

Design of multistable behaviour using speed feedback controller for coupled chaotic systems with mismatch parameters

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Abstract

Multistable behavior typically refers to a system that has multiple stable states. To achieve multistability using speed feedback controller, we consider two different Lorenz and Rossler chaotic systems respectively and also find extreme multistability by considering identical coupled chaotic Chen system with mismatch parameters. Numerical simulation outcomes are explained to exhibit the functional capability of the recommended framework to create highly stable multi-state synchronization performance. This work displays a theoretical base for building utmost multistable system for continuous system.

Subject Classification: 26A18, 28Dxx, 34Cxx, 34Dxx, 35Bxx, 46Lxx, 58Jxx, 70-XX.

Keywords: Speed feedback control (SFC), Lorenz system (LS), Rossler system (RS), Chen system (CS), Bifurcation analysis.

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1. Introduction

Multistability refers to the characteristic of dynamical systems in which solutions can switch among multiple mutually exclusively stable according to Lyapunov criteria, and converging equilibrium states in response to inputs or parameters that change gradually over time. Such behavior has been observed across a broad spectrum of physical models, incorporating laser dynamics [1], matter in condensed phase systems [2], and electronic signal generators [3], as well as biological systems such as population behaviour over time [4], neurobiology [5], and weather climate prediction models [6]. A defining feature of systems with several stable configuration is their extreme sensitivity to starting state. The presence of multiple coexisting stable states means that even a slight perturbation in the starting state can lead to drastically different long-term outcomes. However, the underlying mechanisms driving multistable behavior in many natural and engineered systems remain poorly understood. Investigating and identifying the governing principles of multistability continues to be a fundamental challenge in dynamical systems theory. In cases of extreme multistability, the system demonstrate an uncountable number of stable states. A notable method for designing such systems was proposed by Sun and co-authors [7], who emphasized that the connection methods performs a critical role in producing this behavior. Synchronization among multiple interacting systems with non-linear behaviour is a core principle in non-linear dynamics. Due to the seminal work of Pecora-Carroll [8], various synchronization approaches have been introduced. In 1997, Feudel and Grebogi [9] examined how multistable behavior could emerge by adding weak damping to originally conservative systems. Later, in 2006, Layton, Moore, and Layton [10] investigated co-existing equilibria in TGF dynamics loops, revealing complex frequency structure hypertensive pests. In 2011, Geltrude et al. [11] studied chaotic systems exhibiting multistability to develop a deterministic closed-loop model capable of controlling bursting dynamics. Further, in 2015, Li et al. [12] explored amplitude control techniques to induce multistability in symmetric chaotic systems, particularly those with symmetric under a self-inverse mapping. Dana, and U. Feudel [13] demonstrated limitless attractor coexisting regimes is achievable in two coupled m – dimensional systems when $m - 1$ system states are fully synchronized and the remaining variable maintains a constant offset. Building on this, Pal, Sahoo, and Poria [14] focused that when connecting to two systems with m dimension systems, multistability can emerge if i variables in the coupled systems are fully coherent and the rest j variables differ by fixed values, where $i = m - j$, $1 \leq i \leq m - 1$ and $1 \leq j \leq m - 1$. In recent years, 2017, Bao et al. [15] revealed that in the extended term behavior of memristive hyperchaotic systems is highly dependent at the starting state of the adaptive resistor, resulting in hidden forms of extreme multistability. In the same year, Khan, Nag, and Poria [16] introduced a general framework for constructing multistable systems by coupling two distinct dynamical models. Their approach is based on inducing partial synchronization and identifying constants of motion that are entirely dependent on initial conditions [17-22].

This work delineates a theoretical framework to evaluate the multistable dynamics of bidirectionally coupled chaotic systems using a feedback control mechanism. To validate the broad scope of the method, we implement it in well-known structures including the LS, the RS, and the CS.

