathscr{A}}(\xi)}(\iota)\|\le\Omega\,\iota^{-1+\mathfrak{a}},\quad \iota>0.$$

We can verify the semigroup property.

$$\Gamma_{\bar{\mathscr{A}}(\xi)}(\iota+\jmath)=\Gamma_{\bar{\mathscr{A}}(\tau)}(\iota)\,\Gamma_{\mathscr{A}(\tau)}(\jmath),\quad \iota,\jmath>0,$$

holds. Moreover, since the operator $\bar{\mathscr{A}}(\xi)$ is closed, it follows from (4) that the integral

$$\frac{1}{2\pi i}\int_{\hbar}e^{\xi\iota}(\xi+\bar{\mathscr{A}}(\xi))^{-1}\,d\xi$$

is convergent.

Proposition 2. *([8]) For all* $\iota>0$, *the operator* $\bar{\mathscr{A}}(\tau)\Gamma_{\bar{\mathscr{A}}(\tau)}(\iota)\in L(\mathscr{X})$, *and there exists a constant* $M>0$, *such that*

$$\|\bar{\mathscr{A}}(\jmath)\Gamma_{\bar{\mathscr{A}}(\jmath)}(\iota)\|\le M\,\max\{\iota^{-1},\iota^{-2+\mathfrak{a}}\}.$$

In particular, if $0<\iota<1$, then

$$\|\bar{\mathscr{A}}(\jmath)\Gamma_{\bar{\mathscr{A}}(\jmath)}(\iota)\|\le M\,\iota^{-2+\mathfrak{a}}.$$

Consequently,

$$\Gamma_{\bar{\mathscr{A}}(\tau)}(\iota)\Im_0\in D(\bar{\mathscr{A}}),\quad \iota>0.$$

For $0<1-\mathfrak{a}<\ell<\mathfrak{a}<1$ (with $\mathfrak{a}>\frac{1}{2}$), we define the fractional integral operator

$$\bar{\mathscr{A}}(\iota)^{-\ell}=\frac{\sin(\pi\ell)}{\pi}\int_0^{\infty}s^{-\ell}(s+\bar{\mathscr{A}}(\iota))^{-1}\,ds.$$

We next let $\bar{\mathscr{A}}(\iota)^{\ell}$ denote the inverse of $\bar{\mathscr{A}}(\iota)^{-\ell}$ and put

$$\mathscr{X}^{\ell}=D(\bar{\mathscr{A}}(\iota)^{\ell}),$$

endowed with the graph norm

$$\|x\|_{\ell}=\|\bar{\mathscr{A}}(\iota)^{\ell}x\|,\quad x\in D(\bar{\mathscr{A}}(\iota)^{\ell}).$$

Proposition 3. *Let* $0<\mathfrak{a}<1$ *and* $\ell\in(1-\mathfrak{a},\mathfrak{a})$. *Then, for any* $\Im\in D(\bar{\mathscr{A}}(\iota))$, *there exists* $\varrho>0$ *so that*

$$\|\bar{\mathscr{A}}(\iota)^{\ell}\Im\|\le\varrho\,\|\Im\|^{1-\frac{\ell}{\mathfrak{a}}}\,\|\bar{\mathscr{A}}(\iota)\Im\|^{\frac{\ell}{\mathfrak{a}}}.\tag{6}$$

Using the identity

$$\frac{\pi}{\sin(\pi r)} = \hbar(r)\hbar(1-r)$$

and

$$(\xi + \bar{\mathscr{A}}(\iota))^{-1} = \int_0^\infty e^{-\xi\upsilon}\Gamma_{\bar{\mathscr{A}}(\iota)}(\upsilon)\,d\upsilon, \quad \forall \xi > 0,$$

we have, for $\ell > 1-\mathfrak{a}$,

$$\bar{\mathscr{A}}(\iota)^{-\ell} = \frac{1}{\hbar(\ell)}\int_0^\infty \upsilon^{\ell-1}\Gamma_{\bar{\mathscr{A}}(\iota)}(\upsilon)\,d\upsilon.$$

Lemma 1 ([8])**.** *Let* $0 < \mathfrak{a} < 1$ *and* $\ell \in (0,\mathfrak{a})$*. Then there exists* $C_\ell > 0$*, so that*

$$\|\bar{\mathscr{A}}(\iota)^\ell\Gamma_{\bar{\mathscr{A}}(\iota)}(\jmath)\| \le C_\ell \max\{\jmath^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}, \jmath^{-1+\mathfrak{a}-\ell}\}, \quad \forall \jmath > 0.$$

Proposition 4. *Let* $0 < \mathfrak{a} < 1$*,* $\varepsilon > 0$*, and assume that*

$$\mathfrak{a} + \frac{\varepsilon}{2} > \max\left\{1, \frac{\ell}{\mathfrak{a}}\right\}.$$

