Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 23(1 & 2) May & Nov. 2025, pp. 31–44
DOI : 10.47974/JDSGT-2024-002

Hyper-order of solutions to a class of nonlinear complex differential equations

Manab Biswas *
Department of Mathematics
Kalimpong College
Kalimpong 734301
West Bengal
India

Dilip Chandra Pramanik †
Department of Mathematics
University of North Bengal
Raja Rammohanpur
Darjeeling 734013
West Bengal
India

Debashis Kumar Mandal §
Department of Mathematics
Cooch Behar College
Cooch Behar 736101
West Bengal
India

Abstract

This study focuses on analyzing the hyper-order associated with meromorphic solutions of nonlinear complex differential equations represented by $\mathcal{P}[\boldsymbol{g}] - Q\boldsymbol{g}^{\gamma_{\mathcal{P}}} = R$, where $\mathcal{P}[\boldsymbol{g}]$ denotes a differential polynomial in $\boldsymbol{g}$, $\gamma_{\mathcal{P}}$ indicates for the degree of $\mathcal{P}[\boldsymbol{g}]$, and the coefficients of $\mathcal{P}[\boldsymbol{g}]$, along with $Q = Q(z)$ and $R = R(z)$, satisfy certain prescribed growth conditions.

Subject Classification (2020): *34M03, 34M04, 34M05, 30D20, 30D35.*

Keywords: *Meromorphic solution, Hyper-order, Nonlinear differential equations.*

* *E-mail:* `dr.manabbiswas@gmail.com (Corresponding Author)`
† *E-mail:* `dcpramanik.nbu2012@gmail.com`
§ *E-mail:* `debashis214@gmail.com`

1. Introduction and main results

Let $g(z)$ be a non-constant meromorphic function on the complex plane $\mathbb{C}$, analytic everywhere except for its poles. If $g(z)$ has no poles, it is entire function. We assume familiarity with standard notations in Nevanlinna's value distribution theory, namely $N(r,g)$, $m(r,g)$, $T(r,g)$, and $M(r,g)$, representing respectively the counting function, proximity function, characteristic function, and maximum modulus function. The basic concepts and classical results of this theory may be found in standard references (see [4, 6, 10, 17]).

The measure of a set $E \subset [0,+\infty)$ is defined as $\mu(E) = \int_E dr$. For a subset F of $[1,+\infty)$, its measure in the logarithmic sense is defined as $l\mu(F) = \int_F \frac{dr}{r}$. The lower and upper densities of E are defined repectively by

$$\underline{Dens}\, E = \liminf_{r\to+\infty} \frac{\mu(E\cap[0,r])}{r}, \overline{Dens} E = \limsup_{r\to+\infty} \frac{\mu(E\cap[0,r])}{r}.$$

If H is a set of complex numbers, these densities are defined in terms of the moduli of its elements as

$$\underline{Dens} H = \underline{Dens}\{|z|: z \in H\}, \overline{Dens} H = \overline{Dens}\{|z|: z \in H\}$$

The term $S(r,g)$ stands for any term satisfying

$$S(r,g) = o(T(r,g)(r\to\infty),$$

which implies $\frac{S(r,g)}{T(r,g)} \to 0$ for large r, outside a set of finite measure. A meromorphic function $h(z)$ is referred to as small with respect to a meromorphic function $g(z)$ if

$$T(r,h) = S(r,g),$$

meaning its growth is negligible compared to that of $g(z)$, (see [19], [13]).

In this paper, we also need the following definitions:

Definition 1: The order $\sigma(g)$ and the lower order $\lambda(g)$ corresponding to a non-constant meromorphic function g are expressed as the following limits:

$$\sigma(g) = \limsup_{r\to+\infty} \frac{\log T(r,g)}{\log r}, \lambda(g) = \liminf_{r\to+\infty} \frac{\log T(r,g)}{\log r}.$$

These quantities describe the growth behaviour of $g(z)$ as $|z| \to \infty$.

Definition 2: [19] The hyper order $\sigma_1(g)$ associated with a non-constant meromorphic function g is defined as

$$\sigma_1(g) = \limsup_{r\to+\infty} \frac{\log\log T(r,g)}{\log r}.$$

Definition 3: [1, 12] A differential monomial associated with a non-constant meromorphic function g is expressed as

$$\mathfrak{M}_j[g] = g^{n_{0j}}\left(g^{(1)}\right)^{n_{1j}}\left(g^{(2)}\right)^{n_{2j}} \cdots \left(g^{(k)}\right)^{n_{kj}},$$

where each n_{ij} for $i = 0,1,\dots,k$ is a non-negative integer. The degree $\gamma_{\mathfrak{M}_j}$ and the weight $\Gamma_{\mathfrak{M}_j}$ of $\mathfrak{M}_j[g]$ are given by

$$\gamma_{\mathfrak{M}_j} = \sum_{i=0}^{k} n_{ij}, \quad \Gamma_{\mathfrak{M}_j} = \sum_{i=0}^{k} (i+1)n_{ij}.$$

