

Chaos synchronization of two coupled GLV models using active and adaptive control techniques

Shubhangi Dwivedi *

*Department of Mathematics
School of Advanced Sciences
Vellore Institute of Technology
Vellore 632014
Tamil Nadu
India*

and

*School of Mathematical and Statistical Sciences
Indian Institute of Technology
Mandi
Himachal Pradesh 175005
India*

Nitu Kumari †

*School of Mathematical and Statistical Sciences
Kamand Indian Institute of Technology
Mandi 175005
Himachal Pradesh
India*

Ranjit Kumar Upadhyay §

*Department of Applied Mathematics
Kamand Indian Institute of Technology (ISM)
Dhanbad 826004
Jharkhand
India*

* E-mail: shubhangi.dwivedi@vit.ac.in (Corresponding Author)

† E-mail: nitu@iitmandi.ac.in

§ E-mail: ranjitupadhyay@iitism.ac.in

Abstract

This work applies the generalized Lotka Volterra (GLV) model to study complete replacement synchronization in population dynamics. We examine how three functional responses linear and Holling type II and III-affect the system's overall behaviour. Focusing on the linear functional response, we analyze the model's analytical and dynamical properties for three parameter sets, demonstrating chaotic oscillations and thus stabilize the unstable fixed points using appropriate controllers. Our results show how chaotic ecological models can help forecast populations of competing species in new habitats when information from a similar system is available. To do this, we construct a drive-response setup by coupling predator populations, where the prey of the drive system acts as the driving variable and predators in the response system feed only on that prey. Using active and adaptive control methods, we achieve synchronization between two coupled GLV models and validate the analytical findings through numerical simulations.

Subject Classification: 34H10, 37N25, 37N35.

Keywords: *Lyapunov exponents, Slow manifold equation, Unidirectional coupling, Active and adaptive control methods, Chaos control and synchronization.*

1. Introduction

In ancient philosophy and mythology, the word chaos meant the disordered state of unformed matter that is supposed to have existed before the ordered universe. In the course of its journey to truth, science has met with startling phenomena called chaos. Since the 1960s, with the discovery of chaotic systems, chaos has set a nonlinear dynamics research boom. Chaos theory is attributed to the work of Edward Lorenz. His 1963 paper, "Deterministic Non-periodic Flow" [1], is credited for laying the foundation for chaos theory. Hunt and Ott [2] reviewed the problem and proposed a computationally feasible entropy-based good definition of chaos. They define chaos as "the existence of positive Expansion Entropy (EE) (equal to topological entropy for infinitely differentiable maps) on a given restraining region (bounded positive volume subset)," which confirms both the notions (COS as well as OS). Since EE enjoys the properties of simplicity, computability, and generality, so they call it a "good" definition of chaos. Chaotic systems are sensitive to initial conditions, topologically mixing and with dense periodic orbits [3], [4]. Because of slightest errors, chaotic dynamical systems can lead to entirely different trajectories. The main characteristic of chaos is that the system does not repeat its past behaviour. Mathematically, chaotic dynamical systems are classified as nonlinear dynamical systems having at least one positive Lyapunov exponent [5].

Chaos theory is not just about chaos; instead, it has two aspects: chaos control and chaos synchronization. Interest in the study of chaos control and investigating the qualitative behaviour of chaotic models has increased significantly. It is found to have critical applications in several areas, such as secure communications, biology, neural networks, finance, and several others. On the other side, synchronization means agreement or relationship in time of

different processes. Chaotic systems would seem to be a dynamical system that defies synchronization [6].

This study is based on synchronization of identical chaotic systems that deals with complete synchronization, which aims at using the output of the drive system to control the response system so that the output of the response system tracks the output of the drive system asymptotically [7]. It is equivalent to an asymptotically stable error system. However, attainment of synchrony in coupled ecological system should be dealt with caution as sometimes it is detrimental to construct synchronized systems in ecological context because it has a potential threat to biodiversity. A more ecologically feasible way of attaining complete synchronization between two ecological systems can be followed by techniques of complete replacement synchronization [8]. It can be helpful in a network of patchy ecosystems as it assists in identifying the underlying mechanism that brings the group co-ordination in the presence of severely chaotic oscillations. In ecology, chaotic dynamics may synchronize if populations are coupled through environmental interactions.

The dynamics of multiple interacting species are often complicated, and the forecasting results of a small change in large communities are often difficult to describe. Specific circumstances where the highly synchronizing effects of chaos have been shown to enhance the global persistence of the population model states that even though a large population associated with chaotic dynamics increases the chance of local extinction [9]. From the ecological aspect, it is crucial to obtain conditions that guarantee the permanence and global attractiveness of all species in a multi-species community. Upadhyay and Rai [10] have demonstrated that the two non-identical chaotic ecological systems having different kinds of top-predators are synchronized using an algorithm proposed by Lu and Cao [11]. The idea of this approach is that it takes care of the uncertainties involved in the parameter estimation.

In the control literature, many different methods have been proposed for the complete synchronization. Here, we apply active and adaptive control methods to achieve synchronization. Chaos synchronization using the linear active control technique has attracted the interest of many researchers and has been widely accepted as an efficient technique used for synchronization of identical and non-identical chaotic systems. The active control theory was first used to study chaos synchronization by Bai and Lonngren [12] and Njah and Vincent [13]. In active control method, results on chaos synchronization are derived by using the known parameters of drive and response systems. In this technique, based on the Lyapunov stability theory, novel nonlinear controllers are designed to achieve the synchronization in coupled systems.

Since we will be dealing with an ecological model, the system's parameters cannot be precisely known. The adaptive control is one of the popular techniques used to control and synchronize the solutions of nonlinear systems with unknown or uncertain parameters [14]. In adaptive control method, we want the model to adapt data assimilation along the way that may be useful for predicting the real system's future behaviour [15], [16]. In this method, control law and parameter

like chaotic drive systems. This scheme maintains the consistent performance of a system in the presence of uncertainty, variations in parameters. Consequently, asymptotically global synchronization control of the chaotic system guarantees to converge the error dynamics to the equilibrium point.

In this study, we deal with a three-dimensional chaotic generalized Lotka-Volterra system, which is more general than the competitive predator-prey examples of Lotka-Volterra types and proposed by Samardzija and Grellere[17]. We examine the properties of the model, including equilibrium analysis, dissipative, maximum Lyapunov exponent, and slow manifold analysis. The linear feedback control technique is used for controlling the model to its equilibrium points.

Our goal is to synchronize two identical GLV models with prey as drive species using complete replacement synchronization. Such configurations are advantageous in patchy ecosystems where certain areas contain an abundance of prey. These regions, can be referred to as "hotspots", that naturally attract predators because of their higher prey density compared to surrounding areas. The concentration of prey in these hotspots supports a higher presence of predators, which in turn stabilizes and synchronizes the predator population. Consequently, predators are less likely to disperse in search of food, making their dynamics more stable and predictable. Therefore, in the configuration, prey population in drive system is assumed to be superfluous in such a way that predators from nearby patches also feed upon it. To maintain synchronization with only small adjustments within a two-patch system, we design the active control law (when system parameters are known) and adaptive control law (when system parameters are unknown) mathematically and validate the analytical results through numerical simulation. The organization of this paper is as follows:

In section 2, the GLV model dynamics are discussed with a linear, Holling type II, and Holling type III functional responses. From section 3 onward, GLV model with linear functional response is primarily considered for the study. Properties of the system such as invariance, dissipation, Lyapunov exponents, an equation of slow manifold, the existence and uniqueness of the model's solution are discussed. The local stability of the fixed points of the model is discussed and further, the conditions for stability of unstable equilibrium point are obtained using three linear external controllers. In section 4, the unidirectional coupling scheme is used to synchronize two chaotic GLV models via the active control method and the adaptive control method[18]. Finally, conclusions are drawn in section 5.

A portion of the material in this article is based on the first author's Ph.D. thesis submitted to IIT Mandi [29].

2. The Model

The generalized Lotka-Volterra equations are autonomous and deterministic. The dynamics of the model in a more generalized form are defined as

$$\dot{x}_i(t) = x_i(t)(r_i + f_i(x_1, x_2, \dots, x_n)),$$

with initial condition

$$x_i(0) = x_{(i,0)}, \text{ for } i \in \{1, 2, \dots, n\}.$$

where n represents the number of species, $x_i(t)$ is the size of population i , $\dot{x}_i(t)$ is the time derivative of species i , t is the time variable, $x_{(i,0)}$ is the initial population of species i , r_i is the self-growth of species i , and $f_i(x_1, x_2, x_3, \dots, x_n)$ is the nonlinear multi-variable function with intra and inter-species competition terms for each i . Although the populations are usually measured in integer numbers, $x_i(t)$ is real for each i and can be interpreted as density, biomass or some other measure which correlates with the number of species. Let R^n denotes the Euclidean space and the function f_i is a continuous, smooth and real-valued function for each i . We make the following assumptions on f_i for $i \in \{1, 2, \dots, n\}$ [19].

- (i) f_i , for each $i \in \{1, 2, \dots, n\}$ is bounded on a domain $D \subset R^n$.
- (ii) There exist constant $K_i > 0$ for each $i \in \{1, 2, \dots, n\}$ such that

$$\|f_i(X) - f_i(Y)\| \leq K_i \|X - Y\| \quad \forall \quad X, Y \in D \subset R^n.$$

In this study, we consider three variations of three-dimensional the model, first: first with linear functional response, second with Holling type II functional response and third with Holling type III functional response.