Then, for the evolution process $U(\iota,\tau)$*, there exists a constant* $C_\ell > 0$*, such that*

$$\|\bar{\mathscr{A}}(\iota)^\ell U(\iota,\jmath)\| \le C_\ell \max\left\{(\iota-\jmath)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}, (\iota-\jmath)^{2\mathfrak{a}-\frac{\ell}{\mathfrak{a}}+\varepsilon}\right\}, \quad \forall \iota > \jmath \ge 0.$$

Proof. Let $\Im \in D(\bar{\mathscr{A}}(\tau))$. As in [21], we have

$$U(\iota,\tau)\Im = \Gamma_{\bar{\mathscr{A}}(\iota)}(\iota-\jmath)\Im + \int_\jmath^\iota \Gamma_{\bar{\mathscr{A}}(\iota)}(\iota-\beta)[\bar{\mathscr{A}}(\beta)-\bar{\mathscr{A}}(\iota)]U(\beta,\jmath)\Im\,d\beta.$$

We then estimate

$$\begin{aligned}\|\bar{\mathscr{A}}(\iota)U(\iota,\jmath)\Im\| &\le \|\bar{\mathscr{A}}(\iota)\Gamma_{\bar{\mathscr{A}}(\iota)}(\iota-\jmath)\Im\| + \int_\jmath^\iota \|\bar{\mathscr{A}}(\iota)\Gamma_{\bar{\mathscr{A}}\iota)}(\iota-\beta)\| \\ &\quad \times \|[\bar{\mathscr{A}}(\beta)-\bar{\mathscr{A}}(\iota)]\bar{\mathscr{A}}^{-1}(\beta)\|\,\|\bar{\mathscr{A}}(\beta)U(\beta,\jmath)\Im\|\,d\beta \\ &\le C_\ell(\iota-\jmath)^{-2+\mathfrak{a}}\|\Im\| + C_\ell^2\int_\tau^\iota(\iota-\beta)^{-2+\mathfrak{a}+\varepsilon}\|\bar{\mathscr{A}}(\beta)U(\beta,\jmath)\Im\|\,d\beta.\end{aligned}$$

Applying the generalized Gronwall lemma ([22]), we obtain

$$\|\bar{\mathscr{A}}(\iota)U(\iota,\jmath)\Im\| \le C_\ell(\iota-\jmath)^{-2+\mathfrak{a}}\|\Im\|.$$

Consequently, for $\Im \in D(\bar{\mathscr{A}}(\jmath))$,

$$\begin{aligned}
\|\bar{\mathscr{A}}(\iota)^{\ell}U(\iota,\jmath)\Im\| &\leq \|\bar{\mathscr{A}}(\iota)^{\ell}\Gamma_{\bar{\mathscr{A}}(\iota)}(\iota-\tau)\Im\| \\
&\quad + \int_{\tau}^{\iota} \|\bar{\mathscr{A}}(\iota)^{\ell}\Gamma_{\bar{\mathscr{A}}(\iota)}(\iota-\beta)\| \, \|[\bar{\mathscr{A}}(\beta)-\bar{\mathscr{A}}(\iota)]\bar{\mathscr{A}}^{-1}(\beta)\| \\
&\quad \times \|\bar{\mathscr{A}}(\beta)U(\beta,\tau)\Im\| \, d\beta \\
&\leq C_{\ell}(\iota-\jmath)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\|\Im\| + \bar{B}(\mathfrak{a},\mathfrak{a}-\frac{\ell}{\mathfrak{a}}+\varepsilon)(\iota-\tau)^{2\mathfrak{a}-\frac{\ell}{\mathfrak{a}}+\varepsilon}\|\Im\|,
\end{aligned}$$

where $\bar{B}(\gamma,\chi)$ is the Beta function expressed as

$$\bar{B}(\gamma,\chi) = \int_0^1 \jmath^{\gamma-1}(1-\jmath)^{\chi-1} d\jmath.$$

$\square$

2.1 Existence of Solutions

In this subsection, we prove the existence of a solution to (2) using the Contraction Mapping Principle.

Definition 5. *We call*

$$\Im : (\infty, \Bbbk] \longrightarrow D(\bar{\mathscr{A}}^{\ell}(\iota)), \ \Bbbk > 0,$$

a mild solution of (2) on $[0,\Bbbk]$ *if* $\Im_0 = \wp$ *and*

$$\Im : [0,\Bbbk] \longrightarrow D(\bar{\mathscr{A}}^{\ell}(\iota))$$

is continuous and fulfills the integral identity

$$\Im(\iota) = U(\iota,0)\wp(0) + \int_0^{\iota} U(\iota,s)\bar{\mathscr{B}}(s,\Im_s)\, ds, \quad 0 \leq \iota \leq \Bbbk.$$

In what follows, we will simply refer to such functions as mild solutions.

Theorem 1. *Suppose **(H1)**− **(H3)** hold. Then,* $\forall \wp \in \mathcal{B}$ *and* $\forall \Bbbk > 0$*, there exists a unique mild solution of* (2) *on* $[0,\Bbbk]$.

Proof. The demonstration of Theorem 1 relies on the lemma below. As the argument is direct, we omit the details.