A differential polynomial generated by g is expressed as a finite sum

$$\mathcal{P}[g] = \sum_{j=1}^{t} Q_j(z)\mathfrak{M}_j[g],$$

where each coefficient $Q_j(z)$ is not identically zero and satisfies

$$T(r, Q_j) = S(r, g) \text{ for every } j = 1,2,\dots,t.$$

The quantities $\gamma_{\mathcal{P}}$ and $\Gamma_{\mathcal{P}}$, representing the degree and weight of $\mathcal{P}[g]$, are determined, respectively as

$$\gamma_{\mathcal{P}} = \max\left\{\gamma_{\mathfrak{M}_j}: j = 1,2,\dots,t\right\}, \Gamma_{\mathcal{P}} = \max\left\{\Gamma_{\mathfrak{M}_j}: j = 1,2,\dots,t\right\}.$$

The lower degree of $\mathcal{P}[g]$ is

$$\gamma_{\mathcal{P}}^* = \min\left\{\gamma_{\mathfrak{M}_j}: j = 1,2,\dots,t\right\},$$

and the order of $\mathcal{P}[g]$ is the derivative of g of highest order that appears in $\mathcal{P}[g]$. When $\gamma_{\mathcal{P}} = \gamma_{\mathcal{P}}^*$, $\mathcal{P}[g]$ is is said to be homogeneous. $\mathcal{P}[g]$ is termed linear when $\gamma_{\mathcal{P}} = 1$; otherwise, it is nonlinear.

Define

$$\chi = \max\{\Gamma_{\mathfrak{M}_j} - \gamma_{\mathfrak{M}_j}: j = 1,\dots,t\} = \max\left\{\sum_{i=1}^{k} i.n_{ij}: j = 1,\dots,t\right\}$$

and

$$\psi = \min\{\Gamma_{\mathfrak{M}_j} - \gamma_{\mathfrak{M}_j}: j = 1,\dots,t\} = \min\left\{\sum_{i=1}^{k} i.n_{ij}: j = 1,\dots,t\right\}.$$

Consider a linear differential equation of second order given by

$$g''(z) + \beta_0(z)g'(z) + \beta_1(z)g(z) = 0, \tag{1}$$

with entire coefficients $\beta_0(z)$ and $\beta_1(z)$, where $\beta_1(z)$ is not identically zero. Kwon [9], and Chen et al. [3] respectively obtained the following results regarding the solutions of (1):

Theorem 1: [9] *Let $H \subset \mathbb{C}$ has positive upper density, and let $\beta_0(z)$, $\beta_1(z)$ are entire functions for which, there exist positive constants p and q such that*

$$|\beta_0(z)| \le e^{(o(1)|z|^q)} \text{ and } |\beta_1(z)| \ge e^{(1+o(1))p|z|^q}$$

as $z \to \infty$ on H. Then, any solution $g \not\equiv 0$ of equation (1) possesses infinite order, that is, $\sigma(g) = +\infty$ and its hyper order satisfies $\sigma_1(g) \ge q$.

Theorem 2: [3] *Consider a subset H of $\mathbb{C}$ with positive upper density. Assume $\beta_0(z)$, $\beta_1(z)$ are entire functions with $\sigma(\beta_0) \le \sigma(\beta_1) = \sigma < +\infty$. Also, for* *some fixed positive constant* A *and for every* $\varepsilon > 0$, *the bounds*

$$|\beta_0(z)| \leq e^{(o(1)|z|^{\sigma-\varepsilon})} \text{ and } |\beta_1(z)| \geq e^{(1+o(1))A|z|^{\sigma-\varepsilon})}$$

hold as $z \to \infty$ *along the set* H.*Under these assumptions, each non-zero solution* g *of* (1) *satisfies* $\sigma(g) = +\infty$ *and* $\sigma_1(g) = \sigma(\beta_1) = \sigma$.

Let us take into account the linear differential equation of order n, that is,

$$\beta_n(z)g^{(n)} + \beta_{n-1}(z)g^{(n-1)} + \ldots + \beta_0(z)g = 0, n \geq 2 \tag{2}$$

where each coefficient $\beta_j(z)$ $(j = 0,1,\ldots,n)$ is entire with $\beta_0(z) \not\equiv 0$. Yang [18] considered the linear differential equations (2) and obtained some results on the growth of solutions which generalized Theorem 1 and Theorem 2. He proved the following theorems:

Theorem 3: [18] *If* $H \subset \mathbb{C}$ *has positive upper density and* $\beta_j(z)$ $(j = 0,1,\ldots,n)$ *are entire functions obeying*

$$|\beta_0(z)| \geq e^{p|z|^\eta} \text{ and } |\beta_j(z)| \leq e^{q|z|^\eta}, j = 1,2,\ldots,n,$$

for some constants $0 \leq q < p$ *and* $\eta > 0$ *as* $z \to \infty$ *on* H, *then any non-trivial entire or meromorphic solution* g *of* (2) *possesses infinite order, i.e.,* $\sigma(g) = +\infty$ *and* $\sigma_1(g) \geq \eta$.