2.1 Model with Linear Functional Response

The generalized Lotka Volterra model and its variant have been studied by many authors [20],[21],[22]. Here we are considering the following three-dimensional autonomous system, which has been devised by Samardzija and Greller in 1988 [17]. We assume that the model holds the chaotic dynamics and takes the following form

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_2 + rx_1 - px_3x_1), \\ \dot{x}_2 &= x_2(-1 + x_1), \\ \dot{x}_3 &= x_3(-q + px_1^2). \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are the state variables. Here x_1 is the prey population, x_2 and x_3 are middle and top predator populations respectively. Here p, q, r are positive parameters. This system is found to be chaotic in particular parameters vales as $p = 2.9851$, $q = 3$, $r = 2$ and has very interesting complex dynamical behaviour, as shown in figure 1.

Further, we display the dynamics of GLV system for three different set of parameters to characterize its parameter- sensitivity. Different sets of parameters involved in the system are taken as

$$(p, q, r) \in \{(2.0451, 2.129, 2), (2.9851, 2.99, 2.1), (2.98098, 2.9799, 2)\}.$$

For simulation, we fix the initial condition at $x_1(0) = 1.0023, x_2(0) = 1.0589, x_3(0) = 0.6503$ for the GLV system. Figures 2, 3 and 4 show three dimensional attractor and two dimensional projections of the system on

(x_1, x_2) , (x_1, x_3) and (x_2, x_3) planes. From figures 2 and 3, it can be inferred that the sensitivity of the system on parameters can help in restoring the hidden order out of its chaotic dynamics.

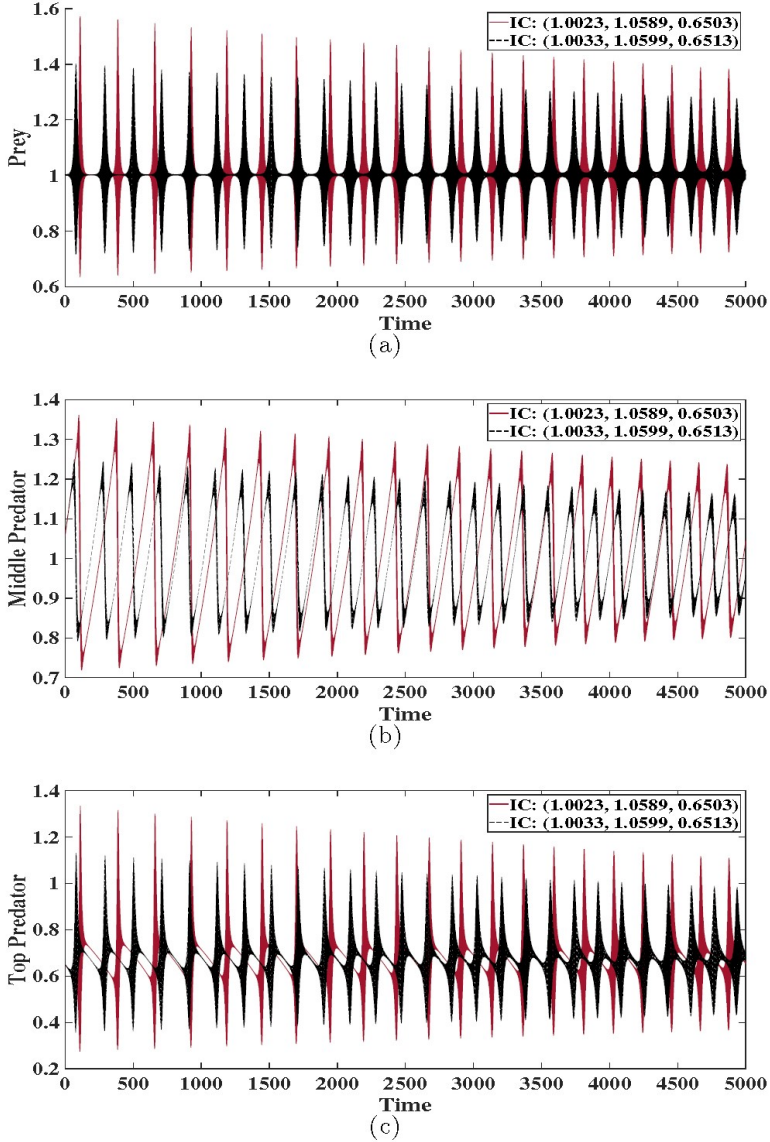


Figure 1

Sensitive dependence on initial condition is shown for all three populations. Time series of x_1 , x_2 , and x_3 are displayed in (a), (b) and (c) for two different initial conditions $(1.1, 1.4, 1)$ and $(1.1 + 10^{-4}, 1.4 + 10^{-4}, 1 + 10^{-4})$.

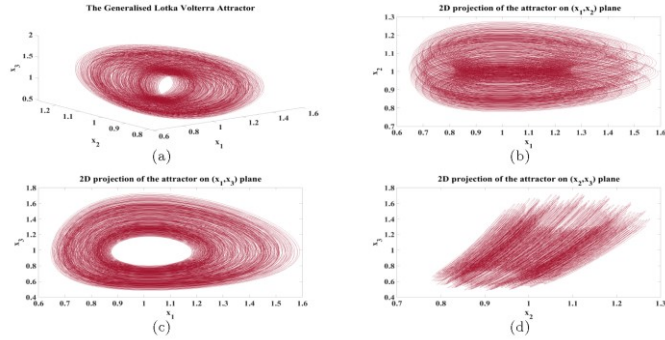


Figure 2

(a): 3D attractor; (b), (c) and (d): 2D projections of the attractor on (x_1, x_2) , (x_1, x_3) and (x_2, x_3) planes respectively for $(p, q, r) = (2.0451, 2.129, 2)$.

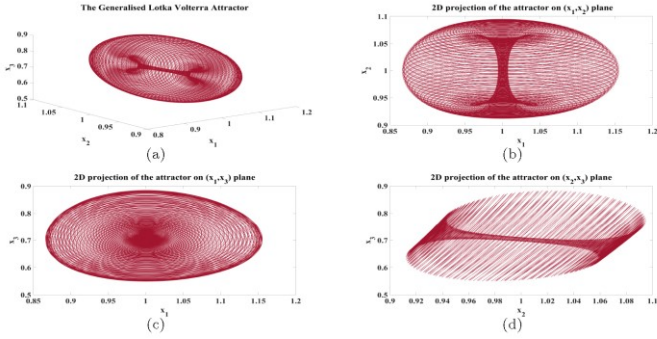


Figure 3

(a): 3D attractor ; (b), (c) and (d): 2D projections of the attractor on (x_1, x_2) , (x_1, x_3) and (x_2, x_3) planes respectively for $(p, q, r) = (2.9851, 2.99, 2.1)$.

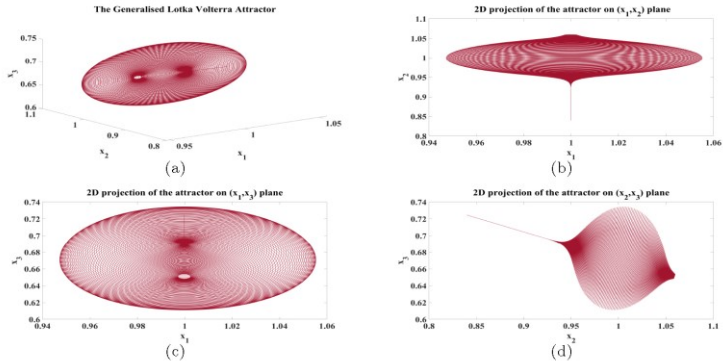


Figure 4

(a): 3D attractor; (b), (c) and (d): 2D projections of the attractor on (x_1, x_2) , (x_1, x_3) and (x_2, x_3) planes respectively for $(p, q, r) = (2.98098, 2.9799, 2)$.

2.2 Model with Cyrtoid type (HT II) Functional Response

Here, we replace the linear prey-middle predator interaction term in prey dynamics by the Holling type II functional response with the term $-\frac{x_1x_2}{x_1+d}$. The meaning of this term is that the prey's contribution to the middle predator growth rate is $\frac{x_1x_2}{x_1+d}$. Using Holling type II functional response, the dynamics of new GLV model are proposed as

$$\begin{aligned}\dot{x}_1 &= x_1 - \frac{x_1x_2}{x_1+d} + rx_1^2 - px_1^2x_3, \\ \dot{x}_2 &= -x_2 + \left(\frac{x_2x_1}{x_1+d}\right), \\ \dot{x}_3 &= -qx_3 + px_3x_1^2.\end{aligned}\tag{2}$$

For simulation, we take the parametric values as $p = 2.0459, q = 8.602, r = 1.1, d = 0.0011$. It is shown in figure 4 that the model (2) has 'stable focus' for these parametric values. The initial value of the state variables with are taken as $x_1(0) = 1.999, x_2(0) = 1.079, x_3(0) = 0.693568$.

The GLV Attractor with HT II Functional Response

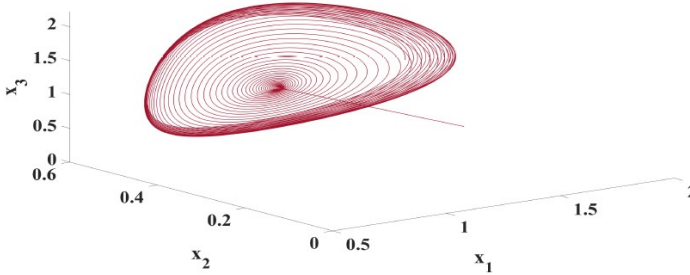


Figure 5

Three-dimensional attractor of GLV model with HT II functional response.