Lemma 2. *Set* $\triangle(\cdot)$ *and* $\mathfrak{p}$ *be given according to **(H3)**, and let* $\Pi(\cdot)$ *follow the definition in* $(A_1)(3)$. *Then the following statements hold:*

1. *The function*

$$\eta(\iota) = [\triangle(\iota)\Pi(\iota)]^{\mathfrak{p}}, \quad \iota \geq 0,$$

is non-negative and belongs to $L^{1}_{\mathrm{loc}}(\mathbb{R}^+)$.

2. *For each real number* $0 < \kappa < 1$, *define*

$$\varphi_\kappa(\iota) = \exp\left(\frac{1}{\kappa}\int_0^\iota \eta(s)\,ds\right).$$

Then $\varphi_\kappa(\cdot)$ *is monotonically increasing and satisfies* $\varphi_\kappa(\iota) \geq 1$ *for all* $\iota \geq 0$.

3. *For all* $\iota \geq 0$,

$$\int_0^\iota \eta(s)[\varphi_\kappa(s)]^{\mathfrak{p}}\,ds \leq \frac{\kappa}{\mathfrak{p}}[\varphi_\kappa(\iota)]^{\mathfrak{p}}.$$

4. *For any* $\Bbbk > 0$, $\mathcal{C}([0,\Bbbk];\mathscr{X})$ *is a Banach space with the norm*

$$|||\omega||| = \sup_{0\leq\gamma\leq\Bbbk}\frac{\|\omega(\gamma)\|}{\varphi_\kappa(\gamma)}, \quad \forall \omega \in \mathcal{C}([0,\Bbbk];\mathscr{X}).$$

Proof. Define

$$\mathcal{C}_\wp = \{\mu \in \mathcal{C}([0,\Bbbk];\mathscr{X}) \mid \mu(0) = \bar{\mathscr{A}}^\ell(0)\wp(0)\}.$$

Thus, $\mathcal{C}_\wp$ is closed in the Banach space $\mathcal{C}([0,\Bbbk];\mathscr{X})$ equipped with $|||\cdot|||$.

For $\mu \in \mathcal{C}_\wp$, define the map

$$(\Delta\mu)(\bar{\iota}) = \bar{\mathscr{A}}^\ell(\bar{\iota})\wp(0) + \int_0^{\bar{\iota}} \bar{\mathscr{A}}^\ell(s)U(\bar{\iota},s)\bar{\mathscr{B}}(s,\bar{\mathscr{A}}^{-\ell}(s)\mu_s)\,ds, \quad 0 \leq \bar{\iota} \leq \Bbbk.$$

Clearly, $\Delta\mu \in \mathcal{C}_\wp$. We now show that Δ is a contraction on $\mathcal{C}_\wp$.

Let $\mu, \vartheta \in \mathcal{C}_\wp$. For all $\bar{\iota} \in [0,\Bbbk]$, it holds that

$$\begin{aligned}
\|(\Delta\mu)(\bar{\iota}) - (\Delta\vartheta)(\bar{\iota})\| &\leq \int_0^{\bar{\iota}} \|\bar{\mathscr{A}}^\ell(s)U(\bar{\iota},s)\|\,\|\bar{\mathscr{B}}(s,\bar{\mathscr{A}}^{-\ell}(s)\mu_s) - \bar{\mathscr{B}}(s,\bar{\mathscr{A}}^{-\ell}(s)\vartheta_s)\|\,ds \\
&\leq \int_0^{\bar{\iota}} C_\ell(\bar{\iota}-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)\|\bar{\mathscr{A}}^{-\ell}(s)\mu_s - \bar{\mathscr{A}}^{-\ell}(s)\vartheta_s\|_{\mathfrak{B}}\,ds \\
&\leq \int_0^{\bar{\iota}} C_\ell(\bar{\iota}-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)\Pi(s)\varphi_\kappa(s)\sup_{0\leq\zeta\leq s}\frac{\|\mu(\zeta)-\vartheta(\zeta)\|}{\varphi_\kappa(s)}\,ds \\
&\leq C_\ell\,|||\mu-\vartheta|||\int_0^{\bar{\iota}}(\bar{\iota}-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\aleph(s)\triangle(s)\Pi(s)\varphi_\kappa(s)\,ds \\
&\leq C_\ell\,|||\mu-\vartheta|||\left(\int_0^{\bar{\iota}}(\bar{\iota}-s)^{\mathfrak{q}(-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}})}[\aleph(s)]^{\mathfrak{q}}\,ds\right)^{\frac{1}{\mathfrak{q}}}\left(\int_0^{\bar{\iota}}\eta(s)[\varphi_\kappa(s)]^{\mathfrak{p}}\,ds\right)^{\frac{1}{\mathfrak{p}}}.
\end{aligned}$$