Theorem 4: [18] *Let* $H \subset \mathbb{C}$ *be a set whose upper density is positive. Suppose the functions* $\beta_j(z)(j = 0,1,\ldots,n-1)$ *are entire and* $\beta_n(z) \equiv 1$, *and assume that*

$$\max\{(\sigma(\beta_j)): j = 1,2,\ldots,n-1\} \leq \sigma(\beta_0) = \sigma < \infty.$$

If, for some constants $0 \leq q < p$ *and all sufficiently small positive* ε,

$$|\beta_0(z)| \geq e^{p|z|^{\sigma-\varepsilon}} \text{ and } |\beta_j(z)| \leq e^{q|z|^{\sigma-\varepsilon}}, j = 1,2,\ldots,n,$$

as $z \to \infty$ *on* H, *then every nonzero meromorphic or entire solution of* (2) *has infinite order and its hyper order* $\sigma_1(g) \geq \sigma(\beta_0)$.

In the same paper, Yang [18] also proved the following theorem:

Theorem 5: [18] *If* $\beta_0(z)$ *is an entire function and* $\beta_1(z)$ *is a transcendental entire satisfying*

$$M(r,\beta_0) = O(M(r,\beta_1))^\xi, 0 < \xi < \infty,$$

then, with at most one exception, every solution g *of the differential equation* $g^n - \beta_1 g = \beta_0$ *satisfies* $\sigma(g) = \sigma(\beta_1)$.

Regarding Theorems 3-5, the following is a valid question:

Question: What can be said about the order and hyper-order of solutions if the linear differential polynomial in Theorems 3-5 be replaced by a non-linear differential polynomial?

To address this question, we consider a class of nonlinear differential equations of the form

$$\mathcal{P}[g] - Qg^{\gamma_{\mathcal{P}}} = R,$$

where $Q = Q(z)$ and $R = R(z)$ are entire functions. In this paper, we establish several results concerning the growth of solutions to this class of equations, thereby generalizing Theorems 3 - 5. Additionally, we provide illustrative examples to support our findings.

The main results are stated in the following theorems:

Theorem 6: *Let $H \subset \mathbb{C}$ be a set with positive upper density. Suppose the entire functions $Q_j(z)$ $(j = 1,2,\dots,t)$ and $Q(z)$ satisfy the growth conditions*

$$|Q(z)| \geq e^{p|z|^{\eta}} \text{ and } |Q_j(z)| \leq e^{q|z|^{\eta}}, j = 1,2,\dots,t$$

for some constants $0 \leq q < p$ and $\eta > 0$ as $z \to \infty$ on H. Then, every nonzero meromorphic (or entire) solution of

$$\mathcal{P}[g] - Qg^{\gamma_{\mathcal{P}}} = 0 \tag{3}$$

must be of infinite order i.e., $\sigma(g) = +\infty$ and its hyper order satisfies $\sigma_1(g) \geq \eta$.

Example 1: Consider the nonlinear differential equation

$$gg'' + (g')^2 + (e^z - 2e^{2z})g = 0.$$

For this equation,

$$\mathcal{P}[g] = Q_1\mathfrak{M}_1[g] + Q_2\mathfrak{M}_2[g] = gg'' + (g')^2$$

is of degree $\gamma_{\mathcal{P}} = 2$,with

$$Q_1(z) = 1, Q_2(z) = 1, \text{ and } Q(z) = e^z - 2e^{2z}.$$

Obviously, the growth conditions

$$|Q_j(z)| = 1 \leq e^{q|z|^{\eta}} (j = 1,2), |Q(z)| \geq e^{2z} \geq e^{p|z|^{\eta}}$$

hold for $p = 1, q = 0$ and $\eta = 1$ along the positive real axis $H = \{x \in \mathbb{R}: x \geq 0\}$ with $\overline{Dens}H > 0$. It can be verified that $g(z) = e^{-e^z}$ is an entire solution of the equation with $\sigma(g) = +\infty$ and $\sigma_1(g) = 1 = \eta$.

Theorem 7: *If $H \subset \mathbb{C}$ has positive upper density and $Q(z), Q_j(z)$ $(j = 1,2,\dots,t-1)$ are entire functions with $Q_t(z) \equiv 1$ and*

$$\max\{\sigma(Q_j) | j = 1,2,\dots,t-1\} \leq \sigma(Q) = \overline{\sigma} < \infty$$

and if there exist constants $0 \leq q < p$ such that for every sufficiently small $\varepsilon > 0$,

$$|Q(z)| \geq e^{p|z|^{\overline{\sigma}-\varepsilon}} \text{ and } |Q_j(z)| \leq e^{q|z|^{\overline{\sigma}-\varepsilon}} (j = 1,2,\dots,t)$$

as $z \to \infty$ on H, then every nontrivial meromorphic (or entire) solution g of (3) has $\sigma(g) = +\infty$ and $\sigma_1(g) = \overline{\sigma}$.