2.3 Model with Sigmoid type (HT III) Functional Response

Here, we present the interaction between prey and top predator population with Holling type III functional response. This functional response occurs in top-predators which increase their search activity with increasing prey density. Using the same assumptions, the dynamics of new GLV model with Holling type III functional response are proposed as

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1x_2 + rx_1^2 - p\frac{x_1^2x_3}{x_1^2+d}, \\ \dot{x}_2 &= -x_2 + x_1x_2, \\ \dot{x}_3 &= -qx_3 + p\frac{x_1^2x_3}{x_1^2+d}.\end{aligned}\tag{3}$$

For simulation, we take parametric values as $p = 7.34, q = 2.0, r = 0.507, d = 3.198$. It is demonstrated in figure 5 that for these parametric values,

model with HT III has 'limit cycle' look-like attractor which means model (3) has stable dynamics.

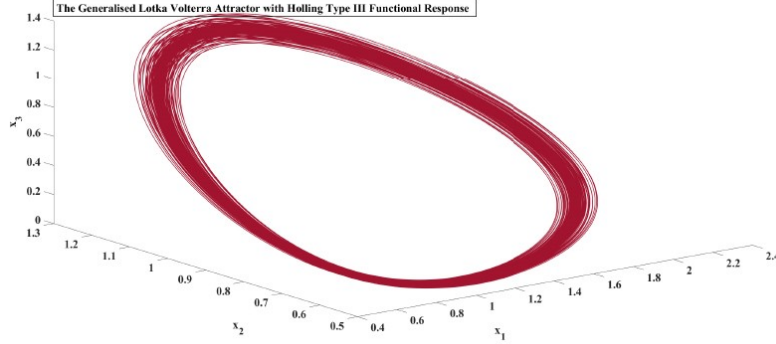


Figure 6
Three-dimensional attractor of GLV model with HT III functional response.

Thus, we find three different ranges of parameters for which models (1), (2), and (3) show completely different dynamics. It shows that the GLV model's unstable dynamics with linear function response can be turned into stable dynamics when functional response changes to HT II or HT III. However, this may not be very effective in a particular scenario. When the prey population is in abundance and predation rate does not follow HT II or HT III functional response. To overcome this problem, we control chaos in the model (1) and achieve complete replacement synchronization in two coupled GLV models. Since GLV system has been shown to have a chaotic attractor for various set of parameters. Out of these sets, we pick the set of parameters $p = 2.9851, q = 3, r = 2$ as in [22] for further study.

3. Mathematical Properties of The Model

We discuss the properties of the model including positive Lyapunov exponent, equation of slow manifold, in-variance and dissipation, stability of feasible equilibrium points, control of instability of unstable equilibrium points.

3.1 Lyapunov exponents

For three-dimensional system, the local behaviour of the dynamics varies along three orthogonal directions in state space. Nearby initial conditions may be moving apart along one axis, and moving together along another. The Lyapunov exponent describes the average rate of separation between two nearby trajectories with different initial conditions subject to a flow [3]. We obtain Lyapunov exponents of model (1) in MATLAB. For simulation we take the parameter values of the system (1) as $p = 2.9851, q = 3, r = 2$ and the initial values are considered as $(x_1(0), x_2(0), x_3(0)) = (1, 1.4, 1)$. The Lyapunov exponents of GLV model are shown in figure (6-7).

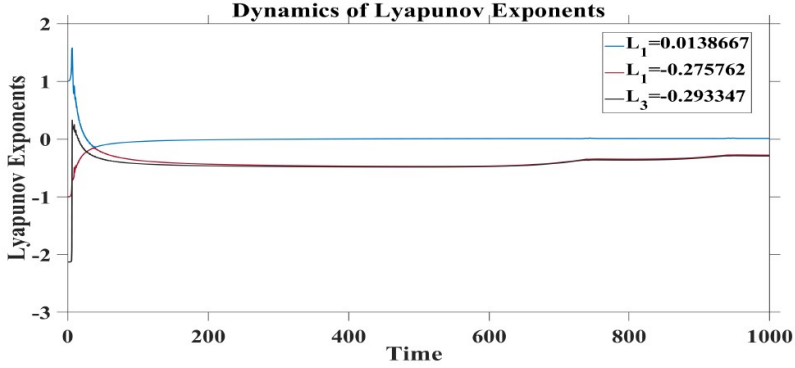


Figure 7
Dynamics of Lyapunov spectrum of system (1).

The Lyapunov exponents of model (1) are as follows

$$\begin{aligned} L_1 &= 0.00272496 > 0, \\ L_2 &= 0.000518866 \approx 0, \\ L_3 &= -0.0000251755 < 0. \end{aligned} \quad (4)$$

The infusion of geometric and topological techniques in chaos theory motivates mathematicians to study the underlying geometric structures. Expression of slow manifold permits to restore a part of the deterministic property of the system that was lost because of SIC (sensitive dependence on initial conditions).

3.2 Equation of Slow Manifold

We consider the system (1) slow-fast autonomous dynamical system (S-FADS). In (S-FADS), variables are separated into two groups: one is group of fast variable and others are slow variables. Slow variables are used to determine the behaviour of whole system. In order to obtain slow manifold equation, we consider that the slow manifold is locally defined by a plane orthogonal to tangent system's left fast eigenvector. It is shown in subsection that the GLV system has a chaotic attractor for the parameters $p = 2.9851, q = 3, r = 2$, therefore, we take these parameter values for further study and write the equations of model (1) as

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_2 + 2x_1 - 2.9851x_3x_1), \\ \dot{x}_2 &= x_2(-1 + x_1), \\ \dot{x}_3 &= x_3(-3 + 2.9851x_1^2). \end{aligned} \quad (5)$$

The Jacobian matrix J at point $\mathbf{x} = (x_1, x_2, x_3)^T$ is obtained as

$$J = \begin{bmatrix} 1 - x_2 + 4x_1 - 5.9702x_1x_3 & -x_1 & -2.9851x_1^2 \\ x_2 & -1 + x_1 & 0 \\ 5.9702x_1x_3 & 0 & -3 + 2.9851x_1^2 \end{bmatrix}$$

Let $\lambda_1(x_1, x_2, x_3)$ be a real, negative and dominant Eigen value (i.e, fast Eigen value) for Jacobian matrix in a large part of attractor's phase space domain. Furthermore, we assume that $\lambda_2(x_1, x_2, x_3)$ and $\lambda_3(x_1, x_2, x_3)$ be two slow Eigen values. Then Eigen vector $Z_{\lambda_1}^T$ corresponding to fast Eigen value λ_1 of $J^T(x_1, x_2, x_3)$ is given by

$$|J - \lambda_1 I|Z_{\lambda_1} = 0. \quad (6)$$

where I is 3×3 identity matrix. Equation (6) gives,

$$Z_{\lambda_1}^T = \begin{bmatrix} (-1 + x_1 - \lambda_1)(-3 + 2.9851x_1^2 - \lambda_1) \\ x_1(-3 + 2.9851x_1^2) \\ 2.9851x_1^2(-1 + x_1 - \lambda_1) \end{bmatrix}.$$

On the attractive parts of phase space (where $J(\mathbf{x})$ has a fast Eigen value λ_1), the equation of the slow manifold is given by

$$\dot{\mathbf{x}}(t) \cdot Z_{\lambda_1}^T = 0. \quad (7)$$

We use the equation (7) to define the equation of slow manifold. With the substitution of $\dot{\mathbf{x}}(t)$ and $Z_{\lambda_1}^T$ in the equation (7), we write the equation of slow manifold as

$$\begin{aligned} &\lambda_1^2(x_1 - x_1x_2 + 2x_1^2 - 2.9851x_1^2x_3) + \lambda_1(-4x_1 + 7x_1^2 - 4.9851x_1^3 - 3x_1x_2 \\ &+ 2.9851x_1^3x_2 - 5.97020x_1^4 - 2.985100x_1^2x_3 + 2.9851x_1^3x_3) + (3x_1 + x_1^2 \\ &- 8.985100x_1^3 - 2.9851x_1^4 + 5.970200x_1^5 + 8.95530x_1x_3 - 17.910600x_1^2x_3 \\ &- 0.044478x_1^3x_3 + 17.821644x_1^4x_3 - 8.91082x_1^3x_3) = 0. \end{aligned} \quad (8)$$

where λ_1 is fast Eigen value of $J(\mathbf{x})$. Because $\lambda_1(x_1, x_2, x_3)$ is uncertain Eigen value, it is not easy to use this implicit equation to draw a slow manifold representation in the three dimensional phase space.

3.3 Invariance Property

Theorem 3.1: Let the system in vector notation is given as

$$\dot{\mathbf{x}}(t) = H(\mathbf{x}(t)) = \begin{bmatrix} H_1(x_1, x_2, x_3) \\ H_2(x_1, x_2, x_3) \\ H_3(x_1, x_2, x_3) \end{bmatrix} \quad (9)$$

$$\begin{aligned} H_1(x_1, x_2, x_3) &= x_1(1 - x_2 + rx_1 - px_3x_1), \\ H_2(x_1, x_2, x_3) &= x_2(-1 + x_1), \\ H_3(x_1, x_2, x_3) &= x_3(-q + px_1^2). \end{aligned}$$

where H_1, H_2 and H_3 are continuously differentiable. Assume that H is locally Lipschitz and generates a flow $\phi_t(\mathbf{x})$. Let

$$L: D \subset R^3 \rightarrow R^3$$

be a continuously differentiable function on a domain $D \subset R^3$ such that $\dot{L}(\mathbf{x}) \leq 0$ in D , then the largest invariant set $\Sigma \subset D$ is the set; where $\nabla L \cdot H(\mathbf{x}) = 0 \forall \mathbf{x} \in \Sigma$.