From the properties of η and φ_κ, there exists $N > 0$, such that

$$\sup_{\bar{\iota}\geq 0}\int_0^{\bar{\iota}}(\bar{\iota}-s)^{\mathfrak{q}(-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}})}[\aleph(s)]^{\mathfrak{q}}\,ds < N$$

and
$$\int_0^{\bar{\iota}} \eta(s)[\varphi_\kappa(s)]^{\mathfrak{p}}\, ds \leq \frac{\kappa}{\mathfrak{p}}[\varphi_\kappa(\bar{\iota})]^{\mathfrak{p}}.$$
Hence,
$$\|(\Delta\mu)(\bar{\iota}) - (\Delta\vartheta)(\bar{\iota})\| \leq C_\ell N \mathfrak{p}^{-\frac{1}{\mathfrak{p}}} \kappa^{\frac{1}{\mathfrak{p}}} \varphi_\kappa(\bar{\iota})|||\mu - \vartheta|||.$$

Choosing $\kappa > 0$ sufficiently small so that
$$\wp := C_\ell N \mathfrak{p}^{-\frac{1}{\mathfrak{p}}} \kappa^{\frac{1}{\mathfrak{p}}} < 1,$$
we obtain
$$|||\Delta\mu - \Delta\vartheta||| \leq \wp\, |||\mu - \vartheta|||.$$

Thus, Δ has a unique fixed point $\mu \in \mathcal{C}_\wp$ satisfying
$$\mu(\bar{\iota}) = \bar{\mathscr{A}}^{\ell}(\bar{\iota})\wp(0) + \int_0^{\bar{\iota}} \bar{\mathscr{A}}^{\ell}(\jmath) U(\bar{\iota}, \jmath) \bar{\mathscr{B}}(\jmath, \bar{\mathscr{A}}^{-\ell}(\jmath)\mu_\jmath)\, d\jmath, \quad 0 \leq \bar{\iota} \leq \Bbbk.$$

Define
$$\Im(\iota) = \begin{cases} \bar{\mathscr{A}}^{-\ell}(t)x(\iota), & 0 \leq \iota \leq \Bbbk, \\ \wp(\iota), & \iota \leq 0. \end{cases}$$
Then $\Im(\iota) \in D(\bar{\mathscr{A}}^{\ell}(\iota))$ for all $\iota \in (\infty, \Bbbk]$, and $\Im \in C([0, \Bbbk]; D(\bar{\mathscr{A}}^{\ell}(\iota)))$. Therefore, $\Im(\iota)$ is the unique mild solution of (2). □

2.2 Stability Analysis: Uniform Asymptotic Case

Suppose $\bar{\mathscr{B}}(\iota, 0) = 0$, $\forall \iota \geq 0$. Then $\Im(\iota) = 0$ is a solution of problem (2) corresponding to the zero initial condition. In this subsection, we provide the exact conditions required to ensure that solutions of (2) remain stable over time.

Theorem 2. *Suppose that conditions* ***(H1)*** *–* ***(H3)*** *hold, and, in addition, the following assumptions are satisfied:*

(H4) *There exists a function $a(\iota)$, defined for all $\iota \geq 0$, such that*
$$\int_0^{\iota} C_\ell(\iota - s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}} \bar{C}(s)\Big[\Pi(s) + \frac{M(s)}{\Lambda}\Big] ds \leq a(\iota),$$
with
$$\sup_{\iota \geq 0} a(\iota) = \bar{\eta} < 1,$$
where $C_\ell > 0$ denotes the constant appearing in fractional-power estimates.

(H5) *For every $\varepsilon > 0$ and $\iota_1 \geq 0$, there exists $\iota_2 > \iota_1$, such that*

$$\|\bar{\mathscr{B}}(\iota, \Im_\iota)\| \leq \triangle(\iota)\Big(\varepsilon + \sup_{\jmath \in [\iota_1, \iota]} \|\Im(\jmath)\|_\ell\Big), \quad \forall \iota \geq \iota_2.$$

It follows that system (2) *exhibits uniform asymptotic stability.*

Remark 4. *Hypothesis **(H4)** imposes a bound on the convolution-type integral of the fractional-power estimates. Although this condition may appear restrictive, it holds in many practical cases where $C(\iota)$, $\Pi(\iota)$, and $M(\iota)$ are bounded or decay rapidly. In applications, one can often verify **(H4)** directly by estimating these functions and ensuring that the resulting supremum η is less than 1. Recall that $\|\cdot\|_\ell$ denotes the graph norm associated with $\bar{\mathscr{A}}(\iota)^\ell$. By Proposition 3, this norm is equivalent to the standard norm on $D(\bar{\mathscr{A}}^\ell(\iota))$, which justifies the estimates in the proof.*

Proof. For given $\bar{l}, \varepsilon > 0$, one can select $\hbar > 0$ satisfying

$$\hbar + \bar{\eta}\bar{l} \leq \max(l, \varepsilon) \quad \text{and} \quad \hbar < \varepsilon.$$

Let $\wp \in \mathfrak{B}$ denote a function satisfying

$$\sup_{0 \leq \iota \leq 1} \iota^{-1+\mathfrak{a}} \|\wp(0)\|_\ell < \hbar, \quad \text{and} \quad \|\wp\|_{\mathfrak{B}} < \frac{\hbar}{\Lambda}.$$