Example 2: Consider the nonlinear differential equation

$$(g'')^3 + g'''(g')^2 = (e^{6z} + 4e^{5z} + 6e^{4z} + 2e^{3z})g^3.$$

In this equation,

$$\mathcal{P}[g] = (g'')^3 + g'''(g')^2$$

is of degree $\gamma_{\mathcal{P}} = 3$, with

$$Q_1(z) = 1, Q_2(z) = 1, \text{and } Q(z) = e^{6z} + 4e^{5z} + 6e^{4z} + 2e^{3z}.$$

Clearly, $Q(z)$ is entire with $\overline{\sigma} = \sigma(Q) = 1$. Therefore, $\sigma(Q_j) = 0 (j = 1,2)$, so that

$$\max\{\sigma(Q_j) | j = 1\} = \sigma(Q_1) \leq \overline{\sigma}.$$

Choose $p = 1, q = 0$,and let $\varepsilon > 0$ be sufficiently small. Then for large $z \in H = \{x \in \mathbb{R}: x \geq 0\}$ with $\overline{Dens H} > 0$, we have

$$|Q(z)| \sim e^{6x} \geq e^{1.|x|^{1-\varepsilon}} = e^{p|z|^{\overline{\sigma}-\varepsilon}},$$

and for $j = 1,2$,

$$|Q_j(z)| = 1 \leq e^{0.|x|^{1-\varepsilon}} = e^{q|z|^{\overline{\sigma}-\varepsilon}}.$$

Hence, all the growth conditions of the theorem are satisfied. A direct computation reveals that $g(z) = e^{e^z}$ is an entire solution of infinite order, and its hyper order $\sigma_1(g) = \overline{\sigma} = 1$.

Theorem 8: *Let $H \subset \mathbb{C}$ has positive upper density, and consider entire functions $Q(z), Q_j(z)$ $(j = 1,2,\dots,t)$, and $R(z)$ satisfying*

$$|Q(z)| \geq e^{p|z|^\eta}, |Q_j(z)| \leq e^{q|z|^\eta} (j = 1,2,\dots,t), |R(z)| \leq e^{q|z|^\eta}$$

for some real constants $0 \leq q < p$ and $\eta > 0$, as $z \to \infty$ on H. Under these conditions, any nonzero meromorphic (or entire) solution g of the equation

$$\mathcal{P}[g] = R(z) + Q(z)g^{\gamma_{\mathcal{P}}} \tag{4}$$

must be of infinite order.

Example 3: Examine the following nonlinear equation:

$$(g')^2 + g'' - (1 + e^{3z})g^2 = e^{3z}e^{-2e^z}.$$

Here,

$$\mathcal{P}[g] = (g')^2 + g''$$

is of degree $\gamma_{\mathcal{P}} = 2$, with

$$Q_j(z) = 1 (j = 1,2), R(z) = e^{3z}e^{-e^z}, \text{ and } Q(z) = e^{3z} + 1.$$

Along the positive real axis $H = \{x \in \mathbb{R}: x \geq 0\}$ with $\overline{Dens H} > 0$, we have

$$|Q_j(z)| \leq e^{1.|x|^1} = e^{q|z|^\eta} (j = 1,2) \text{ and } |Q(z)| \geq e^{3|x|} \geq e^{2.|x|^1} = e^{p|z|^\eta},$$

while the remainder term satisfies

$$|R(z)| = e^{3x - 2e^x} \ll e^{q|x|^\eta} as |z| = x \to \infty,$$

because the term $-2e^x$dominates $3x$ for large x, with the choices $p = 2, q = 1$ and $\eta = 1$. It follows that $g(z) = e^{-e^z}$ is an entire solution with $\sigma(g) = +\infty$.

Theorem 9: *Suppose that* $R(z)$ *and* $Q(z)$ *are entire functions, with* $Q(z)$ *being transcendental and*

$$M(r, R(z)) = O\left\{\left(M(r, Q(z))\right)^{\xi}\right\}, \text{as } r \to \infty$$

for some positive finite number ξ. *Then, any nontrivial meromorphic or entire solution* g *of* (4) *has hyper-order at most* $\sigma(Q)$.

Example 4: Consider the nonlinear equation

$$g'g + (g'')^2 - g^2(e^z + 2) + e^{3z} = 0.$$

In this setting,

$$\mathcal{P}[g] = g'g + (g'')^2$$

is of degree $\gamma_{\mathcal{P}} = 2$, $Q(z) = e^z + 2$ is transcendental with $\sigma(Q) = \sigma(e^z + 2) = \sigma(e^z) = 1$, and $R(z) = -e^{3z}$. The remainder term satisfies the growth condition

$$M(r, R(z)) = O\left\{\left(M(r, Q(z))\right)^{\xi}\right\}, \text{ as } r \to \infty$$

for $\xi = 3$, because $M(r, R(z)) = e^{3r}$ and $M(r, Q(z)) = e^r$ as $r \to \infty$. Therefore, the conditions of Theorem 9 are satisfied. Clearly, $g = e^z$ represents an entire solution with $\sigma_1(g) = \sigma_1(e^z) = 0 \le \sigma(Q)$.

2. Preparatory Lemmas

We introduce a few lemmas needed for the developments in later sections.