Proof: Consider

$$L: D \subset R^3 \rightarrow R^3$$

be a continuously differentiable function on a domain $D \subset R^3$ and defined as

$$L(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{2}. \quad (10)$$

Equation (10) gives,

$$\dot{L}(x_1, x_2, x_3) = x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3. \quad (11)$$

The set $D \subset R^3$ is said to be an invariant set under the flow ϕ_t if for any point $\mathbf{x} \in D$

$$\phi_t(\mathbf{x}) \in D \forall t \in R.$$

Let Σ be a smooth closed surface without boundary in $D \subset R^3$ and suppose that n is a normal vector to the surface Σ at (x_1, x_2, x_3) . If we have

$$n \cdot \langle \dot{x}_1, \dot{x}_2, \dot{x}_3 \rangle = 0 \forall (x_1, x_2, x_3) \in \Sigma. \quad (12)$$

Let us consider Σ be the $x_1 x_2$ plane i.e. $x_3 = 0$. Note that the vector $(0, 0, 1)$ is always normal to Σ and at the point $(x_1, x_2, 0) \in \Sigma$. So we have,

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (x_1(1 - x_2 + rx_1 - px_3x_1), x_2(-1 + x_1), 0).$$

Thus,

$$\langle (0, 0, 1) \cdot (x_1(1 - x_2 + rx_1 - px_3x_1), x_2(-1 + x_1), 0) \rangle = 0.$$

Similar arguments can be verified for x_1 and x_2 planes which directs that each coordinate plane is an invariant subset. It implies that for any given positive initial condition, $x_1(t), x_2(t)$ and $x_3(t)$ are positive for all t that is any trajectory starting in R_+^3 can not cross the co-ordinate planes and it shows that R_+^3 is an invariant set for the system.

3.4 Dissipation

Theorem 3.2: Consider the autonomous vector field

$$\dot{x}(t) = H(x) \text{ for } x \in R^3,$$

and assume that it generates a flow $\phi_t(x)$. Let D_0 is a domain in R^3 which is supposed to have a volume V_0 , and $\phi_t(D_0)$ is its evolution under the flow. If $V(t)$ is the volume of D_t , then the time rate of change of volume is given as

$$\left| \frac{dV}{dt} \right|_{t=0} = \int_{D_0} \nabla \cdot H dx.$$

The system (1) is dissipative if its time- t map decreases volume for all $t > 0$.

Proof: The volume $V(t)$ can be expressed in the following form using the definition of the Jacobian of transformation as

$$V(t) = \int_{D_0} \left| \frac{d\phi_t(\mathbf{x})}{d\mathbf{x}} \right| d\mathbf{x}. \quad (13)$$

Expanding $\phi_t(\mathbf{x})$ in the neighbourhood of $t = 0$. Since the vector field $H(\mathbf{x})$ is smooth enough to have a tangent plane in each point on R^3 so we can expand $\phi_t(\mathbf{x})$ by Taylor series expansion. Hence we get

$$\phi_t(\mathbf{x}) = \mathbf{x} + \dot{\mathbf{x}}t + O(t^2) \text{ for } t \rightarrow 0 \quad (14)$$

Since

$$x(t) = H(\mathbf{x}), \quad (15)$$

The equation (15) gives,

$$\phi_t(\mathbf{x}) = \mathbf{x} + H(\mathbf{x})t + O(t^2) \text{ for } t \rightarrow 0. \quad (16)$$

It follows that

$$\begin{aligned} \frac{\partial \phi}{\partial \mathbf{x}} &= I + \frac{\partial H}{\partial \mathbf{x}} t + O(t^2), \\ \left| \frac{\partial \phi}{\partial \mathbf{x}} \right| &= \left| I + \frac{\partial H}{\partial \mathbf{x}} t \right| + O(t^2). \end{aligned} \quad (17)$$

Here I is 3×3 identity matrix so $\det I$ will be equal to 1. By using expansion of determinant, we get the following

$$\left| \frac{\partial \phi}{\partial \mathbf{x}} \right| = 1 + \text{trace} \left(\frac{\partial H}{\partial \mathbf{x}} \right) t + O(t^2). \quad (18)$$

Note that

$$\text{trace} \left(\frac{\partial H}{\partial \mathbf{x}} \right) = \nabla \cdot H, \quad (19)$$

therefore, we have

$$V(t) = V_0 + \int_{D_0} ((\nabla \cdot H)t + O(t^2)) d\mathbf{x}. \quad (20)$$

It gives

$$\left| \frac{dV}{dt} \right|_{t=0} = \int_{D_0} \nabla \cdot H d\mathbf{x}, \quad (21)$$

i.e. if the volume shrinks then divergence of vector field will be strictly negative [3].

Now considering the equations of model (1) in vector notation and computing its Jacobian

$$J(x_1, x_2, x_3) = \frac{\partial H}{\partial \mathbf{x}}. \quad (22)$$

The Jacobian $J(x_1, x_2, x_3)$ of the model is given by

$$\begin{bmatrix} 1 - x_2 + 2rx_1 - 2px_1x_3 & -x_1 & -px_1^2 \\ x_2 & -1 + x_1 & 0 \\ 2px_1x_3 & 0 & -q + px_1^2 \end{bmatrix}, \quad (23)$$

we take the parameter values as

$$p = 2.9851, \quad q = 3, \quad r = 2. \quad (24)$$

The above argument shows that the G.L.V dynamical system will be dissipative if the generalized divergence should be less than zero, i.e.

$$\sum_i \frac{\partial H_i}{\partial x_i} < 0. \quad (25)$$

where Einstein summation has been used. The divergence of vector field H on R^3 is as follows

$$\begin{aligned}\nabla \cdot H &= \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3}, \\ \nabla \cdot H &= -q - x_2 + (2r + 1 + px_1 - 2px_3)x_1.\end{aligned}\tag{26}$$

Hence system (1) will be dissipative if the following condition is satisfied,

$$(2r + 1 + px_1 - 2px_3)x_1 < q + x_2.\tag{27}$$

3.5 Existence and Uniqueness of Solution

Since we are dealing with a population dynamics model, hence, we are interested in the existence of at least one solution. The uniqueness of existed solutions will give more appropriate results in the context of the ecological system. Here we mention two theorems for which solutions of the system (1) uniquely exist for all $t > 0$.

Theorem 3.3: *If the functions f_1, f_2 and f_3 satisfy assumptions (1) and (2), mentioned in section , then continuity of functions f_i for $i \in \{1,2,3\}$ assures that atleast one solution exists for the dynamics of system (1) in region $D \times I$ where $I = (0, T]$ and the spatial boundary of region $D \subset R^3$ is defined as*

$$D = \{\mathbf{x} = (x_1, x_2, x_3) : \max |x_i| \leq M, \text{ for } i \in \{1,2,3\}\}$$

where $M > 0$.

Theorem 3.4: *Let D be a closed subspace of complete normed linear space R^3 . Consider $H: D \subset R^3 \rightarrow D$ is Lipschitz continuous so that there exist $0 < K < 1$ such that*

$$\|H(\chi) - H(\psi)\| < K\|\chi - \psi\|$$

with

$$K = T \cdot \max(1 + 2M + 2rM + 4pM^2, 1 + 2M, q + 2pM^2)$$

For $0 < K < 1$, $H(t)$ will be a contraction map. With the help of Banach fixed point theorem, it can be ensured that $0 < K < 1$ is sufficient condition for uniqueness of solution of the system (1).

Complete detail of the proof can be seen in [22].

3.6 Stability of Feasible Equilibrium Points

The equilibrium points of system (1) are solutions of following algebraic equations

$$\begin{aligned}xx_1(1 - x_2 + rx_1 - px_3x_1) &= 0, \\ x_2(-1 + x_1) &= 0, \\ x_3(-q + px_1^2) &= 0.\end{aligned}\tag{28}$$

We obtain five equilibrium points by solving the system (28),

$$X_0^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X_1^* = \begin{bmatrix} 1 \\ 1+r \\ 0 \end{bmatrix}, \quad X_2^* = \begin{bmatrix} \sqrt{\frac{q}{p}} \\ 0 \\ \frac{1+r\sqrt{\frac{q}{p}}}{\sqrt{pq}} \end{bmatrix}, \quad X_3^* = \begin{bmatrix} -\frac{1}{r} \\ 0 \\ 0 \end{bmatrix}, \quad X_4^* = \begin{bmatrix} -\sqrt{\frac{q}{p}} \\ 0 \\ \frac{-1+r\sqrt{\frac{q}{p}}}{\sqrt{pq}} \end{bmatrix}. \quad (29)$$

From an ecological point of view, negative population density is not realistic as the population can not be negative, therefore, we take the vector $\mathbf{x} = (x_1, x_2, x_3)$ as an element of R_+^3 . R_+^3 is defined as

$$R_+^3 = \{X \in R^3: x_i \geq 0 \text{ for } i \in \{1, 2, 3\}\}. \quad (30)$$

Since equilibrium points X_0^* , X_1^* and X_2^* are elements of the set $Int(R_+^3)$, therefore, we study the local stability of ecologically feasible equilibrium points X_0^* , X_1^* and X_2^* .

(I) Stability of Trivial Equilibrium Point X_0^* .

Theorem 3.5: Consider the dynamics of the model (1) in the following form

$$\dot{\mathbf{x}} = H(\mathbf{x}) = A\mathbf{x} + f(\mathbf{x}) \text{ for } \mathbf{x} \in R^3. \quad (31)$$

If following three conditions are satisfied

- (i) Constant matrix $A_{3 \times 3}$ has 3 Eigen-values with non-zero real part,
- (ii) $f(\mathbf{x})$ is smooth and
- (iii) $\lim_{\|\mathbf{x}\| \rightarrow 0} \frac{\|f(\mathbf{x})\|}{\|\mathbf{x}\|} = 0$,

then in a neighbourhood of the critical point $X_0^* = (0, 0, 0)$, there exists stable and unstable manifolds W_s and W_u with the same dimensions n_s and n_u as the stable and unstable manifolds E_s and E_u of the system

$$\dot{Z}(t) = AZ.$$

In $\mathbf{x} = 0$, E_s and E_u are tangent to W_s and W_u [23].