This manuscript is organized using the following approach: In 2nd section, significance of research is presented. Scheme for designing multistability system using SFC for different chaotic system is discussed in section-3. In section-4, performance evaluation using Lorenz and Rössler system simulations for the proposed method in section-3. In section-5, extreme multistability with mismatch parameters for identical chaotic system is discussed. In section-6, numerical results of identical Chen systems are used to verify the validity of the proposed theorem in section-5. Simulation results and conclusion are discussed in section-7 and section-8 respectively and finally acknowledgement is drawn in section-9.

2. Significance of Research

Research on extreme multistability behaviour using SFC is significant for different reasons like understanding complex systems, it provides insights into the behaviour of complex dynamical system, particularly those with multiple stable states. This understanding is crucial in different areas like physics, biology and engineering systems. In the realm of robotics, understanding extreme multistability can lead to advancements in the robot control and decision making algorithms. This could enhance the adaptability and resilience of robots operating in uncertain or dynamic environments and also many biological systems exhibit multistability such as neuronal networks and genetic regulatory networks. Research in this area could provide insights into the functioning of these systems, potentially leading to advancements in biotechnology and medicine. Extreme multistability often leads to emergent phenomena, where the collective behaviour of a system can not be easily predicted from the behaviour of its individual components. Studying these phenomena can shed light on fundamental principles governing complex systems.

Overall, research on extreme multistability behaviour using SFC has broad implications across various disciplines and could pave the way for innovations in control theory, robotic, biology and other fields.

3. Scheme for designing multistable systems using SFC

In this section, we introduce a method to construct multistable behaviour arising from the coupling of two distinct state-space systems. We explore 2-dynamical systems as of dimension n as $\dot{X} = \Psi(X)$ and $\dot{Y} = \Phi(Y)$ where $X \in R^n$ and $Y \in R^n$ are n – dimensional states. $\Psi, \Phi: R^n \rightarrow R^n$ are continuous functions. We interconnected the systems through SFC as described below.

$$\begin{aligned}\dot{X} &= \Psi(X, Y), \\ \dot{Y} &= \Phi(Y) + \mu(X, Y)\Psi(X, Y).\end{aligned}\tag{1}$$

[illegible]

4. Numerical Example

$$\begin{aligned}\dot{x}_1 &= -(x_2 + x_3), \\ \dot{x}_2 &= ay_2 + x_1, \\ \dot{x}_3 &= -cx_3 + x_1x_3 + b, \\ \dot{y}_1 &= \sigma(y_2 - y_1) - \mu_1(x, y)(x_2 + x_3) \\ \dot{y}_2 &= ry_1 - y_1y_3 - y_2 + \mu_2(x, y)(x_1 + ay_2), \\ \dot{y}_3 &= y_1y_2 - \beta y_3 + \mu_3(x, y)(b - cx_3 + x_1x_3).\end{aligned}\tag{6}$$
$$\begin{aligned}\dot{e}_1 &= (\mu_1 - 1)(x_2 + x_3) - \sigma(y_2 - y_1), \\ \dot{e}_2 &= -(\mu_2 - 1)(x_1 + ay_2) - ry_1 + y_2 + y_1y_3, \\ \dot{e}_3 &= -(\mu_3 - 1)(b - cx_3 + x_1x_3) - y_1y_2 + \beta y_3.\end{aligned}\tag{7}$$
$$\begin{aligned}\mu_1 &= 1 + \frac{\sigma(y_2 - x_1)}{x_2 + x_3}, \\ \mu_2 &= \frac{(\sigma - 1)x_1 - \beta x_3 + (\sigma - r)y_1 + (a + 1)y_2 + \beta y_3 + y_1 y_3}{x_1 + a y_2}, \\ \mu_3 &= \frac{b + x_1 x_3 - y_1 y_3 + (\beta - c)y_3}{b - c x_3 + x_1 x_3},\end{aligned}\tag{8}$$
$$\begin{aligned}\dot{e}_1 &= -\sigma e_1, \\ \dot{e}_2 &= \sigma e_2 + \beta e_3, \\ \dot{e}_3 &= -\gamma e_3.\end{aligned}\tag{9}$$