Lemma 1: [10, 8, 2] *Let* $g(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function. The maximum term associated with* g *is expressed as*

$$\pi(r, g) = \max\{|a_k| r^k, k = 0,1, \ldots\}$$

and the central index is given by

$$\nu(r, g) = \max\{m : \pi(r, g) = |a_m| r^m\}$$

Then

$$\sigma(g) = \limsup_{r \to +\infty} \frac{\log \nu(r,g)}{\log r}.$$

For a transcendental entire function g,

$$\sigma_1(g) = \limsup_{r \to +\infty} \frac{\log \log \nu(r,g)}{\log r}.$$

Lemma 2: [18, 15] *Suppose that* $p(r)$ *and* $q(r)$ *are non-decreasing functions defined on* $(0, \infty)$ *and* $p(r) \ge q(r)$ *for every* $r \notin E \cup [0,1]$, *where* $E \subset (1, +\infty)$ *has finite logarithmic measure. Consequently, for each* $\kappa > 1$, *one can* *find a constant* $r_0 > 0$ *ensuring that* $p(\kappa r) \geq q(r)$ *whenever* $r > r_0$.

Lemma 3: [8, 17] Let g be a non-constant meromorphic function on $\mathbb{C}$, and let $k > 0$ be an integer. Then for any pair of radii r and R satisfying $1 \leq r < R < \infty$, for some positive constant A determined by g and k, we have

$$m\left(r, g^{(k)}/g\right) \leq A\left(\log^{+} \tfrac{RT(R,g)}{R-r} + 1\right).$$

Lemma 4: [5, 1] *Consider a transcendental meromorphic function* g *on* $\mathbb{C}$ *having finite order* σ. *Let*

$$\Delta = \{(i_1, k_1), (i_2, k_2), \ldots, (i_m, k_m)\}$$

denote a set of distinct ordered integer pairs such that $i_j > k_j \geq 0$ $(j = 1, \ldots, m)$. *For any prescribed* $\varepsilon > 0$, *there exists a subset* $E \subset (1, +\infty)$ *having finite logarithmic measure such that, for all* z *with* $|z| \notin E \cup [0,1]$ *and for every pair* $(i, k) \in \Delta$, *we have*

$$\left|\tfrac{g^{(i)}(z)}{g^{(k)}(z)}\right| \leq |z|^{(i-k)(\sigma-1+\varepsilon)}.$$

Lemma 5: [5, 1] *Let* g *be a transcendental meromorphic function, and choose a real number* $\phi > 1$. *Then one can find a subset* $E \subset (1, +\infty)$ *having finite logarithmic measure and a positive constant B depending only on* ϕ *and the integer* $k = 1, 2, \ldots, m$ *such that whenever* $z \in \mathbb{C}$ *with* $|z| = r \notin E \cup [0,1]$, *the following estimate holds:*

$$\left|g^{(k)}(z)\right| \leq B\left\{\frac{(\log r)^{\phi} T(\phi r, g) \log T(\phi r, g)}{r}\right\}^{k} |g(z)|,$$

for each k.

Lemma 6: [7, 10, 16] *Let* $g(z)$ *be a transcendental entire function, and let* $E \subset (1, +\infty)$ *be a subset of finite logarithmic measure. Then for any point* z *satisfying* $|z| = r \notin [0,1] \cup E$ *and* $|g(z)| = M(r, g)$, *we have*

$$g^{(i)}(z) = g(z)\left\{\tfrac{\nu(r,g)}{z}\right\}^{i}(1 + o(1)), i \in \mathbb{N}.$$

Lemma 7: [16 11, 14] *Consider a transcendental entire function* $g(z)$, *and suppose that* $E \subset (1, +\infty)$ *is of finite logarithmic measure. In this case, we can find a sequence* $\{z_n = r_n e^{i\theta_n}\}$ *such that*

$$|z_n| = r_n \notin E, \theta_n \in [0, 2\pi), \lim_{n \to +\infty} \theta_n = \theta_0 \in [0, 2\pi),$$

and $|g(z_n)| = M(r_n, g)$. *If* $0 < \sigma(g) < +\infty$, *then for every* $\varepsilon > 0$ *and all sufficiently large* r_n,

$$r_n^{\sigma(g)-\varepsilon} < \nu(r_n, g) < r_n^{\sigma(g)+\varepsilon}.$$

If $\sigma(g) = +\infty$, *then for any large* $K > 0$ *and sufficiently large* r_n,

$$\nu(r_n, g) > r_n^{K}.$$

3. Proofs of main results

Proof of Theorem 6. Suppose $g \not\equiv 0$ is a meromorphic solution of (3) with finite order $\sigma(g) = \sigma < +\infty$. Then by (3), we obtain

$$1 \leq \sum_{j=1}^{t} \left|\frac{Q_j(z)}{Q(z)}\right| \left|\frac{\mathfrak{M}_j(g)}{g^{\gamma_{\mathfrak{M}_j}}}\right| \leq \sum_{j=1}^{t} \left|\frac{Q_j(z)}{Q(z)}\right| \left(\prod_{i=1}^{k} \left|\frac{g^{(i)}}{g}\right|^{n_{ij}}\right) \tag{5}$$

Aplying Lemma 4, we can identify a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure such that, for all z with $|z| \notin E_2 = [0,1] \cup E_1$ and for any $\varepsilon > 0$, $i = 1,2,\dots,k$, the following estimate holds:

$$\left|\frac{g^{(i)}(z)}{g(z)}\right| \leq |z|^{i(\sigma-1+\varepsilon)}. \tag{6}$$

