

On R-complex Finsler space with cubic (α, β) -metric

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Abstract

In this paper we study about the basic geometrical properties for R-complex Finsler space which is equipped with Cubic (α, β) -metric and the metric is defined by $\frac{(\alpha+\beta)^3}{\alpha^2}$. We obtained the fundamental metric field $\widetilde{g}_{ij}$, $\widetilde{g}_{ij}$ and the inverse of these tensor fields. Further we study the properties of non-Hermitian space for the R-complex Finsler space equipped with Cubic (α, β) -metric.

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1. Introduction

The concept of complex Finsler space introduced by Rizza [19] as a generalization of the homogeneity of the fundamental function for any complex scalars λ with respect to the fibre variables and the first example of the complex hyperbolic geometry was introduced by Kobayashi [11] and that metric function known as Kobayashi metric which gives an impulse to the study of complex Finsler geometry. Hermitian metrics on the base manifold, the locally Minkowski complex metrics, η -Einstein metrics etc are some interesting examples of complex Finsler metric. Further Munteanu and Purcure [7] extended the concept of complex Finsler space and introduced the concept of R-complex Finsler spaces by reducing the scalars to $\lambda \in \mathbb{R}$ i.e. in a real scalar. Alda and Munteanu [16] studied R-complex Finsler spaces for the Finsler (α, β) -metric and determined the basic fundamental metric function and properties of Hermitian space with respect to the R-complex Finsler spaces with (α, β) -metric. Further some authors [7, 8, 9, 14, 15, 17, 20, 21], are also studied R-complex Finsler spaces with special metric and obtained interesting geometric properties and various results about R-complex Finsler space.

A Finsler metric L in a smooth manifold M^n is said to be an (α, β) -metric, if L is a positively homogeneous function of degree one in $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i(x)y^i$. The interesting and important examples of an (α, β) -metric is Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto-metric $\frac{\alpha^2}{\alpha - \beta}$. Further Yang [10] generalized a special form of an (α, β) -metric on smooth manifold M^n which is known as the class p-power (α, β) -metric and given by

$$F = \alpha \left(1 + \frac{\beta}{\alpha} \right)^p, \quad (1)$$

where $p \neq 0$ is a real constant. So, if we consider $p = 1$ then equation (1) reduces to a well-known form which is known as Randers metric and this metric have very interesting curvature properties in the field of physics and it was introduced by Ingarden in 1957. If $p = 2$ then it becomes square metric and it also known as Z. Shen square metric. If $p = -1$ it reduces to Matsumoto type metric which can be used in measurement of the slope of a mountain. If $p = \frac{1}{2}$ it reduces to square root metric i.e. $F = \sqrt{\alpha(\alpha + \beta)}$, If $p = 3$ in equation (1) and get special class of (α, β) -metric in the form of

$$F = \frac{(\alpha + \beta)^3}{\alpha^2}, \quad (2)$$

and the importance of this metric studied by Tripathi, Khan and Chaubey [4] and termed this metric as cubic (α, β) -metric and proved under which condition it is projectively flat. Further, Tripathi and et al. [1, 2, 3, 5] studied its weakly berwald properties for the Finsler space with Cubic (α, β) -metric, Locally Dually Flatness Properties in Cubic (α, β) -metric, Projectively Flat Cubic

(α, β) -metric with constant Flag Curvature $K = 0$ and On the L- Duality of a Finsler Space with Special (α, β) -metric.

In the present paper we considered an R-complex Finsler spaces with Cubic (α, β) -metric and obtained the fundamental metric tensor fields $\widetilde{g}_{ij}, \widetilde{g}^{ij}$, and the inverses of these tensor fields. Further, we study the properties of non-Hermitian space for R-complex Finsler space equipped with Cubic (α, β) -metric.