It follows from (8) that $e_1, e_3 \rightarrow 0$ as $t \rightarrow \infty$ i.e., $y_1 = x_1, y_3 = x_3$, which indicates that $\dot{e}_2 = 0$ i.e., $e_2 = \text{constant} = -k$, i.e., hence $y_{20} = x_{20} + k$, k is arbitrary constant term. It relies atop the initial restrictions of system. The dynamics (6) is equal to the following multistable system

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3, \\
\dot{x}_2 &= x_1 + a(x_2 + y_{20}), \\
\dot{x}_3 &= b - cx_3 + x_1x_3.
\end{aligned} \tag{10}$$

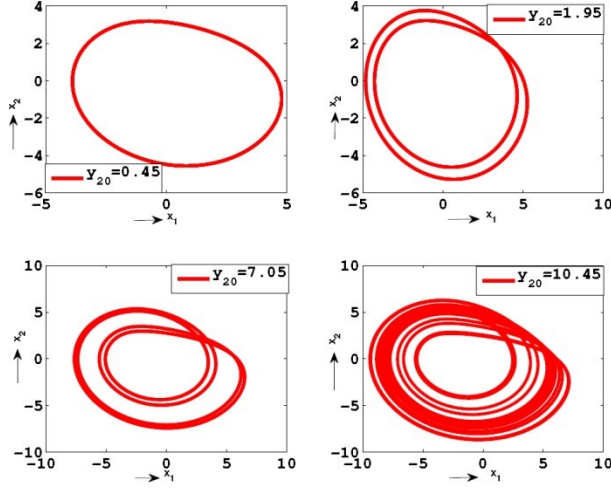


Figure 1

Phase diagram of the system (10) in $x_1 - x_2$ plane for different initial restrictions $y_{20} (= 0.45, 1.95, 7.05, 10.45)$ with $a = b = \frac{1}{5}$, and $c = \frac{5}{2}$.

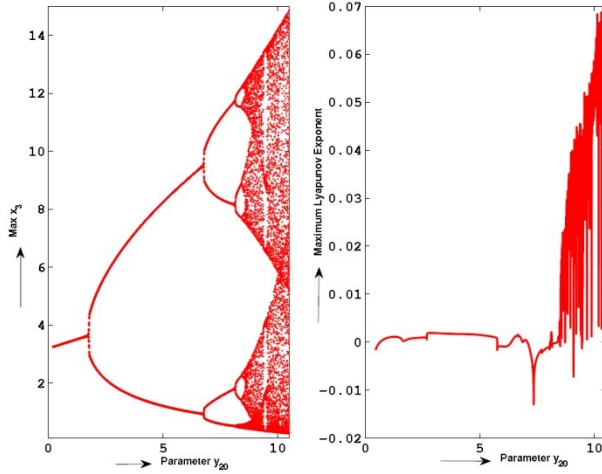


Figure 2

Bifurcation diagram and LLE with respect to the initial condition y_{20} of the system (10) for $a = \frac{1}{5}$, $b = \frac{1}{5}$, and $c = \frac{5}{2}$.

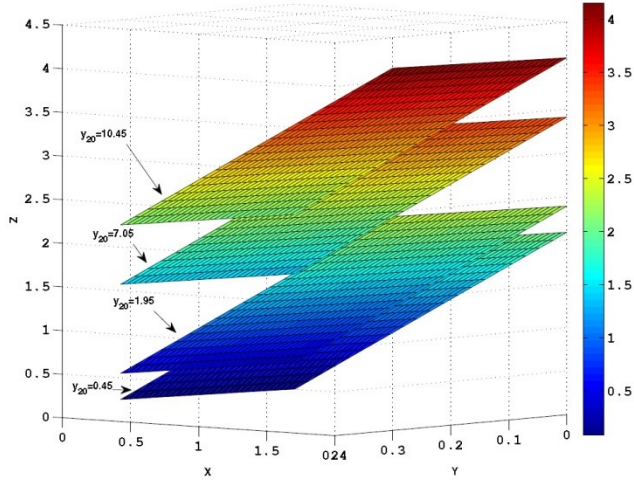


Figure 3
Surface plot of the system (10) with different variation of y_{20} ($= 0.45, 1.95, 7.05$ and 10.45).