Proof: For GLV system, $X_0^* = (0, 0, 0)$ is trivial equilibrium point. Here we check all three mentioned conditions of theorem 5.

- (i) Note that for model (1), the constant matrix $A_{3 \times 3}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -q \end{pmatrix}$$

The determinant of the matrix A is non zero if $q \neq 0$. Since, we have taken $q = 3$, therefore, all Eigen values of A have non zero real part.

- (ii) Since functions $f_1(x_1, x_2, x_3)$, $f_2(x_1, x_2, x_3)$ and $f_3(x_1, x_2, x_3)$ for model (1) are considered as

$$\begin{aligned} x_1(-x_2 + rx_1 - px_3x_1) &= f_1(x_1, x_2, x_3), \\ x_2x_1 &= f_2(x_1, x_2, x_3), \\ x_3(px_1^2) &= f_3(x_1, x_2, x_3). \end{aligned}$$

All three functions are continuous and have continuous partial derivative for all $x \in R^3$ which implies that $f(x)$ is smooth on R^3 . Hence, the second condition also holds.

- (iii) For any $x \in R^3$, converting the Cartesian coordinates into spherical coordinates by making the following transformation

$$\begin{bmatrix} x_1 = r \sin \theta \cos \phi \\ x_2 = r \sin \theta \sin \phi \\ x_3 = r \cos \theta \end{bmatrix},$$

where $r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$.

Using this transformation in model (1), we have

$$\lim_{\|x\| \rightarrow 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

Hence the critical point $(0,0,0)$ of system (1) is of the same type of critical point of the system

$$\dot{Z}(t) = AZ.$$

The Eigen values of A are $1, -1$ and $-q$ which implies that $(0,0,0)$ is saddle node for the system $\dot{Z}(t) = AZ$. Therefore, the trivial steady state $X_0^* = (0,0,0)$ of the model (1) is a saddle point.

- (II) Stability of Axial Equilibrium Point X_1^* .

The Jacobian matrix of model (1) for parameter values $p = 2.9851, q = 3, r = 2$ is given as

$$J = \begin{bmatrix} 1 - x_2 + 4x_1 - 5.9702x_1x_3 & -x_1 & -2.9851x_1^2 \\ x_2 & -1 + x_1 & 0 \\ 5.9702x_1x_3 & 0 & -3 + 2.9851x_1^2 \end{bmatrix}. \quad (32)$$

Jacobian matrix (32) of the model (1) about $X_1^* = (1,3,0)$ yields the following Jacobian matrix

$$J_{X_1^*} = \begin{bmatrix} 2 & -1 & -2.9851 \\ 3 & 0 & 0 \\ 0 & 0 & -0.0149 \end{bmatrix}. \quad (33)$$

The characteristic equation $|J_{X_1^*} - \lambda I| = 0$ of matrix (33) is given as

$$\lambda^3 - (1.9851)\lambda^2 + (2.9702)\lambda + 0.0447 = 0. \quad (34)$$

The characteristic equation (34) has the following Eigen values

$$\lambda_1 = -0.014900, \lambda_2 = 1 + \sqrt{2}i, \lambda_3 = 1 - \sqrt{2}i. \quad (35)$$

Since λ_2 and λ_3 have positive real parts, it implies that $X_1^* = (1,3,0)$ is unstable equilibrium point.

- (III) Stability of Planer Equilibrium Point X_2^* .

The Jacobian matrix of model (1) for parameter values $p = 2.9851, q = 3, r = 2$ is given as

$$J = \begin{bmatrix} 1 - x_2 + 4x_1 - 5.9702x_1x_3 & -x_1 & -2.9851x_1^2 \\ x_2 & -1 + x_1 & 0 \\ 5.9702x_1x_3 & 0 & -3 + 2.9851x_1^2 \end{bmatrix}. \quad (36)$$

Jacobian matrix (36) of the model (1) about $X_2^* = (1.002493, 0, 1.4159)$ yields the following Jacobian matrix

$$J_{X_2^*} = \begin{bmatrix} 3.004986 & -1.002493 & 3.000002 \\ 0 & .002493 & 0 \\ 6.00887 & 0 & .000002 \end{bmatrix}. \quad (37)$$

The characteristic equation $|J_{X_2^*} - \lambda I| = 0$ of matrix (37) is given as

$$\lambda^3 + (0.997507)\lambda^2 + (18.027423)\lambda - 0.044949 = 0. \quad (38)$$

The characteristic equation (38) has the following Eigen values

$$\lambda_1 = -0.002493, \lambda_2 = -0.5 + 4.216i, \lambda_3 = -0.5 - 4.216i. \quad (39)$$

Since all three Eigen-values of matrix (37) have negative real parts, it shows that $X_2^* = (1.002493, 0, 1.4159)$ is locally stable equilibrium point.

Thus, we find that planer equilibrium point is stable while trivial and axial equilibrium points are unstable equilibrium points. Since trivial equilibrium point refers the zero density of all three species, therefore, we neglect the instability of trivial equilibrium point. From an ecological point of view, we mainly focus on non-trivial unstable equilibrium point X_1^* and try to stabilize it by adding some external control inputs.

3.7 Control of Instability of Axial Equilibrium Point

In order to suppress instability to $X_1^* = (1, 3, 0)$, we consider the controlled GLV system in the following form

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_2 + rx_1 - px_3x_1) + u_1, \\ \dot{x}_2 &= x_2(-1 + x_1) + u_2, \\ \dot{x}_3 &= x_3(-q + px_1^2) + u_3. \end{aligned} \quad (40)$$

We introduce the external control law

$$\begin{aligned} u_1 &= -\mu_1(x_1 - 1), \\ u_2 &= -\mu_2(x_2 - 3), \\ u_3 &= -\mu_3(x_3 - 0). \end{aligned} \quad (41)$$

with x_1, x_2, x_3 as the feedback variable and μ_1, μ_2, μ_3 as the positive feedback gains. We substitute control law (41) into (40) and hence, the controlled system (40) takes the following form

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_2 + rx_1 - px_3x_1) - \mu_1(x_1 - 1), \\ \dot{x}_2 &= x_2(-1 + x_1) - \mu_2(x_2 - 3), \\ \dot{x}_3 &= x_3(-q + px_1^2) - \mu_3(x_3 - 0). \end{aligned} \quad (42)$$

Theorem 3.6: *The equilibrium point $X_1^* = (1,3,0)$ of the model (1) will be asymptotically stable if positive gains μ_1, μ_2 and μ_3 satisfy the following inequalities [24]*

$$\begin{aligned}\mu_1 &> 2, \\ \mu_1\mu_2 &> 1 + 2\mu_2, \\ \mu_1\mu_2(\mu_3 + 0.0149) &> \mu_2(2\mu_3 + 2.2528) + \mu_3 + 0.0149.\end{aligned}\quad (43)$$

Proof: The Jacobian matrix J of the system (41) is given by

$$\begin{bmatrix} 1 - x_2 + 2x_1(r - px_3) - \mu_1 & -x_1 & -px_1^2 \\ x_2 & -1 + x_1 - \mu_2 & 0 \\ 2px_1x_3 & 0 & -q + px_1^2 - \mu_3 \end{bmatrix} \quad (44)$$

Let us consider that

$$\begin{aligned}e_1 &= (x_1 - 1), \\ e_2 &= (x_2 - 3), \\ e_3 &= (x_3 - 0).\end{aligned}\quad (45)$$

From (45), we get the error system as

$$\begin{aligned}\dot{e}_1 &= (2 - \mu_1)e_1 - e_2 - 2.9851e_3, \\ \dot{e}_2 &= 3e_1 - \mu_2e_2, \\ \dot{e}_3 &= -(0.014900 + \mu_3)e_3.\end{aligned}\quad (46)$$

The system (1) with constant and known parameters, will be stabilized to steady state $X_1^* = (1,3,0)$, if error system (45) stabilized to $(0,0,0)$.

To study the stability of equilibrium point $(0,0,0)$ of error system, we consider the Lyapunov function $L(e_1, e_2, e_3)$ as:

$$L = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \quad (47)$$

The time derivative of L in the neighbourhood of $(0,0,0)$ is given as

$$\dot{L} = (2 - \mu_1)e_1^2 + 2e_1e_2 - \mu_2e_2^2 - 2.9851e_1e_3 - (0.0149 + \mu_2)e_3^2. \quad (48)$$

The time derivative of Lyapunov function can be re-written in the following form

$$\dot{L} = \mathbf{e}^T M \mathbf{e}. \quad (49)$$

where $\mathbf{e} = ((x_1 - 1), (x_2 - 3), (x_3 - 0))$ is the error vector in R^3 , $\mathbf{e}^T$ is the transpose of error vector $\mathbf{e}$ and the matrix M is 3×3 is given as

$$M = \begin{bmatrix} 2 - \mu_1 & 1 & -1.49250 \\ 1 & -\mu_2 & 0 \\ -1.49250 & 0 & -(0.0149 + \mu_3) \end{bmatrix}$$

According to Lyapunov stability theory, the equilibrium point $(0,0,0)$ of the system will be asymptotically stable if $\dot{L} < 0$. And $\dot{L} < 0$ if Matrix M will be negative definite. Considering this, we find that mentioned condition will be fulfilled if positive feedback gains μ_1 , and μ_2 satisfy the following inequalities,

$$\begin{aligned}\mu_1 &> 2, \\ \mu_1\mu_2 &> 1 + 2\mu_2, \\ \mu_1\mu_2(\mu_3 + 0.0149) &> \mu_2(2\mu_3 + 2.2528) + \mu_3 + 0.0149.\end{aligned}\quad (50)$$

In the next section, we target to achieve synchronization of chaos in two chaotic generalized Lotka Volterra models. After pioneering work of Pecora and Carroll [7], synchronization of chaos has attracted considerable interest among researchers during the last decade. A decent number of algorithms have been suggested so far in the literature for synchronization of chaos. Active and adaptive control methods are two of the popular techniques used for complete synchronization.