Meanwhile, the hypothesis of Theorem 6 ensures the existence of a set $H \subset \mathbb{C}$ with $\overline{DensH} > 0$, for which every $z \in H$ satisfies,

$$|Q(z)| \geq e^{p|z|^{\eta}} \text{ and } |Q_j(z)| \leq e^{q|z|^{\eta}} (j = 1,2,\dots,t) \text{ as } z \to \infty. \tag{7}$$

Combining the results from (5), (6) and (7) we arrive at a subset $E_3 \subset (1, +\infty)$ with positive upper density for which, when $|z| \in E_3$ as $|z| \to \infty$,

$$1 \leq te^{(q-p)|z|^{\eta}} |z|^{(\sigma-1+\varepsilon)\chi}.$$

Since

$$e^{(q-p)|z|^{\eta}} |z|^{(\sigma-1+\varepsilon)\chi} \to 0 \text{ as } |z| \to +\infty,$$

letting $|z| \to \infty, |z| \in E_3$, leads to a contradiction. Therefore, every meromorphic solution $g \not\equiv 0$ of equation (3) must be of infinite order.

Now, let f be a meromorphic solution of equation (3) with infinite order. Using (3) we obtain

$$|Q(z)| \leq \sum_{j=1}^{t} |Q_j(z)| \left(\prod_{i=1}^{k} \left|\frac{g^{(i)}}{g}\right|^{n_{ij}}\right) \tag{8}$$

Applying Lemma 5 one can obtain a set $E_4 \subset (1, +\infty)$ having finite logarithmic measure, such that whenever z meets the condition $|z| \notin E_5 = [0,1] \cup E_4$, we have

$$\left|g^{(i)}(z)\right| \leq B|g(z)|\{T(2r, g)\}^{i+1}, i = 1,2,\dots,k, \tag{9}$$

where $B > 0$ is constant.

Combining (7), (8) and (9) one can obtain a set $E_6 \subset (1, +\infty)$ with positive upper density such that

$$e^{(p-q)|z|^{\eta}} \leq tB^{\gamma_{\mathcal{P}}}\{T(2|z|, g)\}^{\Gamma_{\mathcal{P}}}$$

as $|z| \to \infty$, $|z| \in E_6$. Therfore, it follows that

$$\sigma_1(g) = \limsup_{r\to\infty} \frac{\log\log T(r,g)}{\log r} \geq \eta.$$

Hence, the theorem.

Proof of Theorem 7. Suppose that $g \not\equiv 0$ satisfies (3). By applying the reasoning used in Theorem 6, it follows that $\sigma(g) = +\infty$ and for any $\varepsilon > 0$,

$\sigma_1(g) \geq \overline{\sigma} - \varepsilon$. Since ε is arbitrary, we deduce that $\sigma_1(g) \geq \overline{\sigma} = \sigma(Q)$.

Again Lemma 6 yields a subset $E_7 \subset (1, +\infty)$ having finite logarithmic measure, such that whenever z satisfies $|z| = r \notin E_8 = [0,1] \cup E_7$ and $|g(z)| = M(r, g)$, we have

$$g^{(i)}(z) = g(z) \left\{\frac{\nu(r,g)}{z}\right\}^i (1 + o(1)), i = 1,2, \ldots, k.$$

In view of the hypothesis of Theorem 7, for any fixed $\varepsilon > 0$, one can find a constant $r_0 > 0$ such that whenever $r \geq r_0$

$$|Q(z)| \leq e^{r^{\overline{\sigma}+\varepsilon}} \text{and} \left|Q_j(z)\right| \leq e^{r^{\overline{\sigma}+\varepsilon}} (j = 1,2, \ldots, t-1),$$

as $z \in H, z \to \infty$.

Without affecting the generality of the result, suppose the degree of $\mathfrak{M}_t[g]$ is $\gamma_{\mathcal{P}}$. Then, from equation (3), we obtain

$$|Q_t(z)| \left|\frac{\mathfrak{M}_t[g]}{g^{\gamma_{\mathcal{P}}}}\right| \leq \sum_{j=1}^{t-1} \left|Q_j(z)\right| \left|\frac{\mathfrak{M}_j[g]}{g^{\gamma_{\mathcal{P}}}}\right| + |Q(z)|$$

$$\Rightarrow \left(\prod_{i=1}^{k} \left|\frac{g^{(i)}}{g}\right|^{n_{it}}\right) \leq \sum_{j=1}^{t-1} e^{r^{\overline{\sigma}+\varepsilon}} \left(\prod_{i=1}^{k} \left|\frac{g^{(i)}}{g}\right|^{n_{ij}}\right) + e^{r^{\overline{\sigma}+\varepsilon}}$$

$$\Rightarrow \left\{\frac{\nu(r,g)}{r}\right\}^A \leq \left\{(t-1) \left\{\frac{\nu(r,g)}{r}\right\}^\chi (1 + o(1)) + 1\right\} e^{r^{\overline{\sigma}+\varepsilon}}, \tag{10}$$

for $z \in H, r = |z|$ lying outside E_8, where $A = \sum_{i=1}^{k} i.n_{it}$.