Define

$$S_{\bar{l},\wp} := \Big\{\Im : \mathbb{R} \to D(\bar{\mathscr{A}}^\ell(\iota)) : \Im \in C(\bar{l}),\ \Im(\iota) = \wp(\iota) \text{ if } \iota \leq 0,\ \Im(\iota) \to 0 \text{ as } \iota \to \infty\Big\},$$

where

$$C(\bar{l}) := \Big\{\Im : \mathbb{R}^+ \to D(\bar{\mathscr{A}}^\ell(\iota)) \text{ continuous with } \|\Im(\iota)\|_\ell \leq \bar{l}\Big\}.$$

Define the operator $\Upsilon : S_{l,\wp} \to S_{l,\wp}$ by

$$(\Upsilon\Im)(\bar{\iota}) := \begin{cases} U(\bar{\iota}, 0)\wp(0) + \displaystyle\int_0^{\bar{\iota}} U(\bar{\iota}, s)\bar{\mathscr{B}}(s, \Im_s)\, ds, & \bar{\iota} \geq 0, \\ \wp(\bar{\iota}), & \bar{\iota} \leq 0. \end{cases}$$

Clearly, $\Upsilon\Im : \mathbb{R}^+ \to D(\bar{\mathscr{A}}^\ell(\iota))$ is continuous and $(\Upsilon\Im)(\iota) = \wp(\iota)$ for $\iota \leq 0$. We now show that $\|(\Upsilon\Im)(\iota)\|_\ell \leq \bar{l}$, $\forall \iota \geq 0$ and $(\Upsilon\Im)(\iota) \to 0$ as $\iota \to \infty$.

For all $\iota \geq 0$, we derive

$$\begin{aligned}
\|(\Upsilon\Im)(\iota)\|_\ell &\leq \|U(\iota,0)\wp(0)\|_\ell + \int_0^\iota \|U(\iota,\beta)\bar{\mathscr{B}}(\beta,\Im_\beta)\|_\ell \, d\beta \\
&\leq \|U(\iota,0)\wp(0)\|_\ell + \int_0^\iota C_\ell(\iota-\beta)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(\iota)\|u_\beta\|_{\mathfrak{B}} \, d\beta \\
&\leq \hbar + \int_0^\iota C_\ell(\iota-\beta)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(\iota)\Big[\Pi(\beta)\sup_{0\leq\tau\leq\beta}\|\Im(\tau)\|_\ell + M(\beta)\|\Im_0\|_{\mathfrak{B}}\Big] d\beta \\
&\leq \hbar + \int_0^\iota C_\ell(\iota-\beta)^{\mathfrak{a}-\frac{\ell}{\mathfrak{a}}-1}\bar{C}(\iota)\Big[\Pi(\beta)\bar{l} + M(\beta)\frac{\hbar}{\Lambda}\Big] d\beta,
\end{aligned}$$

since $\Im_0 = \wp$ on $(\infty,0]$.

Hence,

$$\begin{aligned}
\|(\Upsilon\Im)(\iota)\|_\ell &\leq \hbar + \bar{l}\int_0^\iota C_\ell(\iota-\beta)^{\mathfrak{a}-\frac{\ell}{\mathfrak{a}}-1}\bar{C}(\iota)\left[\Pi(\beta) + \frac{M(\beta)}{\Lambda}\right] d\beta \\
&\leq \hbar + \bar{l}\eta \\
&\leq \bar{l}.
\end{aligned}$$

Because $\lim_{\iota\to\infty}\Im(\iota) = 0$, there exists $\iota_1 > 0$ for which

$$\|\Im(\iota)\|_\ell \leq \varepsilon, \quad \text{for each} \iota \geq \iota_1.$$

By Hypothesis **(H5)**, for this ε and ι_1, there exists $\iota_2 > \iota_1$, such that for all $\iota \geq \iota_2$, we have

$$\|\bar{\mathscr{B}}(\iota,\Im_\iota)\| \leq \triangle(\iota)\left(\|\Im(\iota)\|_\ell + \sup_{s\in[\iota_1,\iota]}\|\Im(s)\|_\ell\right) \leq 2\,\varepsilon\,\bar{C}(\iota).$$

Therefore, for $\iota \geq \iota_2$, we have

$$\begin{aligned}
\int_0^\iota \|U(\iota,s)\bar{\mathscr{B}}(s,\Im_s)\|_\ell ds &\leq \int_0^\iota C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\|\bar{\mathscr{B}}(s,\Im_s)\|ds \\
&\leq \int_0^{\iota_2} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\|\bar{\mathscr{B}}(s,\Im_s)\|ds \\
&\qquad + \int_{\iota_2}^\iota C_\ell(\iota-s)^{(-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}})}\|\bar{\mathscr{B}}(s,\Im_s)\|ds \\
&\leq \int_0^{\iota_2} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)\|\Im_s\|_{\mathfrak{B}}ds \\
&\qquad + 2\varepsilon\int_{\iota_2}^\iota C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)ds \\
&\leq l\int_0^{\iota_2} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)(\Pi(s) + \frac{M(s)}{\Lambda})ds + 2\varepsilon\eta \\
&\leq (l+2\varepsilon)\bar{\eta}.
\end{aligned}$$

We observe that
$$\int_{\iota_2}^{\iota} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)ds \leq \bar{\eta}.$$
As a result, for $\iota \geq \iota_2$, it holds that
$$\|(\Upsilon\Im)(\iota)\|_\ell \leq \varepsilon(1+2\eta).$$
Hence, $(\Upsilon\Im)(\iota)$ tends to zero as $\iota \to \infty$, and consequently $(\Upsilon\Im) \in S_{l,\wp}$.