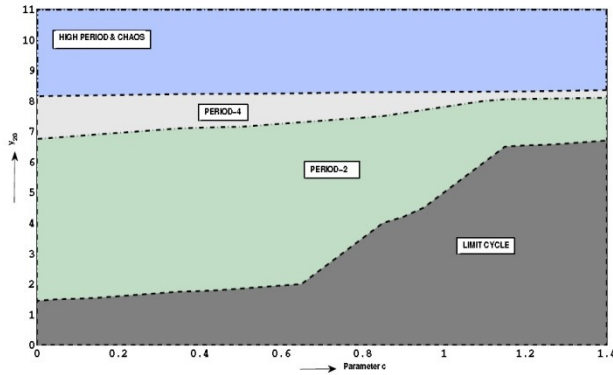


Figure 4
Qualitative behaviour of the model in the $c - y_{20}$ parameter plane.

5. Extreme multistability with mismatch parameter

Here, discuss consequences of variation of parameter upon extreme multistability. No two systems in real world phenomenon are equally likely. Obviously there should be a mismatch in characteristics, therefore the study of robustness of extreme multistability of two systems with parameter mismatch is necessary. This poses a situation for the experimental realization of extreme multistability by two systems with a mismatch which may lead to the non-appearance of extreme multistability. To achieve extreme multistability with parameter mismatch as $\dot{X} = \Psi(X, Y, \alpha)$, $X \in R^n$ and $Y \in R^n$, n – dimensional states. $\Psi: R^n \rightarrow R^n$ are the continuous vector functions and α depicts all the

$$\begin{aligned}
\mu_1 &= \frac{a(x_3 - y_3)}{(a + \Delta a)(y_2 - y_1)}, \\
\mu_2 &= \frac{c(e_1 + e_2 - e_3) - (a + \Delta a)e_1 - \Delta c(y_1 + y_2) - x_1 x_3 + y_1 y_3}{c(x_1 + x_2) - (c - a - \Delta a - x_3)x_1}, \\
\mu_3 &= \frac{\Delta b y_3 + x_1 x_2 - y_1 y_2}{x_1 x_2 - b x_3},
\end{aligned} \tag{15}$$

then the error system (14) transformed into

$$\begin{aligned}
\dot{e}_1 &= a e_3, \\
\dot{e}_2 &= c e_3, \\
\dot{e}_3 &= -b e_3.
\end{aligned} \tag{16}$$

From (16), it is clear that $e_3 \rightarrow 0$ as $t \rightarrow \infty$, i.e., $e_1 = \text{constant} = k_1$ and $e_2 = \text{constant} = k_2$, where k_1, k_2 are arbitrary constants. For simplicity here we choose $k_1 = k_2 = k$. Hence $y_{10} = x_{10} + k$ and $y_{20} = x_{20} + k$, where k is controlled by the starting restrictions of differential equations. The outcome is influenced by the starting conditions of the entire system. Consequently, the system of equation (13), be represented by the following three-dimensional systems.

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + \Delta(x_2 - x_1), \\
\dot{x}_2 &= c(x_1 + x_2) - (a + \Delta a + x_3)x_1, \\
\dot{x}_3 &= -b x_3 + x_1 x_2.
\end{aligned} \tag{17}$$

The equation (13) is a systems of several stable configurations if the system of equation (17) changes with the slight difference of the mismatch parameter Δa .