4. Complete Replacement Synchronization

Here, we target to achieve complete replacement synchronization in two GLV models. We consider the model (1) and start with two identical chaotic GLV systems having the same parameter but not the same initial conditions. Since the coupling between models is needed to maintain the synchronous state and the computational process of data assimilation, therefore, we couple the states of both models with two controllers. In order to achieve complete replacement synchronization in coupled GLV models, we drive the response system with prey species x_1 . In response system, we remove x_1 of the drive system with its counterpart in the response system. Here we can think of prey species x_1 as a driving variable for response system with an assumption that it is superfluous in the system of two coupled GLV models. This construction gives us a new five-dimensional drive-response system having drive and response variables as x_{1d}, x_{2d}, x_{3d} and x_{2r}, x_{3r} respectively. The coupled chaotic system with x_{1d} drive configuration is as follows:

$$\text{Drive – Response System} \begin{cases} \dot{x}_{1d} = x_{1d}(1 - x_{2d} + rx_{1d} - px_{3d}x_{1d}), \\ \dot{x}_{2d} = x_{2d}(-1 + x_{1d}), \\ \dot{x}_{3d} = x_{3d}(-q + px_{1d}^2), \\ \dot{x}_{2r} = x_{2d}(-1 + x_{1d}) + u_1, \\ \dot{x}_{3r} = x_{3d}(-q + px_{1d}^2) + u_2, \\ x_{2d}(0) \neq x_{2r}(0) \text{ and } x_{3d}(0) \neq x_{3r}(0). \end{cases} \quad (51)$$

4.1 Active Controllers and Stability of Synchronization Manifold

Theorem 4.1: *The identical synchronization manifold $\Omega = [x_{2d} = x_{2r}, x_{3d} = x_{3r}]$ is globally asymptotically stable for the coupling between drive and response system in set of equations (51) for positive μ_1 and μ_2 , where μ_1 and μ_2 are large enough such that*

$$\mu_1 + 1 > x_{1d}, \quad \mu_2 + q > px_{1d}^2$$

Proof: To prove the theorem 7, we consider drive-response system given by equations (51) simultaneously and add the uni-directional controllers through the linear positive constants μ_1 and μ_2 . We choose two controllers for response system as

$$\begin{aligned} u_1 &= -\mu_1(x_{2r}(t) - x_{2d}(t)) \\ u_2 &= -\mu_2(x_{3r}(t) - x_{3d}(t)). \end{aligned} \quad (52)$$

Existence of all forms of identical synchronization in any dynamical system (chaotic or not), are really manifestations of dynamical behaviour restricted to a flat hyper-plane in the phase space i.e. to say motion is continually confined to a hyper-plane and this hyper-plane can be referred as synchronization manifold [7]. We consider the identical synchronization manifold of the systems equation (51) as

$$\Omega = [x_{2d} = x_{2r}, x_{3d} = x_{3r}].$$

Further, we consider the errors between states of drive and response systems of system (51) as

$$\begin{aligned} e_2(t) &= x_{2r}(t) - x_{2d}(t), \\ e_3(t) &= x_{3r}(t) - x_{3d}(t). \end{aligned} \quad (53)$$

The dynamics of error system is as follows

$$\begin{aligned} \dot{e}_2 &= (-1 - \mu_1 + x_{1d})e_2, \\ \dot{e}_3 &= (-q - \mu_2 + px_{1d}^2)e_3. \end{aligned} \quad (54)$$

where, we are now interested in the stability of origin. The Jacobian of right side of (54) is given by

$$J(e_2, e_3) = \begin{bmatrix} -1 + \mu_1 + x_{1d} & 0 \\ 0 & -q + \mu_2 + px_{1d}^2 \end{bmatrix} \quad (55)$$

We treat the response system $(x_{2d}(t), x_{3r}(t))$ as a separate system driven by x_{1d} , then the solutions of equation (55) convey us about convergence and divergence of two initially nearby trajectories of $\{x_{2r}(t), x_{2d}(t)\}$ and $\{x_{3r}(t), x_{3d}(t)\}$.

Next, We analyse the possibility of synchronization using the Lyapunov function construction method. We consider the Lyapunov function as

$$L(e_2, e_3) = \frac{1}{2}(e_2^2 + e_3^2). \quad (56)$$

$$\frac{dL}{dt} = e_2\dot{e}_2 + e_3\dot{e}_3. \quad (57)$$

Plugging dynamics of errors (54) into (56), we get

$$\frac{dL}{dt} = -[(\mu_1 + 1 - x_{1d})e_2^2 + (\mu_2 + q - px_{1d}^2)e_3^2], \quad (58)$$

which will be strictly negative for following conditions on μ_1 and μ_2

$$\begin{aligned} \mu_1 + 1 &> x_{1d}, \\ \mu_2 + q &> px_{1d}^2. \end{aligned} \quad (59)$$

for all $t > 0$. Condition (59) ensures that we consider the bounded density of prey species, then we can bound the positive feedback gains.

Thus, if μ_1 and μ_2 satisfy (59), then it can be assured that $\frac{dL}{dt} < 0$ for all $t > 0$ or in other words, the complete replacement synchronization follows as e_2 and $e_3 \rightarrow 0$ as $t \rightarrow \infty$.

For the coupled chaotic systems, it is pre-arranged that the coupling tends to make the state vectors of chaotic systems coming together and coupling does not

perturb the chaotic state which has been synchronized. Very strong coupling tends to make the state of two chaotic system identical. It influences not only frequency but also the chaotic amplitude. As a result, the states coincide (or nearby coincide) and regime of complete synchronization sets in [25].

4.1.1 Numerical Simulation

We numerically integrate system (51) and display results in figure 8, where drive and response systems are shown to synchronize identically when considered positive feed-back gain are chosen as $\mu_1 = 0.000024, \mu_2 = 1.345$.

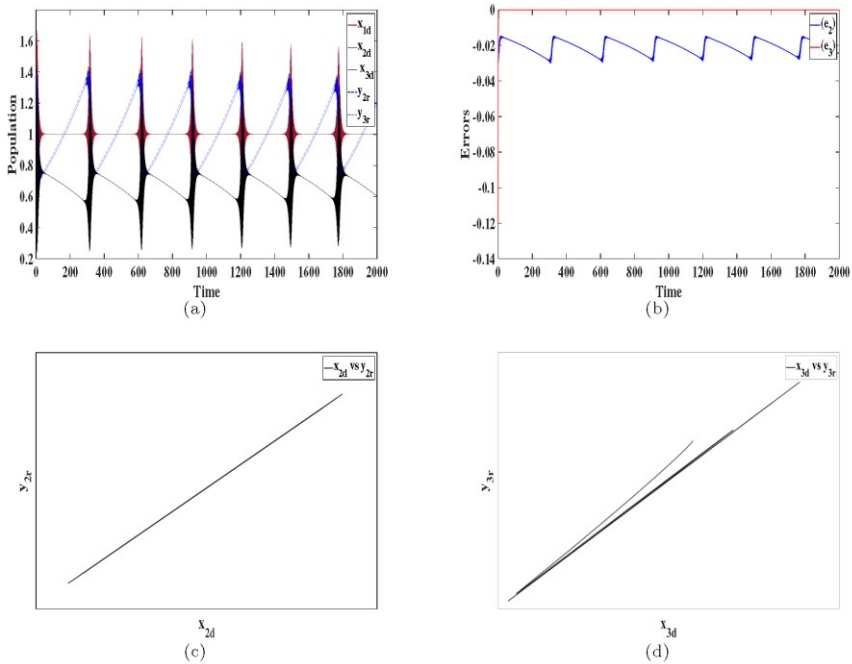


Figure 8

(a): solutions of drive and response systems plotted over time, (b): log of absolute values of errors over time between drive and response systems, (c): synchronization plot of $\{x_{2d}, x_{2r}\}$, (d): synchronization plot of $\{x_{3d}, x_{3r}\}$ for coupling strengths $\mu_1 = 0.000024$ and $\mu_2 = 1.345$.

Since the necessary condition for the stability of the synchronization manifold is the negative largest transverse Lyapunov exponent. In the case of complete replacement synchronization, the transverse Lyapunov exponents are also known as conditional Lyapunov exponents. It is because Lyapunov exponents for the new system depend on the coupling from the drive[6]. The Lyapunov spectrum is a global indicator of the system's state, aggregating over the behavior of the entire system trajectory in phase space. The typical approach for observing these transitions is to see a change in sign of the Lyapunov

exponents of the system, as obtained from ensemble averages of the eigenvalues of the Jacobian matrix of system (51) [26].