2. Preliminaries

Let us consider a smooth complex manifold M of dimension η and z^k be the local complex co-ordinate in a chart $(U, \emptyset)$ then the complexified of the real tangent bundle $T_c M$ will be splits into the sum of holomorphic tangent bundle $T'M$ of $(1, 0)$ -type vector and its dual tangent bundle $T''M$. The bundle $T'M$ is an complex manifold and the local coordinate of the chart $u \in T'M$ will be denoted by (z^k, η^k) . Now the change of the co-ordinate in the respective local chart will be determined by

$$z'^k = z'^k(z), \quad \eta'^k = \frac{\partial z'^k}{\partial z^j} \eta^j. \quad (3)$$

Roughly we can say that the geometry of a complex Finsler space is the study of the geometric objects of smooth complex manifold $T'M$ which is equipped with the Hermitian metric structure defined by g_{ij} . Since the complexified tangent bundle of $T'M$ is decomposed as $T_c(T'M) = T'(T'M) \oplus T''M$. A natural local frame for $T'(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$ and it changes according to the rules given below:

$$\frac{\partial}{\partial z^k} = \frac{\partial z'^k}{\partial z^h} \frac{\partial}{\partial z'^k} + \frac{\partial^2 z'^k}{\partial z^j \partial z^h} \eta^j \frac{\partial}{\partial \eta'^k}; \quad \partial \eta^k = \frac{\partial z'^k}{\partial z^j} \partial \eta'^k.$$

From above it is clear that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

A complex nonlinear connection briefly (c.n.c) is a supplementary distribution $H(T'M)$ to a vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\frac{\partial}{\partial \eta^k}$ and an adapted frame in $H(T'M)$ is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are the coefficient of the complex nonlinear connection and they have a certain rule of change at (3), so that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M . Next, we use the abbreviations $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\partial_{\bar{k}}, \dot{\partial}_{\bar{k}}, \delta_{\bar{k}}$ for their conjugates. The dual adapted basis of $\delta_k, \dot{\partial}_k$ are $\{dz^k, \delta \eta^k = d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, \delta \bar{\eta}^k\}$ their conjugates.

We recall, that the homogeneity of the metric function of a complex Finsler space [6, 12, 18, 20, 22] is with respect to complex scalars and the metric tensor of the space is a Hermitian one. In [7] slightly changed the definition of complex Finsler spaces as:

Definition 2.1: An R-complex Finsler metric on M is a continuous function $F : T'M \rightarrow \mathbb{R}$ which satisfying following conditions:

1. The metric $L = F^2$ is 2-degree homogeneous function on $T'M/0$;
2. $F(z, \eta) \geq 0$, this equality holds if and only if $\eta=0$;
3. $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda|F(z, \eta, \bar{z}, \bar{\eta})$, for all $\lambda \in \mathbb{R}$.
4. The Hermitian matrix $g_{i\bar{j}}(z, \eta) = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is positive definite on $T'M$ and it is called the fundamental metric tensor.

It follows that L is (2, 0) homogeneous with respect to the real scalar λ and is proved that the following identities are fulfilled [14];

$$\frac{\partial L}{\partial \eta^i} + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i = 2L; g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \bar{\eta}^j}, \quad (4)$$

$$\frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0; \frac{\partial g_{i\bar{k}}}{\partial \eta^j} \eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0, \quad (5)$$

$$2L = g_{ij}\eta^i\eta^j + g_{\bar{j}i}\bar{\eta}^i\bar{\eta}^j + 2g_{ij}\eta^i\bar{\eta}^j, \quad (6)$$

where,

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}; \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}; \quad g_{\bar{j}i} = \frac{\partial^2 L}{\partial \bar{\eta}^j \partial \eta^i}.$$

Definition 2.2: An R-complex Finsler space (M, F) is called R-complex Finsler space with (α, β) -metric if the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is R-homogeneous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$ -depends on $z^i, \eta^i, \bar{z}^i$ and $\bar{\eta}^i$, ($i = 1, 2, \dots, n$) by means of $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$. That is

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \quad (7)$$

where,

$$\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2} (a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j) = \text{Re} \{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\}, \quad (8)$$

$$\beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2} (b_i\eta^i + b_{\bar{i}}\bar{\eta}^i) = \text{Re}(b_i\eta^i),$$

With $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, $b_i = b_i(z)$. We denote

$$L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \quad (9)$$

Definition 2.3: An R-complex Finsler space (M, F) is called Hermitian space, if the tensor $g_{ij}=0$ and the Hermitian metric $g_{i\bar{j}}$ is invertible and an R-complex Finsler space (M, F) is called non-Hermitian space if the metric tensor $g_{i\bar{j}} = 0$ and the Hermitian metric g_{ij} is invertible. Where, g_{ij} and $g_{i\bar{j}}$ are the metric tensors of the space and are given by, $g_{ij} = \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \eta^j} L$ and $g_{i\bar{j}} = \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \bar{\eta}^j} L$.