For $\iota \geq 0$, we show that Υ defines a contraction mapping as follows:
$$\begin{aligned}
\|(\Upsilon\Im)(\iota)-\Upsilon(v)(\iota)\|_\ell &\leq \int_0^{\iota} \|U(\iota,s)(\bar{\mathscr{B}}(s,\Im_s)-\bar{\mathscr{B}}(s,v_s))\|_\ell ds\\
&\leq \int_0^{\iota} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\|\bar{\mathscr{B}}(s,\Im_s)-\bar{\mathscr{B}}(s,v_s)\|ds\\
&\leq \int_0^{\iota} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)\|\Im_s-v_s\|_{\mathfrak{B}}ds\\
&\leq \int_0^{\iota} C_\ell(\iota-s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(s)\Pi(s)\sup_{0\leq\tau\leq s}\|\Im(\tau)-v(\tau)\|_\ell ds\\
&\leq \sup_{0\leq\tau\leq\Bbbk}\|\Im(\tau)-v(\tau)\|_\ell \int_0^{\iota} C_\ell(\iota-\beta)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\bar{C}(\iota)\left[\Pi(\beta)+\frac{M(\beta)}{\Lambda}\right]d\beta\\
&\leq \bar{\eta}\sup_{0\leq\tau\leq\Bbbk}\|\Im(\tau)-v(\tau)\|_\ell.
\end{aligned}$$
Or,
$$\sup_{0\leq s\leq\Bbbk}\|(\Upsilon\Im)(s)-(\Upsilon v)(s)\|_\ell \leq \bar{\eta}\sup_{0\leq s\leq\Bbbk}\|\Im(s)-v(s)\|_\ell.$$
Applying the contraction mapping principle, we conclude that Υ admits a unique fixed point $\Im$ in $S_{\bar{l},\wp}$, which corresponds to a solution of (2). Given $\varepsilon > 0$ with $\varepsilon < \bar{l}$, select $\hbar > 0$ so that
$$\hbar + \bar{\eta}\varepsilon \leq \varepsilon.$$
Let $\Im(\iota) = \Im(\iota,\wp)$ denote a solution of (2) satisfying $\sup_{0\leq\bar{\iota}\leq 1}\bar{\iota}^{-1+\mathfrak{a}}\|\wp(0)\|_\ell < \hbar$ and $\|\wp\|_{\mathfrak{B}} < \dfrac{\hbar}{\Lambda}$, then
$$u(\iota) = U(\iota,0)\wp(0) + \int_0^{\iota} U(\iota,s)\bar{\mathscr{B}}(s,\Im_s)ds.$$
We assert that $\|\Im(\iota)\|_\ell < \varepsilon, \forall\iota > 0$. Indeed, we have
$$\|\Im(0)\|_\ell = \|\wp(0)\|_\ell \leq \Lambda\|\wp\|_{\mathfrak{B}} < \Lambda\cdot\frac{\hbar}{\Lambda} = \hbar < \varepsilon.$$

Assume, toward a contradiction, that there exists $\iota^\star > 0$ such that

$$\|\Im(\iota^\star)\|_\ell = \varepsilon, \quad \text{while} \quad \|\Im(s)\|_\ell < \varepsilon \text{ for all } 0 \le s < \iota^\star.$$

We then estimate

$$\begin{aligned}\|\Im(\iota^\star)\|_\ell &\le \hbar + \int_0^{\iota^\star} C_\ell(\iota^* - \beta)^{\mathfrak{a}-\frac{\ell}{\mathfrak{a}}-1}\bar{C}(\beta)\|\Im_\beta\|_{\mathfrak{B}}\, d\beta \\ &\le \hbar + \bar{\eta}\varepsilon \\ &< \varepsilon,\end{aligned}$$

which is in contradiction with the choice of $\iota^\star$. Hence,

$$\|\Im(\iota)\|_\ell < \varepsilon, \quad \text{for all } \iota > 0,$$

which establishes the uniform asymptotic stability of system (2). □

3 Some Special Cases

3.1 Abstract Equation Without Delay

Consider:

$$\frac{d\Im(\iota)}{d\iota} + \bar{\mathscr{A}}(\iota)\Im = \aleph(\iota, \Im), \quad \Im(0) = x \in D(\bar{\mathscr{A}}^\ell(\iota)), \tag{7}$$