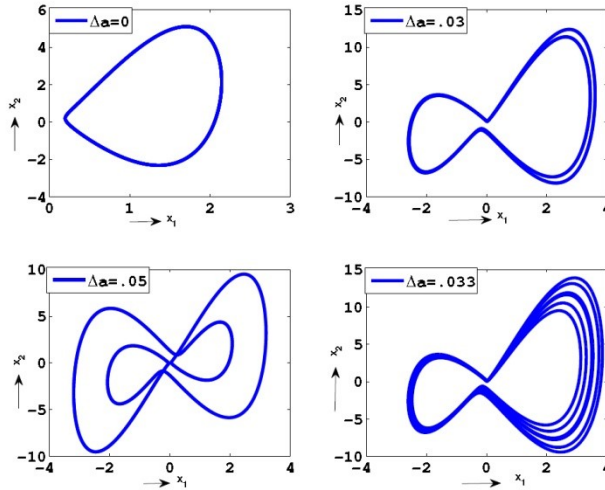
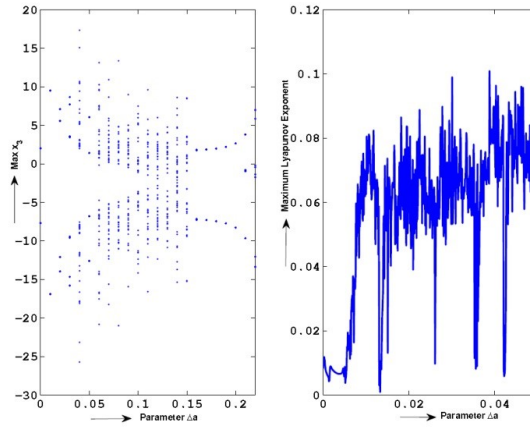
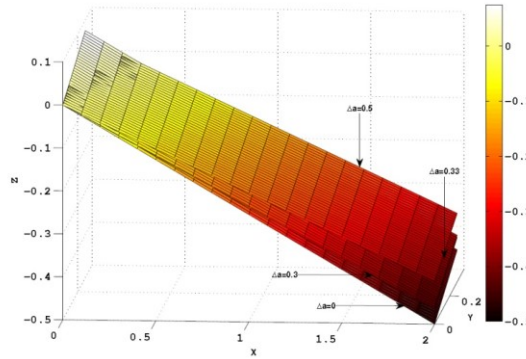


Figure 5

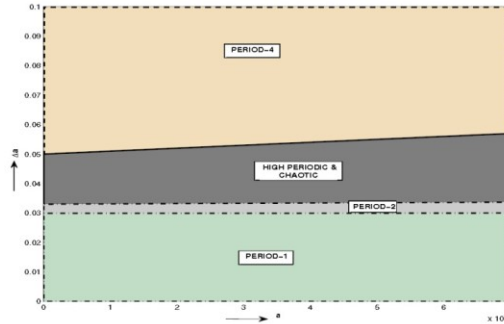
Phase diagram of the system (14) in $x_1 - x_2$ plane for different mismatch parameter $\Delta a(0, 0.03, 0.05, 0.033)$ with $a = 0.34, b = -0.3$, and $c = -1$.

**Figure 6**

Bifurcation diagram, maximum lyapunov exponent, respect to the starting condition Δa of (14) with $a = 0.34$, $b = -0.3$, $c = -1$.

**Figure 7**

Surface plot of the system (14) with different mismatch parameters $\Delta a = (0, 0.03, 0.05 \text{ and } 0.033)$.

**Figure 8**

Qualitative behaviour of the model in the $a - \Delta a$ parameter plane.