3. R-complex cubic (α, β) -metric

The R-complex Finsler space produce two type of fundamental metric tensor fields one is g_{ij} other is $g_{i\bar{j}}$ and in Hermitian geometry the tensor field $g_{i\bar{j}}$ must be invertible. A space is said to be Hermitian space if $\det(g_{i\bar{j}} \neq 0)$ in R-complex Finsler spaces and it will be non-Hermitian space if $\det(g_{ij} \neq 0)$ in R-complex Finsler spaces. In present section, we determine the fundamental tensor for R-complex Finsler space equipped with Cubic (α, β) -metric and obtained condition for property of non-Hermitian R-complex Finsler spaces. Now consider R-complex Finsler space with the metric

$$L(\alpha, \beta) = \frac{(\alpha+\beta)^6}{\alpha^4}. \quad (10)$$

Then, it follows that $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ i.e. Cubic (α, β) -metric. Now, we find the following quantities on R-complex Finsler spaces with Cubic (α, β) -metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$.

$$\alpha L_\alpha + \beta L_\beta = 2L, \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \quad (11)$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \quad (12)$$

where,

$$L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \quad (13)$$

$$L_\alpha = \frac{6\alpha(\alpha+\beta)^5 - 4(\alpha+\beta)^6}{\alpha^5}, \quad (14)$$

$$L_\beta = \frac{6(\alpha+\beta)^5}{\alpha^4}, \quad (15)$$

$$L_{\alpha\alpha} = \frac{-48\alpha(\alpha+\beta)^5 + 30\alpha^2(\alpha+\beta)^4 + 20(\alpha+\beta)^6}{\alpha^6}, \quad (16)$$

$$L_{\beta\beta} = \frac{30(\alpha+\beta)^4}{\alpha^4}, \quad (17)$$

$$L_{\alpha\beta} = \frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{\alpha^5}, \quad (18)$$

$$L_{\beta\alpha} = \frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{\alpha^5}, \quad (19)$$

$$\begin{aligned} \alpha L_\alpha + \beta L_\beta &= \alpha \left(\frac{6\alpha(\alpha+\beta)^5 - 4(\alpha+\beta)^6}{\alpha^5} \right) + \left(\frac{6\beta(\alpha+\beta)^5}{\alpha^4} \right), \\ &= \frac{(\alpha+\beta)^5(2\alpha + 2\beta)}{\alpha^4} = \frac{2(\alpha+\beta)^6}{\alpha^4} = 2L, \end{aligned} \quad (20)$$

$$\begin{aligned} \alpha L_{\alpha\beta} + \beta L_{\beta\beta} &= \alpha \left(\frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{\alpha^5} \right) + \left(\frac{30\beta(\alpha+\beta)^4}{\alpha^4} \right), \\ &= \frac{(\alpha+\beta)^4(6\alpha + 6\beta)}{\alpha^4} = \frac{6(\alpha+\beta)^5}{\alpha^4} = L_\beta. \end{aligned} \quad (21)$$

$$\begin{aligned} \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} &= \frac{1}{\alpha^5} [(-48\alpha(\alpha+\beta)^5 + 30\alpha^2(\alpha+\beta)^4 + 20(\alpha+\beta)^6) + \\ &\beta(30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5)] = \frac{(\alpha+\beta)^4}{\alpha^5} [2\alpha^2 - 2\alpha\beta - 4\beta^2] = L_\alpha. \end{aligned} \quad (22)$$

$$\begin{aligned} \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} &= \frac{1}{\alpha^4} (-48\alpha(\alpha + \beta)^5 + 30\alpha^2(\alpha + \beta)^4 + 20(\alpha + \beta)^6) \\ &+ \frac{2\beta}{\alpha^4} (30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5) + \frac{30\beta^2(\alpha + \beta)^4}{\alpha^4} = \frac{2(\alpha + \beta)^6}{\alpha^4} = 2L. \end{aligned} \quad (23)$$