where $\iota > 0$, $\bar{\mathscr{A}}(\iota) : D(\bar{\mathscr{A}}(\iota)) \subset \mathscr{X} \to \mathscr{X}$ is uniformly almost sectorial, associated with a linear evolution process of growth $1 - \mathfrak{a}$ $(0 < \mathfrak{a} < 1)$, and a family of fractional power spaces $D(\bar{\mathscr{A}}^\ell(\iota))$. The function $f : \mathbb{R}^+ \times D(\bar{\mathscr{A}}^\ell(\iota)) \to \mathscr{X}$ satisfies

$$\|\aleph(\iota, \varsigma_1) - \aleph(\iota, \varsigma_2)\| \le C(\iota)\|\bar{\mathscr{A}}^\ell(\iota)(\varsigma_1 - \varsigma_2)\|, \quad \aleph(\iota, 0) = 0,$$

for all $\varsigma_1, \varsigma_2 \in D(\bar{\mathscr{A}}^\ell(\iota))$, where $\bar{C}(\iota) = \triangle(\iota)\aleph(\iota)$, with $\triangle(\cdot)$ a nonnegative measurable function in $L^{\mathfrak{p}}_{\text{loc}}(\mathbb{R}^+)$ and

$$\sup_{\iota \ge 0} \int_0^\iota (\iota - s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}}\aleph(s)^{\mathfrak{q}}\, ds < \infty, \quad \frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1.$$

In the phase space C_μ with norm $\|\cdot\|_\mu$, we note that

$$\aleph(\iota, \Im(\iota)) = \aleph(\iota, \Im(\iota + 0)) = \aleph(\iota, \Im_\iota(0)),$$

we define

$$\bar{\mathscr{B}}(\iota, \wp) = \aleph(\iota, \wp(0)) \quad \text{and} \quad \wp(\iota) = x \text{ for all } \iota \le 0.$$

Then, problem (7) can be rewritten as

$$\frac{d\Im(\iota)}{d\iota} + \bar{\mathscr{A}}(\iota)\Im = \bar{\mathscr{B}}(\iota, \Im_\iota), \quad \Im(0) = \wp, \quad \iota \geq 0, \tag{8}$$

For all $\wp, \psi \in C_\mu$, we have

$$\|\bar{\mathscr{B}}(\iota, \wp) - \bar{\mathscr{B}}(\iota, \psi)\| \leq \bar{C}(\iota)\|\wp(0) - \psi(0)\|_\ell \leq \bar{C}(\iota)\|\wp - \psi\|_\mu.$$

By Theorem 1, a unique mild solution of (7) exists on $[0, \infty)$. Theorem 2, together with hypotheses **(H4)** and **(H5)**, then concludes the proof.

3.2 Semilinear Parabolic Equations with Infinite Delay

We recall (see [18]) that for $\mu \in (0, 1)$, the Hölder space $C_0^\mu(\Theta)$ consists of all uniformly continuous functions $\nu : \Theta \to \mathbb{R}$ and forms a Banach space under the norm

$$\|\nu\|_\mu := \sup_{\sigma \in \Theta} |\nu(\sigma)| + \sup_{\substack{\sigma, \phi \in \Theta \\ 0 < |\sigma - \phi| \leq 1}} \frac{|\nu(\sigma) - \nu(\phi)|}{|\sigma - \phi|^\mu} = \|\nu\|_{C(\Theta)} + H^\mu(\nu),$$

where $H^\mu(\nu)$ denotes the Hölder semi-norm of ν.

Consider the semilinear parabolic problem

$$\begin{cases} \dfrac{d\Im(\iota)}{d\iota} = L(\iota, \mu)\Im(\iota, \mu) + \displaystyle\int_\infty^0 G_1(\iota + s)G_2(\iota + s)\Im(\iota + s, \mu)\, ds, & \iota > 0, \ \mu \in \Omega, \\ \Im(\iota, \mu) = 0, & \mu \in \partial\Omega, \ \iota \geq 0, \\ \Im(\iota, \mu) = \Im_0(\iota, \mu), & \iota \leq 0, \ \mu \in \Omega, \end{cases} \tag{9}$$

where
$bar\mathscr{A}(\iota) : D(\bar{\mathscr{A}}(\iota)) \subset C_0^\mu(\Theta) \to C_0^\mu(\Theta)$ is defined by

$$\bar{\mathscr{A}}(\iota)\nu = L(\iota, \cdot)\nu, \quad \nu \in D(\bar{\mathscr{A}}(\iota)),$$

with

$$L(\iota, \mu) = \sum_{i,j=1}^n \varpi_{ij}(\iota, \mu) \frac{\partial^2 \Im}{\partial\mu_i \partial\mu_j},$$

for each $\iota \geq 0$. Assume that the functions $G_1(\iota + \cdot)G_2(\iota + \cdot)$ are positive and integrable on $(\infty, 0]$, and let $\Im_0$ be a suitable continuous initial function.