4.1.2 Lyapunov Spectrum For Drive-Response System

We solve equations of system (51) and get the Jacobian matrix as

$$J(x_{1d}, x_{2d}, x_{3d}, x_{2d}, x_{3r}) = A_{5 \times 5}, \text{ where } A = [a_{ij}]. \quad (60)$$

The entries of the matrix A are given as

$$\begin{aligned} a_{11} &= 1 - x_{2d} + 2(r - px_{3d})x_{1d}, \\ a_{12} &= -x_{1d}, \quad a_{13} = -px_{1d}^2, \\ a_{21} &= x_{2d}, \quad a_{22} = -1 + x_{1d}, \\ a_{31} &= 2px_{1d}x_{3d}, \quad a_{33} = -q + px_{1d}^2, \\ a_{41} &= x_{2d}, \quad a_{42} = \mu_1, \quad a_{44} = -1 + x_{1d} - \mu_1x_{2d}, \\ a_{51} &= 2px_{3r}x_{1d}, \quad a_{53} = \mu_2, \quad a_{55} = -q + px_{1d}^2 - \mu_2, \\ a_{14} &= a_{15} = a_{23} = a_{24} = a_{25} = a_{32} = 0, \\ a_{34} &= a_{35} = a_{43} = a_{45} = a_{52} = a_{54} = 0. \end{aligned}$$

Averaging the eigenvalues of Jacobian J over all phase space configurations set-up by the chaotic trajectory, we get the five Lyapunov exponents of the system of two coupled GLV models. A simulation is run and all Lyapunov exponents are displayed in figure 9 for set of parameter values $\{p, q, r\} = \{2.9851, 3, 2\}$.

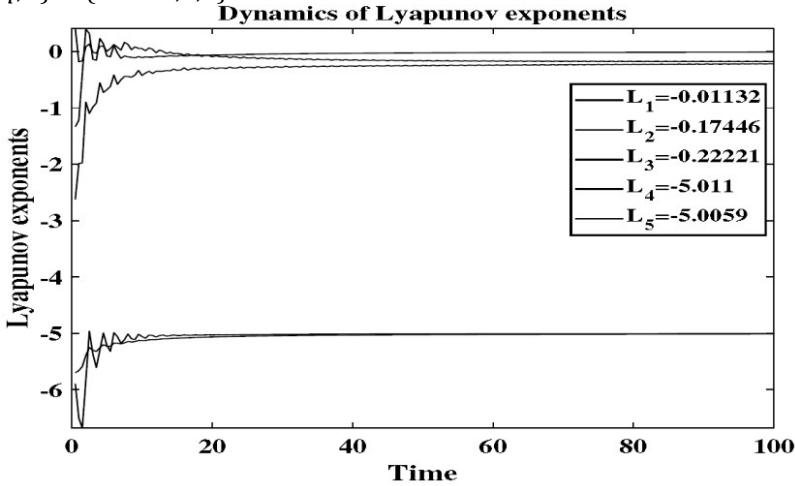


Figure 9
Lyapunov Exponents of two GLV models coupled with positive feedback gains $\mu_1 = 0.000024$ and $\mu_2 = 1.345$.

The Lyapunov exponents of the drive-response system are obtained as

$$L_1 = -0.011320, L_2 = -0.174464, L_3 = -0.22221, L_4 = -5.011, L_5 = -5.0059. \quad (61)$$

Since, all Lyapunov exponents are negative which ensures the stable synchronization manifold, therefore, it can be concluded that the states of coupled GLV systems are synchronized.

4.2 Adaptive Controllers and Stability of Synchronization Manifold

We design non-linear adaptive controller for global complete-Replacement synchronization of identical chaotic GLV systems with unknown parameters. We consider the drive system as

$$\begin{aligned}\dot{x}_{1d} &= x_{1d}(1 - x_{2d} + rx_{1d} - px_{3d}x_{1d}), \\ \dot{x}_{2d} &= x_{2d}(-1 + x_{1d}), \\ \dot{x}_{3d} &= x_{3d}(-q + px_{1d}^2).\end{aligned}\tag{62}$$

The response system is given by controlled chaotic system

$$\begin{aligned}\dot{x}_{2r} &= x_{2r}(-1 + x_{1d}) + u_1, \\ \dot{x}_{3r} &= x_{3r}(-q + px_{1d}^2) + u_2.\end{aligned}\tag{63}$$

The synchronization error between the identical chaotic system is defined as

$$\begin{aligned}e_2(t) &= x_{2r}(t) - x_{2d}(t) \\ e_3(t) &= x_{3r}(t) - x_{3d}(t).\end{aligned}\tag{64}$$

The error dynamics between the identical chaotic system is calculated as :

$$\begin{aligned}\dot{e}_2 &= -e_2 + x_{1d}e_2 + u_1, \\ \dot{e}_3 &= -qe_3 + px_{1d}^2e_3 + u_2.\end{aligned}\tag{65}$$

In (65), unknown parameters p , q are to be determined by using parameter estimates $P(t)$ and $Q(t)$ respectively. For this purpose, we consider adaptive control laws u_1 and u_2 with positive feedback gains μ_1 and μ_2 as

$$\begin{aligned}u_1 &= e_2 - x_{1d}e_2 - \mu_1e_2, \\ u_2 &= Q(t)e_3 - P(t)x_{1d}^2e_3 - \mu_2e_3.\end{aligned}\tag{66}$$

Using control law (66) into the error dynamics (65), we get

$$\begin{aligned}\dot{e}_2 &= -\mu_1e_2, \\ \dot{e}_3 &= -(q - Q(t))e_3 + (p - P(t))x_{1d}^2e_3 - \mu_2e_3.\end{aligned}\tag{67}$$

We define parameter estimation error as

$$\begin{aligned}e_p(t) &= (p - P(t)), \\ e_q(t) &= (q - Q(t)).\end{aligned}\tag{68}$$

Using (68), we can simplify the error dynamics (67) as

$$\begin{aligned}\dot{e}_2 &= -\mu_1e_2, \\ \dot{e}_3 &= -e_q(t)e_3 + e_p(t)x_{1d}^2e_3 - \mu_2e_3.\end{aligned}\tag{69}$$

Differentiating (68) with respect to t , we get

$$\begin{aligned}\dot{e}_p &= -\dot{P}(t), \\ \dot{e}_q &= -\dot{Q}(t)\end{aligned}\tag{70}$$

Theorem 4.2: *The identical synchronization manifold $\Omega = [x_{2d} = x_{2r}, x_{3d} = x_{3r}]$ is globally asymptotically stable for the coupling between derive and response system in equation (51) for positive μ_1 and μ_2 .*

Proof: The identical synchronization manifold for the system (51) can be written as

$$\Omega = [x_{2d} = x_{2r}, x_{3d} = x_{3r}].$$

Using change of coordinates

$$e = \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} x_{2d} - x_{2r} \\ x_{3d} - x_{3r} \end{bmatrix}, \quad (71)$$

such that Ω can be written

$$\Omega = (0,0).$$

Next, we use Lyapunov stability theory for finding an update law for the parameter estimates. we consider the quadratic Lyapunov function as

$$L = \frac{1}{2}(e_2^2 + e_3^2 + e_p^2 + e_q^2) \quad (72)$$

Note that Lyapunov function L is positive definite on R^4 . Differentiating L along the trajectories of (67) and (72). We get,

$$\begin{aligned} \frac{dL}{dt} &= e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_p \dot{e}_p + e_q \dot{e}_q, \\ \frac{dL}{dt} &= -\mu_1 e_2^2 - \mu_2 e_3^2 + e_p(t)(-\dot{P}(t) + x_{1d}^2 e_3) + e_q(t)(-\dot{Q}(t) - e_3^2). \end{aligned} \quad (73)$$

We want error system to be asymptotically stable i.e.

$$\frac{dL}{dt} < 0. \quad (74)$$

In view of (73), we take the parameter update law as

$$\begin{aligned} \dot{P}(t) &= x_{1d}^2 e_3, \\ \dot{Q}(t) &= -e_3^2. \end{aligned} \quad (75)$$

By substituting the parameter update law (75) into Lyapunov function, we obtain time derivative of L as

$$\frac{dL}{dt} = -\mu_1 e_2^2 - \mu_2 e_3^2, \quad (76)$$

From (76), it is clear that $\dot{L}$ is negative semi-definite function on R^4 . Thus, we can conclude that the synchronization error vector $e(t)$ and the parameter estimation error are globally bounded, i.e.

$$[e_2, e_3, e_p, e_q]^T \in L_\infty. \quad (77)$$

We define $\mu = \min\{\mu_1, \mu_2\}$, then it follows from (76) that

$$\frac{dL}{dt} \leq -\mu \|e\|^2. \quad (78)$$

Integrating the inequality (78) with respect to τ from 0 to t . We get,

$$\int_0^t \mu \|e(\tau)\|^2 d\tau \leq L(0) - L(t). \quad (79)$$

From (79) it follows that $e \in \mathbf{L}_2$ and hence, $\dot{e}(t) \in \mathbf{L}_\infty$.

With the help of Barbalat's lemma [27],[28], we conclude that $e(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$ for all initial conditions $\mathbf{e}(0) \in \mathbb{R}^2$. Therefore, the origin corresponding to identical synchronization manifold is asymptotically stable between the chaotic systems with unknown parameters for all initial conditions and large enough coupling strength.

4.2.1 Numerical Simulation

We use the classical fourth-order Runge-Kutta method to solve the GLV system and perform all simulations in MATLAB. The initial value of parameter estimates are taken as $P(0) = 3.9, Q(0) = 4$. The initial values of states of drive and response systems are taken as $(x_{1d}(0), x_{2d}(0), x_{3d}(0)) = (4, 1.4, 1.41)$ and $(x_{2r}(0), x_{3r}(0)) = (1, 1.414)$ respectively. The effectiveness of complete replacement synchronization via adaptive control method is verified through simulation results which are shown in figure 10. Figure 10(a) shows the solutions of system (51). Figure 10(b) shows the time history of synchronization errors $e_2(t)$ and $e_3(t)$. Figures 10(c) and (d) show the synchronization between the states $\{x_{2d}(t), x_{2r}(t)\}$ and $\{x_{3d}(t), x_{3r}(t)\}$. It is clear from figure 10(b) that although the initial values are different, the error dynamics approach to zero as time goes to ∞ . Therefore, our numerical results confirm that the achieved synchronization is complete replacement synchronization and, hence the amplitude and frequency of state variables of response system become same as the drive system.