We propose to determine the metric tensors of an R-complex Finsler space with using the following equalities as:

$$g_{ij} = \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta^i \partial \bar{\eta}^j}.$$

Each of these being of interest in the following :

We consider,

$$\begin{aligned} \frac{\partial \alpha}{\partial \eta^i} &= \frac{1}{2\alpha} (a_{ij} \eta^j + a_{i\bar{j}} \bar{\eta}^j) = \frac{1}{2\alpha l_i}, \quad \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i, \\ \frac{\partial \alpha}{\partial \bar{\eta}^i} &= \frac{1}{2\alpha} (a_{i\bar{j}} \bar{\eta}^j + a_{ij} \eta^j) = \frac{\partial \beta}{\partial \bar{\eta}^i} = \frac{1}{2} b_{\bar{i}}, \end{aligned}$$

where, $l_i = (a_{ij} \eta^j + a_{i\bar{j}} \bar{\eta}^j)$, $l_{\bar{j}} = (a_{i\bar{j}} \bar{\eta}^j + a_{ij} \eta^i)$. We find immediately, $l_i \eta^i + l_{\bar{j}} \bar{\eta}^j = 2\alpha^2$.

We denote:

$$\begin{aligned} \eta^j &= \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i} F^2 = 2F \frac{\partial}{\partial \eta^i} \left(\frac{(\alpha + \beta)^3}{\alpha^2} \right), \\ \eta_i &= \rho_0 l_i + \rho_1 b_i, \end{aligned}$$

where,

$$\rho_0 = \frac{1}{2} \alpha^{-1} L_{\alpha}, \quad (24)$$

and

$$\rho_1 = \frac{1}{2} L_{\beta}. \quad (25)$$

Differentiating ρ_0 and ρ_1 with respect to η^j and $\bar{\eta}^j$ respectively, which yields:

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2} l_j + \rho_{-1} b_j,$$

and

$$\frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2} l_{\bar{j}} + \rho_{-1} b_{\bar{j}}.$$

$$\text{Similarly } \frac{\partial \rho_1}{\partial \eta^i} = \eta_{-1} l_i + \mu_0 b_i, \quad \frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-1} l_{\bar{i}} + \mu_0 b_{\bar{i}},$$

where,

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_{\alpha}}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \quad (26)$$

By direct computation, using (24), (25), and (26). We obtain the following result.

Theorem 3.1: *The invariants of R-complex Finsler space with Cubic (α, β) -metric: $\rho_0, \rho_1, \rho_{-2}, \rho_{-1}$ and μ_0 are given by:*

$$\begin{aligned}
\rho_0 &= \frac{1}{2}\alpha^{-1}L_\alpha = \frac{1}{2\alpha^6}(6\alpha(\alpha+\beta)^5 - 4((\alpha+\beta)^6)), \\
\rho_1 &= \frac{1}{2}L_\beta = \frac{3(\alpha+\beta)^5}{\alpha^4}, \\
\rho_{-2} &= \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3} = \frac{-27\alpha(\alpha+\beta)^5 + 15\alpha^2(\alpha+\beta)^4 + 12(\alpha+\beta)^6}{2\alpha^8}, \\
\rho_{-1} &= \frac{L_{\alpha\beta}}{4\alpha} = \frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{4\alpha^6}, \\
\mu_0 &= \frac{L_{\beta\beta}}{4} = \frac{30(\alpha+\beta)^4}{4\alpha^4}.
\end{aligned}$$

Subscript 0, 1, -2, -1 and 0 gives us the degree of homogeneity of these invariants.