Let $\Xi > 0$ be a constant such that the coefficients $\varpi_{ij} : \mathbb{R} \times \Theta \to \mathbb{R}$, together with their derivatives, are uniformly continuous and fulfill:

1. Each ϖ_{ij} is uniformly continuous in $\mu \in \Theta$ and equicontinuous in ι,

2. For all $|\iota - \Upsilon| < 1$,

$$\sup_{x\in\Omega} |\varpi_{ij}(\iota,\mu) - \varpi_{ij}(\Upsilon,\mu)| \le \Xi|\iota - \Upsilon|,$$

3. For all $|\iota - \Upsilon| < 1$ and $0 \le k \le 1$,

$$\sup_{\mu\in\Omega} \left| \frac{\partial \varpi_{ij}}{\partial x_k}(\iota,\mu) - \frac{\partial \varpi_{ij}}{\partial \mu_k}(\Upsilon,\mu) \right| \le \Xi|\iota - \Upsilon|,$$

and uniformly with respect to ι, the ellipticity condition: for any $\xi \in \mathbb{R}^n$, there exist constants $a > 0$ and $A > 0$ such that

$$a|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le A|\xi|^2.$$

As shown in [8], $\bar{\mathscr{A}}(\iota)$ exhibits the uniform almost sectorial property with growth $1 - \mathfrak{a}$ $(0 < \mathfrak{a} < 1)$, with fractional power spaces $D(\bar{\mathscr{A}}^{\ell}(\iota)) = C_0^{\mu}(\Theta)$, and it verifies hypothesis **(H2)**.

Set

$$\wp(s)(\mu) = \Im_0(\beta,\mu),\ \beta \le 0,\ \mu \in \Theta,$$

$$\bar{\mathscr{B}}(\iota,\wp)(\mu) = \int_{\infty}^{0} G_1(\iota+\beta)G_2(\iota+\beta)\wp(s)(\mu)\,d\beta, \quad \wp \in C_{\upsilon}.$$

Then problem (9) can be rewritten as

$$\begin{cases} \dfrac{d\Im}{d\iota} = \bar{\mathscr{A}}(\iota)\Im + \bar{\mathscr{B}}(\iota,\Im_{\iota}), \quad \iota > 0,\ x \in \Omega, \\ \Im_0(\iota,\mu) = \wp \in C_{\upsilon}. \end{cases}$$

We assume:

1. For all $\iota \ge 0$, the function $s \mapsto G_1^2(\iota+s)e^{-2\upsilon s}$ is integrable on $(\infty, 0]$.
2. $\displaystyle \sup_{\iota\ge 0} \int_0^{\iota} (\iota - s)^{-1+\mathfrak{a}-\frac{\ell}{\mathfrak{a}}q} \left(\int_{\infty}^{0} G_2^2(t+\jmath)\,d\jmath \right)^2 ds < \infty.$
3. $\displaystyle \lim_{\jmath\to\infty} (e^{\upsilon\jmath}\|\Im_0(\jmath,\cdot)\|)$ exists, and $\Im(0,x) = 0$ for $\mu \in \partial\Omega$.

For every $\wp, \psi \in C_{\upsilon}$, we have

$$\begin{aligned} \int_{\infty}^{0} G_1(\iota+\jmath)G_2(\iota+\jmath)\|\wp(\jmath)(\cdot) - \psi(\jmath)(\cdot)\|d\jmath &= \int_{\infty}^{0} e^{-\upsilon\jmath}G_1(\iota+\jmath)G_2(\iota+\jmath)e^{\upsilon\jmath} \\ &\quad \times \|\wp(\jmath)(\cdot) - \psi(\jmath)(\cdot)\|d\jmath \\ &\le \int_{\infty}^{0} e^{-2\upsilon\jmath}G_1^2(\iota+\jmath)d\jmath \\ &\quad \times \int_{\infty}^{0} G_2^2(\iota+\jmath)d\jmath\, \|\wp - \psi\|_{\mu}. \end{aligned}$$

Hence, we obtain

$$\|\bar{\mathscr{B}}(\iota, u_\iota) - \bar{\mathscr{B}}(\iota, v_\iota)\| \leq \bar{C}(\iota)\|\wp - \psi\|_\mu,$$

where $\bar{C}(\iota) = \triangle(\iota)\mho(\iota)$ with

$$\triangle(\iota) = \int_\infty^0 e^{-2\upsilon\jmath} G_1^2(\iota + \jmath)d\jmath, \quad \mho(\iota) = \int_\infty^0 G_2^2(\iota + \jmath)d\jmath.$$

Therefore, assumptions (1)–(3) imply hypothesis **(H3)**. By Theorem 1, there exists a unique integral solution $\Im(t, x)$ on $\mathbb{R} \times \Omega$. Moreover, under hypotheses **(H4)** and **(H5)**, Theorem 2 establishes the result for system (9).

4 Conclusion

We have identified several gaps in the stability theory of differential equations with infinite delay and almost sectorial operators. The results presented here establish a rigorous foundation for future investigations. Subsequent research might generalize these findings to broader classes of nonlinear perturbations, analyze stochastic or impulsive variants of such systems, or design numerical methods to approximate solutions while ensuring stability. Additionally, exploring these concepts in infinite-dimensional control problems such as with distributed parameters or partial differential equations with memory effects, offers a compelling direction for future work.

Conflict of interest The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

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