Consequently, one can identify a subset $E_9 \subset (1, +\infty)$ having positive upper density for which

$$\left\{\frac{\nu(r,g)}{r}\right\}^\chi (1 + o(1)) \leq e^{r^{\overline{\sigma}+\varepsilon}},$$

as $r = |z| \to \infty, r \in E_9$.

So letting $r \to \infty$, we get

$$\limsup_{r\to\infty} \frac{\log\log \nu(r,g)}{\log r} \leq \overline{\sigma} + \varepsilon.$$

Since ε is arbitrary, combining Lemma 1 with the above inequality gives $\sigma_1(g) \leq \overline{\sigma}$, which completes the proof.

Proof of Theorem 8. Let g be a meromorphic solution of equation (4) with finite order $\sigma(g) = \sigma < +\infty$. Then (4) gives

$$|Q(z)| \leq |g|^{-\gamma_{\mathcal{P}}}|R(z)| + \sum_{j=1}^{t} \left|Q_j(z)\right| \left(\prod_{i=1}^{k} \left|g^{(i)}\right|^{n_{ij}} |g|^{-n_{ij}}\right). \tag{11}$$

According to Lemma 4, a set $E_{10} \subset (1, +\infty)$ of finite logarithmic measure can be found such that for every z with $|z| \notin E_{11} = [0,1] \cup E_{10}$, we have

$$\left|\frac{g^{(i)}(z)}{g(z)}\right| \leq |z|^{i(\sigma-1+\varepsilon)} \text{ for each } i=1,2,\ldots,k. \tag{12}$$

Again, by the condition of Theorem 8, for every z belonging to a set $H \subset \mathbb{C}$ with positive upper density, we have

$$|Q(z)| \geq e^{p|z|^{\eta}}, \left|Q_j(z)\right| \leq e^{q|z|^{\eta}} (j=1,2,\ldots,t), \text{ and } |R(z)| \leq e^{q|z|^{\eta}} \tag{13}$$

as $z \to \infty$. Using (11), (12) and (13), we get for all $z \in H$ with $|z| \notin E_{11}$,

$$1 \leq te^{(q-p)|z|^{\eta}}|z|^{(\sigma-1+\varepsilon)\chi} + e^{(q-p)|z|^{\eta}} \frac{1}{|g|^{\gamma_{\mathcal{P}}}}.$$

for all $z \in H$ with $|z| \notin E_{11}$. As $|z| \to +\infty$, the expression

$$te^{(q-p)|z|^{\eta}}|z|^{(\sigma-1+\varepsilon)\chi} + e^{(q-p)|z|^{\eta}} \frac{1}{|g|^{\gamma_{\mathcal{P}}}}$$

tends to zero, which leads to a contradiction. Hence, every meromorphic solution g of equation (4) must be of infinite order.

Proof of Theorem 9. Since $Q(z)$ is transcendental, any solution g of (4) is necessarily transcendental entire function.

Using Lemma 6, we can identify a subset $E \subset (1,+\infty)$ with finite logarithmic measure, such that whenever z satisfies with $r=|z|$ outside E and $|g(z)| = M(r,g)$, it holds that

$$g(z)\left\{\frac{\nu(r,g)}{z}\right\}^{i}\{1+o(1)\} = g^{(i)}(z), \text{ for i} = 1,2,\ldots,\text{k}.$$

Consequently, we obtain

$$\begin{aligned}\left|\frac{\mathcal{P}[g]}{g^{\gamma_{\mathcal{P}}}}\right| &\leq \sum_{j=1}^{t}\left|Q_j\right|\left|\frac{\nu(r,g)}{z}\right|^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1+o(1)) \\ &\leq \left|\frac{\nu(r,g)}{z}\right|^{\Omega}\left(\sum_{j=1}^{t}\left|Q_j\right|\right)(1+o(1))\end{aligned}$$

Hence,

$$\left|\frac{\mathcal{P}[g]}{g^{\gamma_{\mathcal{P}}}}\right| = \left|\frac{\nu(r,g)}{z}\right|^{\Omega}\left(\sum_{j=1}^{t}\left|Q_j\right|\right)\big(1+o(1)\big), \tag{14}$$

where $\Omega = \chi$ if $\nu(r,g) > r$ and $\Omega = \psi$ if $\nu(r,g) < r$.

Then, equations (4) and (14) give

$$\left(\sum_{j=1}^{t}\left|Q_j\right|\right)\left|\frac{\nu(r,g)}{z}\right|^{\Omega}(1+o(1)) = \left|\frac{R(z)}{g^{\gamma_{\mathcal{P}}}} + Q(z)\right|,$$

which leads to

$$\{\nu(r,g)\}^{\Omega} \leq r^{\Omega}\left(\sum_{j=1}^{t} |Q_j|\right)^{-1}\left(M(r,Q)+\frac{M(r,R)}{|g|^{\gamma_{\mathcal{P}}}}\right), r\to\infty, r\notin E.$$