4. Fundamental tensor of R-complex Finsler Space with Cubic (α, β) -Metric

The fundamental metric tensors of R-complex Finsler space with Cubic (α, β) -metric given by

$$g_{ij} = \rho_0 a_{ij} + \rho_{-2} l_i l_j + \mu_0 b_i b_j + \rho_{-1} (b_j l_i + b_i l_j) \quad (27)$$

From Theorem (3.1), we have

$$\begin{aligned}
g_{ij} &= \frac{1}{2\alpha^6} (6\alpha(\alpha+\beta)^5 - 4((\alpha+\beta)^6)) a_{ij} + \left(\frac{-27\alpha(\alpha+\beta)^5 + 15\alpha^2(\alpha+\beta)^4 + 12(\alpha+\beta)^6}{2\alpha^8} \right) l_i l_j \\
&\quad + \frac{30(\alpha+\beta)^4}{4\alpha^4} b_i b_j + \frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{4\alpha^6} (b_j l_i + b_i l_j). \quad (28)
\end{aligned}$$

$$\begin{aligned}
\widetilde{g}_{\bar{i}\bar{j}} &= \frac{1}{2\alpha^6} (6\alpha(\alpha+\beta)^5 - 4(\alpha+\beta)^6) a_{\bar{i}\bar{j}} + \left(\frac{-27\alpha(\alpha+\beta)^5 + 15\alpha^2(\alpha+\beta)^4 + 12(\alpha+\beta)^6}{2\alpha^8} \right) l_{\bar{i}} l_{\bar{j}} \\
&\quad + \frac{30(\alpha+\beta)^4}{4\alpha^4} b_{\bar{i}} b_{\bar{j}} + \frac{30\alpha(\alpha+\beta)^4 - 24(\alpha+\beta)^5}{4\alpha^6} (b_{\bar{j}} l_{\bar{i}} + b_{\bar{i}} l_{\bar{j}}). \quad (29)
\end{aligned}$$

Or, equivalently,

$$\widetilde{g}_{ij} = \rho_0 [a_{ij} + p l_i l_j + q b_i b_j + r \eta_i \eta_j], \quad (30)$$

$$\widetilde{g}_{\bar{i}\bar{j}} = \rho_0 [a_{\bar{i}\bar{j}} + p l_{\bar{i}} l_{\bar{j}} + q b_{\bar{i}} b_{\bar{j}} + r \eta_{\bar{i}} \eta_{\bar{j}}], \quad (31)$$

where, $\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha$.

$$p = \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \quad (32)$$

$$q = \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \quad (33)$$

$$r = \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \quad (34)$$

The next objectives is to obtain the determinant and the inverse of the tensor field $\widetilde{g}_{ij}$. The solution of the non-singular non-Hermitian metric $\widetilde{Q}_{ij}$ as follows. The following proposition is proved by [13].

Proposition 4.1: Suppose:

1. (Q_{ij}) is a non-singular $n \times n$ complex matrix with inverse Q^{ji} ;
2. C_i and $C_{\bar{i}} = \bar{C}_i$, $i = 1, 2, 3, \dots, n$ are complex numbers;

3. $C^i := Q^{ji}C_j$ and its conjugates: $C^2 := C_iC^i = \tilde{C}^iC_i$; $H_{ij} := Q_{ij} \pm C_iC_j$.

Then,

1. $\det(H_{ij}) = (1 \pm C^2)\det(Q_{ij})$,
2. Whenever $(1 \pm C^2) \neq 0$, the metrix (H_{ij}) is invertible and in this case its inverse is $H^{ji} = Q^{ji} \pm \frac{1}{1 \pm C^2} C^iC^j$.

Using the above proposition we prove the following theorem:

Theorem 4.2: *For a non-Hermitian R-complex Finsler space with Cubic (α, β) -metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$, following identities holds good:*