7. Simulation Results

To demonstrate the effectualness of multistability of different chaotic systems using SFC, we fix $a = b = \frac{1}{5}, c = \frac{5}{2}$ (for RS) and $r = 28.0, \sigma = 10.0, b = 2.66$ (for LS). Here we change only one initial condition y_{20} and observe the different periodic behaviour of the coupled system by simulation. Except the initial condition y_{20} , all other initial restrictions are fixed. Figure 1. shows the phase diagram belonging to the system of differential equation for different values of initial state y_{20} . Figure 1 displays the presence of different attractors in the state space such as period-1, period-2, period-4 cycles, and chaotic behavior that occur for varying values of y_{20} . This highlights extreme multistability when the initial condition y_{20} is altered. We constructed the bifurcation diagram and largest Lyapunov exponent against varying initial conditions y_{20} for the above set of parameter values which depicts in Figure 2. From Figure 2, it is clear that several periodic and chaotic attractors exist for different initial condition values. Surface plot of the system (10) with different values of y_{20} ($= 0.45, 1.95, 7.05, 10.45$) is plotted in Figure 3. Qualitative behaviour of the said model in $c - y_{20}$ plane is depicted in Figure 4.

In section 5, for robustness of extreme multistability of two identical Chen systems using SFC with parameter mismatch, we vary a single mismatch parameter Δa . For simplicity we choose $\Delta b = \Delta c = 0$. In Figure 5. we derive phase diagram of (17) with different variation of mismatch parameter Δa . Fig5. shows the coexistence of different types of attractors in state space such as period-1, period-2, period-4 and chaos for different values of $\Delta a = 0, 0.03, 0.05$ and $\Delta a = 0.033$ respectively. Lastly, for extreme multistability under the variation of single mismatch parameter Δa , we plotted the bifurcation diagram and largest Lyapunov exponent for the above set of parameter values which is described earlier depicts in Figure 6. Surface plot of the system (17) with different mismatch parameters are shown in Figure 7. and finally qualitative behaviour of the model (17) in the $a - \Delta$ parameter is depicted in Figure 8. Therefore, the results confirm that proposed theorem design multistability, using different initial conditions as well as different mismatch parameter.

8. Conclusions

An antagonistic feature that a challenging aspect of dynamical systems is the necessity for the coupled systems to be identical or closely matched. Additionally, it is essential to develop a method for precisely adjusting the initial conditions to induce extreme multistability. We have successfully overcome these situations by considering coupled Rossler and Lorenz system. And again we able to find extreme multistability by consider equivalent coupled chaotic Chen system with mismatch parameter. Moreover, it is clear that extreme multistability rarely occur in most physical, chemical and biological systems. But we accomplish to find extreme multistability with the change of initial conditions and mismatch parameters. To validate the proposed schemes, we present phase, bifurcation diagrams, and the LLE as initial conditions and mismatch parameters change. Understanding the core principles of multistability can be valuable for managing many multistable systems.