Since $R(z)$ and $Q(z)$ satisfy the condition $M(r,R(z)) = O\left\{\left(M(r,Q(z))\right)^{\xi}\right\}$, with $0<\xi<\infty$ and $g(z)$ is transcendental entire, we have

$$\{\nu(r,g)\}^{\Omega} \leq 2r^{\Omega}\left(\sum_{j=1}^{t} |Q_j|\right)^{-1}\{M(r,Q)\}^{s},$$

where $s=\max\{1,\xi-1\}$ denotes a constant. Therefore, by Lemma 2 one can find a positive real number $r_0>0$ for which every $r>r_0$ satisfies

$$\{\nu(r,g)\}^{\Omega}\sum_{j=1}^{t} |Q_j| \leq 2^{\Omega+1}r^{\Omega}\{M(2r,Q)\}^{s},$$

which consequently leads to the conclusion

$$\sigma_1(g)\leq\sigma(Q).$$

Hence, the theorem.

4. Conclusions

From the foregoing discussion, it is evident that we have not examined the growth metrics of the solutions of

$$\mathcal{P}[g]-Qg^{n}=R,$$

for any non-negative integral powers $n\neq\ \gamma_{\mathcal{P}}$. Furthermore, the growth conditions on the coefficients $Q_j(z)$ $(j=1,2,\ldots.,t)$ in $\mathcal{P}[g], Q(z)$, and $R(z)$ were appropriately chosen to align with our computations. This naturally raises an important question: Is it possible to relax such restrictions further without slowing the growth of the solutions? Consequently, these remain unresolved issues.

The present work provides a basic framework for studying the growth characteristics of solutions to nonlinear differential equations involving functions of transcendental meromorphic and entire nature. In the future, one may consider relaxing the conditions on the coefficients or allowing them to grow more rapidly. It is also possible to investigate more general forms of nonlinear differential equations involving coefficients rather than small functions. Another direction is to obtain sharper growth estimates under weaker assumptions on the coefficients, thereby making the results more flexible and applicable to a broader class of problems. A deeper understanding of the asymptotic behavior of solutions may also help bridge classical growth theory with recent developments in the study of complex differential equations and their applications.

References

[1] M. Biswas and D. K. Mandal, " On the growth of infinite order solutions of non-linear differential equations with entire coefficients," *Bull. Calcutta Math. Soc.,* vol. 112, no. 6, pp. 525-538 (2020).

[2] M. Biswas and D. C. Pramanik, " Uniqueness of entire function and its linear differential polynomial," *Serdica Math. J.,* vol. 50, pp. 173-182 (2024).

[3] Z. X. Chen and C. C. Yang, " Some further results on the zeros and growths of entire solutions of second order linear differential equations," *Kodai Math. J.,* vol. 22, pp. 273-285 (1999).

[4] W. Cherry and Z. Ye, Nevanlinna's Theory of Value Distribution. Berlin, Germany: Springer-Verlag (2001).

[5] G. Gundersen, " Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates," *J. London Math. Soc.*, vol. 37, no. 2, pp. 88-104 (1998).

[6] W. K. Hayman, Meromorphic Functions. Oxford, U.K.: Clarendon Press (1964).

[7] W. K. Hayman, " The local growth of power series: a survey of the Wiman-Valiron method," *Can. Math. Bull.,* vol. 17, pp. 317-358 (1974).

[8] Y. Z. He and X. Z. Xiao, Algebroid Functions and Ordinary Differential Equations. Beijing, China: Science Press (1988).

[9] K. H. Kwon, " On the growth of entire functions satisfying second order linear differential equations," *Bull. Korean Math. Soc.,* vol. 33, no. 3, pp. 487-496 (1996).

[10] I. Laine, Nevanlinna Theory and Complex Differential Equations. Berlin, Germany: Walter de Gruyter (1993).

[11] Z. Q. Mao, " Uniqueness theorems on entire functions and their linear differential polynomials," *Results Math.,* vol. 55, pp. 447-456 (2009).

[12] D. C. Pramanik and M. Biswas, " On solutions of some non-linear complex differential equations in connection to Brück conjecture," Tamkang J. Math., vol. 48, no. 4, pp. 365-375 (2017).

[13] D. C. Pramanik, M. Biswas, and R. Mandal, " On the study of Brück conjecture and some non-linear complex differential equations," *Arab J. Math. Sci.,* vol. 23, pp. 196-204 (2017).

[14] D. C. Pramanik and M. Biswas, " Growth of entire solutions of non-linear differential equations," *Appl. Math. E-Notes, v*ol. 23, pp. 203-208 (2023).

[15] D. C. Pramanik and M. Biswas, " Growth of solutions of non-homogeneous linear differential equations and its applications," *Korean J. Math.*, vol. 29, no. 1, pp. 65-73 (2021).

[16] G. Valiron, Lectures on the General Theory of Integral Functions. New York, USA: Chelsea (1949).

[17] L. Yang, Value Distribution Theory. Berlin, Germany: Springer-Verlag (1993).

[18] L. Z. Yang, " The growth of linear differential equations and their applications," *Israel J. Math.,* vol. 147, pp. 359-370 (2005).

[19] H. X. Yi and C. C. Yang, Uniqueness Theory of Meromorphic Functions. Beijing, China: Science Press (1995).

Received February, 2024