(1) The contravariant tensor $\tilde{g}^{ij}$ of the fundamental tensor $\tilde{g}_{ij}$ is:

$$\tilde{g}^{ji} = \frac{2\alpha^6}{(6\alpha(\alpha+\beta)^5 - 4(\alpha+\beta)^6)} a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j + \frac{A_4}{A_5},$$

where,

$$A_1 = \frac{h_1}{h_2} + \frac{h_3}{h_4},$$

$$h_1 = -3\alpha\beta + 12\beta^2,$$

$$h_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$h_3 = 15\alpha^2\epsilon^2(-3\alpha\beta + 12\beta^2)^2,$$

$$h_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_2 = \frac{h_5}{h_6},$$

$$h_5 = 15\alpha^2(-3\alpha\beta + 12\beta^2)\epsilon,$$

$$h_6 = (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_3 = \frac{h_7}{h_8},$$

$$h_7 = 15\alpha^2,$$

$$h_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}).$$

$$A_4 = (1 + \gamma k_1 + \epsilon k_2)^2 \eta^i \eta^j + (1 + \gamma k_1 + \epsilon k_2) \times \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right) (\eta^i b^j + \eta^j b^i) + \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right)^2 b^i b^j,$$

$$A_5 = 1 + (\gamma(1 + \gamma k_1 + \epsilon k_2) + \epsilon \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right)) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}},$$

$$k_1 = \frac{r_1}{r_2} + \frac{r_3}{r_4},$$

$$r_1 = -3\alpha\beta + 12\beta^2,$$

$$\begin{aligned}
r_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\
r_3 &= 15\alpha^2 \epsilon^2 (-3\alpha\beta + 12\beta^2)^2, \\
r_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times \\
&\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - \\
&\quad 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2 (-3\alpha\beta + 12\beta^2)\}], \\
k_2 &= \frac{r_5}{r_6}, \\
r_5 &= (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2, \\
r_6 &= [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\
&\quad 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\
&\quad \epsilon^2 (-3\alpha\beta + 12\beta^2)\}]] \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2, \\
r_7 &= 15\epsilon\alpha^2, \\
r_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}), \\
r_9 &= \epsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2, \\
r_{10} &= \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma})\} \times \\
&\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma). \\
(2) \det \left(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j + \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \eta_i \eta_j \right) = \\
&\quad [1 + (\gamma(1 + \gamma k_1 + \epsilon k_2)) + \epsilon \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}] \times [1 + \omega + \\
&\quad \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}] (1 + 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + \gamma(-3\alpha\beta + 12\beta^2)) \det(a_{ij}).
\end{aligned}$$

Proof: Step 1: We claim of this theorem proved by following three steps:

We write $\widetilde{g}_{ij}$ from (30) in the form.

$$\widetilde{g}_{ij} = \rho_0[a_{ij} + pl_i l_j + qb_i b_j + r\eta_i \eta_j], \quad (35)$$

we take $\widetilde{Q}_{ij} = a_{ij}$ and $\widetilde{C}_i = \sqrt{\frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}} l_i$. By applying the Proposition

(4.1) we obtain $\widetilde{Q}^{ij} = a^{ij}$,

$$\begin{aligned}
\widetilde{C}^2 &= \widetilde{C}^i \widetilde{C}_i = \sqrt{\frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}} l_i \widetilde{Q}^{ij} \widetilde{C}_j = \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j a^{ji} = \\
&\quad \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma, \text{ and } 1 + \widetilde{C}^2 = \left(1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma\right).
\end{aligned}$$

So, the matrix $\widetilde{H}_{ij} = a_{ij} - \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j$, is invertible with

$$\widetilde{H}^{ij} = a^{ji} + \frac{(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \eta^i \eta^j$$

$$\det(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j) = (1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma) \det(a_{ij}).$$

Step 2: Now, we consider

$$\widetilde{Q}^{ij} = a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j, \text{ and } \widetilde{C}_i = \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b_i,$$

By applying the Proposition (4.1) we have,

$$\begin{aligned} \widetilde{Q}^{ij} &= a^{ji} + \frac{(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \eta^i \eta^j, \\ \widetilde{C}^2 &= \widetilde{C}^i \widetilde{C}_i = \widetilde{Q}^{ji} \times \widetilde{C}_j \times \widetilde{C}_i = \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b_i \\ &\left[a^{ij} + \frac{(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \eta^i \eta^j \times \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b^j \right]. \\ \widetilde{C}^2 &= \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \left[\omega + \frac{(-3\alpha\beta + 12\beta^2)\epsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \right] \end{aligned}$$