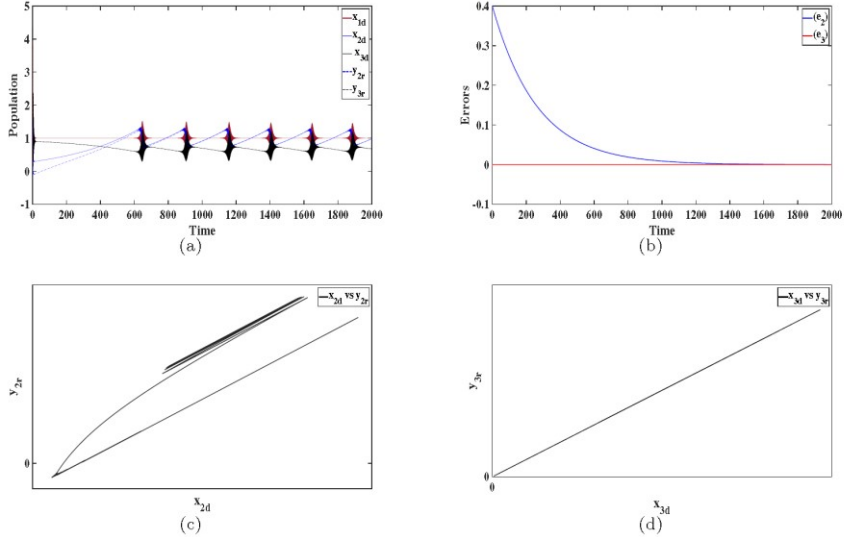


Figure 10

(a): Solutions of drive and response systems over time, (b): log of absolute values of errors between drive and response systems are shown over time, (c): synchronization plot for $\{x_{2d}, x_{2r}\}$, (d): synchronization plot for $\{x_{3d}, x_{3r}\}$ for coupling strengths $\mu_1 = 0.0038$ and $\mu_2 = 2$.

5. Conclusion

In an ecosystem, to maintain sustainable development, predator-prey systems demonstrate a sophisticated balance. In our work, we mainly focus on GLV equations with three species. These equations can be direct model competition and the trophic relationship between an arbitrary number of species. Since GLV equations are capable of a wide variety of dynamics, including limit cycles and chaos as well as point attractors, therefore, we write three various forms of three-dimensional GLV model. For each form, we consider different function responses. We consider linear, Holling type II and III functional responses for considered forms of GLV models. Three different parameter ranges for three various forms of the GLV model have been found. The model with the linear functional response shows chaotic dynamics, and model with Holling type II and II show stable dynamics. For the study, we primarily consider the GLV model with the linear functional response and analyze its basic properties such as dissipation, in-variance, Lyapunov exponents, slow-manifold equation, the stability of feasible equilibrium points, and the suppression of instability equilibrium point. The model is found to be chaotic because of one positive Lyapunov exponent and model has two unstable equilibrium points for known and constant parameters $p = 2.9851, q = 3, r = 2$.

Further, the complete replacement synchronization has been achieved in two GLV models using two control schemes: active and adaptive control techniques. In the configuration, the prey population in the drive system is considered to be superfluous. It acts as a driving variable for the response system in such a way that the other two predator populations can depend only on the prey population. We consider the drive system with known parameters and with different initial conditions and coupled it with response system under unidirectional complete replacement technique. We then apply the active control method to synchronize two chaotic GLV models. For this purpose, two simple linear controllers have been presented to synchronize the states of GLV systems, and these controllers are more straightforward than those of the current results.

Since, in many real applications, it may be challenging to determine the values of system parameters, so we focus on a much better complete replacement synchronization of two chaotic GLV systems in which the parameters of response are unknown. The main adaptive results for stabilization and synchronization are established using the Lyapunov stability theory, and numerical simulations are used to verify the effectiveness of proposed controllers. We design controllers and adaptive laws of parameters by using the Lyapunov stability theory. These controllers ensure global and exponential synchronization of two chaotic systems with unknown parameters.

Our findings indicate that the approaches of active and adaptive control techniques are efficient for global synchronization. Also, these techniques can

have wide applicability to chaotic dynamical systems.

Funding

The second author's research was funded by the Science and Engineering Research Board (SERB), under two separate grants with grant numbers MTR/2018/000727 and EMR/2017/005203.

Disclosure statement

The authors declare that they have no conflict of interest.

References

- [1] E. N. Lorenz, "Deterministic nonperiodic flow," *Journal of the atmospheric sciences*, vol. 20, no. 2, pp. 130–141 (1963).
- [2] B. R. Hunt and E. Ott, "Defining chaos," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 25, no. 9, pp. 097618 (2015).
- [3] H. G. Solari, M. A. Natiello, and G. B. Mindlin, *Nonlinear dynamics: a two-way trip from physics to math*. CRC Press (1996).
- [4] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. CRC Press (2018).
- [5] T. Janaki and G. Rangarajan, "Lyapunov exponents for continuous-time dynamical systems," Ph.D. dissertation, Msc, Thesis, Indian Institute of Science of education, India (2003).
- [6] M. Frisk, "Synchronization in chaotic dynamical systems," (2016).
- [7] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical review letters*, vol. 80, no. 10, pp. 2109 (1998).
- [8] —, "Synchronization of chaotic systems," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 25, no. 9, pp. 097611 (2015).
- [9] A. Singh and S. Gakkhar, "Synchronization of chaos in a food web in ecological systems," *World Academy of Science, Engineering and Technology*, vol. 70, pp. 94–98 (2010).
- [10] R. K. Upadhyay and V. Rai, "Complex dynamics and synchronization in two non-identical chaotic ecological systems," *Chaos, Solitons & Fractals*, vol. 40, no. 5, pp. 2233–2241 (2009).
- [11] J. Lu and J. Cao, "Adaptive complete synchronization of two identical or different chaotic (hyperchaotic) systems with fully unknown parameters,"

- Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 15, no. 4, pp. 043901 (2005).
- [12] E.-W. Bai and K. E. Lonngren, "Synchronization of two lorenz systems using active control," *Chaos, Solitons & Fractals*, vol. 8, no. 1, pp. 51–58 (1997).
 - [13] A. Njah and U. Vincent, "Synchronization and anti-synchronization of chaos in an extended bonhöffer–van der pol oscillator using active control," *Journal of Sound and Vibration*, vol. 319, no. 1-2, pp. 41–49 (2009).
 - [14] S. Chen and J. L"u, "Synchronization of an uncertain unified chaotic system via adaptive control," *Chaos, Solitons & Fractals*, vol. 14, no. 4, pp. 643–647 (2002).
 - [15] D. Dai and X.-K. Ma, "Chaos synchronization by using intermittent parametric adaptive control method," *Physics Letters A*, vol. 288, no. 1, pp. 23–28 (2001).
 - [16] J. H. Park, "Adaptive synchronization of a unified chaotic system with an uncertain parameter," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 6, no. 2, pp. 201–206 (2005).
 - [17] N. Samardzija and L. D. Greller, "Explosive route to chaos through a fractal torus in a generalized lotka-volterra model," *Bulletin of Mathematical Biology*, vol. 50, no. 5, pp. 465–491 (1988).
 - [18] G. Cai, J. Huang, L. Tian, and Q. Wang, "Adaptive control and slow manifold analysis of a new chaotic system," *International Journal of Nonlinear Science*, vol. 2, no. 1, pp. 50–60 (2006).
 - [19] V. S. H. Rao and B. R. Phaneendra, "Global dynamics of bidirectional associative memory neural networks involving transmission delays and dead zones," *Neural networks*, vol. 12, no. 3, pp. 455–465 (1999).
 - [20] A. S. Ackleh, D. F. Marshall, and H. E. Heatherly, "Extinction in a generalized lotka-volterra predator-prey model," *International Journal of Stochastic Analysis*, vol. 13, no. 3, pp. 287–297 (1900).
 - [21] O. Malcai, O. Biham, P. Richmond, and S. Solomon, "Theoretical analysis and simulations of the generalized lotka-volterra model," *Physical Review E*, vol. 66, no. 3, pp. 031102 (2002).
 - [22] A. Elsadany, A. Matouk, A. Abdelwahab, and H. Abdallah, "Dynamical analysis, linear feedback control and synchronization of a generalized lotka-volterra system," *International Journal of Dynamics and Control*, vol. 6, no. 1, pp. 328–338 (2018).

- [23] G. F. Simmons, *Differential equations with applications and historical notes*. CRC Press (2016).
- [24] M. Yassen, “Adaptive chaos control and synchronization for uncertain new chaotic dynamical system,” *Physics Letters A*, vol. 350, no. 1-2, pp. 36–43 (2006).
- [25] A. C. Luo, “A theory for synchronization of dynamical systems,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 1901–1951 (2009).
- [26] N. Lahav, I. Sendiña-Nadal, C. Hens, B. Kshirim, B. Barzel, R. Cohen, and S. Boccaletti, “Synchronization of chaotic systems: A microscopic description,” *Physical Review E*, vol. 98, no. 5, pp. 052204 (2018).
- [27] Y.-Y. Min and Y.-G. Liu, “Barbalat lemma and its application in analysis of system stability,” *Journal of Shandong University (engineering science)*, vol. 37, no. 1, pp. 51–55 (2007).
- [28] M. Sun, “A barbalat-like lemma with its application to learning control,” *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2222–2225 (2009).
- [29] S. Dwivedi, Optimal parameters and control design for synchronization of chaos in ecological systems, Ph.D. dissertation, Indian Institute of Technology Mandi (2023).

Received October, 2021