References

- [1] F. T. Arecchi, R. Meucci, G. Puccioni, and J. Tredicce, “Experimental evidence of subharmonic bifurcations, multistability and turbulence in Q-switched gas laser,” *Phys. Rev. Lett.*, vol. 49, pp. 1217-1220 (1982).
- [2] S. A. Beznosyuk, B. F. Minaev, and Z. M. Muldakhmetov, “Informative energetic structure and electronic multistability of condensed state,” *J. Mol. Struct. THEOCHEM*, vol. 227, pp. 125-129 (1991).
- [3] M. S. Patel, U. Patel, A. Sen, G. C. Sethia, C. Hens, S. K. Dana, and R. E. Amritkar, “Experimental observation of extreme multistability in an electronic system of two coupled Rossler oscillators,” *Phys. Rev. E*, vol. 89, no. 2, p. 022918 (2014).
- [4] E. Ullner, A. Zaikin, E. I. Volkov, and J. Garcia-Ojalvo, “Multistability and clustering in a population of synthetic genetic oscillators via phase-repulsive cell-to-cell communication,” *Phys. Rev. Lett.*, vol. 99, no. 14, p. 148103 (2007).
- [5] M. I. Rabinovich, P. Varona, A. I. Selverston, and H. D. Abarbanel, “Dynamical principles in neuroscience,” *Rev. Mod. Phys.*, vol. 78, no. 4, p. 1213 (2006).
- [6] R. Calov and A. Ganopolski, “Multistability and hysteresis in the climate cryosphere system under orbital forcing,” *Geophys. Res. Lett.*, vol. 32, no. 21 (2005).
- [7] H. Sun, S. K. Scott, and K. Showalter, “Uncertain destination dynamics,” *Phys. Rev. E*, vol. 60, pp. 3876-3880 (1999).
- [8] L. M. Pecora and T. L. Carroll, “Synchronization in chaotic systems,” *Phys. Rev. Lett.*, vol. 64, pp. 821-824 (1990).
- [9] U. Feudel and C. Grebogi, “Multistability and the control of complexity,” *Chaos*, vol. 7, no. 4, pp. 597-604 (1997).
- [10] A. T. Layton, L. C. Moore, and H. E. Layton, “Multistability in tubuloglomerular feedback and spectral complexity in spontaneously hypertensive rats,” *Am. J. Physiol. Renal Physiol.*, vol. 291, no. 1, pp. F79-F97 (2006).
- [11] A. Geltrude, K. Al Naimee, S. Euzzor, R. Meucci, F. T. Arecchi, and B. K. Goswami, “Feedback control of bursting and multistability in chaotic systems,” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 17, no. 7, pp. 3031-3039 (2012).
- [12] C. Li, W. Hu, J. C. Sprott, and X. Wang, “Multistability in symmetric chaotic systems,” *Eur. Phys. J. Spec. Top.*, vol. 224, no. 8, pp. 1493-1506 (2015).

- [13] C. Hens, S. K. Dana, and U. Feudel, "Extreme multistability: Attractor manipulation and robustness," *Chaos*, vol. 25, no. 5, p. 053112 (2015).
- [14] S. Pal, B. Sahoo, and S. Poria, "Multistable behaviour of coupled Lorenz–Stenflo systems," *Physica Scripta*, vol. 89, 045202 (2014).
- [15] B. C. Bao, H. Bao, N. Wang, M. Chen, and Q. Xu, "Hidden extreme multistability in memristive hyperchaotic system," *Chaos Solitons Fractals*, vol. 94, pp. 102-111 (2017).
- [16] M. A. Khan, M. Nag, and S. Poria, "Design of multistable systems via partial synchronization," *Pramana*, vol. 89, no. 2, p. 19 (2017).
- [17] M. F. Abdul Rahim, H. Natiq, N. A. A. Fataf, and S. Banerjee, "Dynamics of a new hyperchaotic system and multistability," *Eur. Phys. J. Plus*, vol. 134, pp. 1-9 (2019).
- [18] H. Chang, Y. Li, G. Chen, and F. Yuan, "Extreme multistability and complex dynamics of a memristor-based chaotic system," *Int. J. Bifurcation Chaos*, vol. 30, no. 08, p. 2030019 (2020).
- [19] A. N. Pisarchik and U. Feudel, "Control of multistability," *Phys. Rep.*, vol. 540, no. 4, pp. 167-218 (2014).
- [20] F. Yuan, G. Wang, and X. Wang, "Extreme multistability in a memristor-based multi-scroll hyper-chaotic system," *Chaos*, vol. 26, no. 7 (2016).
- [21] S. Zhang, Y. Zeng, Z. Li, M. Wang, and L. Xiong, "Generating one to four-wing hidden attractors in a novel 4D no-equilibrium chaotic system with extreme multistability," *Chaos*, vol. 28, no. 1 (2018).
- [22] X. Wang, N. V. Kuznetsov, and G. Chen, Eds., *Chaotic Systems with Multistability and Hidden Attractors*, vol. 40. Cham, Switzerland: Springer, pp. 149-150 (2021).

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