Therefore,

$$1 + \widetilde{C}^2 \neq 0.$$

Where, $\epsilon = b_j \eta^j, \omega = b_j b^j$. It results that the inverse $\widetilde{H}_{ij} = a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j$ exists and it is

$$\widetilde{H}^{ji} = Q^{ji} + \frac{1}{1 + \widetilde{C}^2} C^i C^j,$$

$$\widetilde{H}^{ji} = a^{ji} + \frac{p\eta^i \eta^j}{1 + p\gamma} + \frac{q \left[b^i + \frac{p\epsilon \eta^i}{1 + p\gamma} \right] \left[b^j + \frac{p\epsilon \eta^j}{1 + p\gamma} \right]}{\tau} \quad (36)$$

$$\widetilde{H}^{ji} = a^{ji} + \left(\frac{p}{1 + p\gamma} + \frac{qp^2\epsilon^2}{\tau(1 + p\gamma)^2} \right) \eta^i \eta^j + \frac{pq\epsilon}{\tau(1 + p\gamma)} (b^i \eta^j + b^j \eta^i) + \frac{q}{\tau} b^i b^j. \quad (37)$$

Where,

$$\tau = 1 + q \left[\omega + \frac{p\epsilon^2}{1 + p\gamma} \right].$$

Now, putting value of p, q, τ and we get following equation

$$\widetilde{H}^{ji} = a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j, \quad (38)$$

where,

$$\begin{aligned} A_1 &= \frac{h_1}{h_2} + \frac{h_3}{h_4}, \\ h_1 &= -3\alpha\beta + 12\beta^2, \\ h_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\ h_3 &= 15\alpha^2\epsilon^2(-3\alpha\beta + 12\beta^2)^2, \\ h_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times \\ &\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - \\ &\quad 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2)\}], \\ A_2 &= \frac{h_5}{h_6}, \\ h_5 &= 15\alpha^2(-3\alpha\beta + 12\beta^2)\epsilon, \end{aligned}$$

$$h_6 = (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_3 = \frac{h_7}{h_8},$$

$$h_7 = 15\alpha^2,$$

$$h_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right). \quad (39)$$

$$\det \left[a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j \right] = \left[1 + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \right) \right] \left(1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma \right) \det(a_{ij}).$$

Step 3: We put

$$\tilde{Q}^{ij} = a^{ji} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j. \quad (40)$$

And $\tilde{C}_i = \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} \eta_i$, clearly observe that and obtain

$$\tilde{Q}^{ij} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\epsilon^2}{\tau(1+p\gamma)^2} \right) \eta^i \eta^j + \frac{pq\epsilon}{\tau(1+p\gamma)} (b^i \eta^j + b^j \eta^i) + \frac{q}{\tau} b^i b^j. \quad (41)$$

And $\tilde{C}_i = M\eta^i + N b^j$, where

$$M = \left[1 + \left(\frac{p}{1+p\gamma} + \frac{qp^2\epsilon^2}{\tau(1+p\gamma)^2} \right) \right] \gamma + \frac{pq\epsilon}{\tau(1+p\gamma)^3}, \quad (42)$$

$$N = \frac{q\epsilon}{\tau} + \frac{pq\epsilon\gamma}{\tau(1+p\gamma)}. \quad (43)$$

Now putting value of p, q, τ and we get,

$$M = 1 + \gamma k_1 + \epsilon k_2.$$

Where,

$$k_1 = \frac{r_1}{r_2} + \frac{r_3}{r_4},$$

$$r_1 = -3\alpha\beta + 12\beta^2,$$

$$r_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$r_3 = 15\alpha^2 \epsilon^2 (-3\alpha\beta + 12\beta^2)^2,$$

$$r_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$k_2 = \frac{r_5}{r_6},$$

$$r_5 = (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2,$$

$$r_6 = (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}] \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2.$$

Now putting value of p, q, τ and we get,

$$N = \frac{r_7}{r_8} + \frac{r_9}{r_{10}}.$$

Where,

$$\begin{aligned} r_7 &= 15\epsilon\alpha^2, \\ r_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right), \\ r_9 &= \epsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2, \\ r_{10} &= \left\{ 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right) \right\} \times \\ &\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma). \end{aligned}$$

And

$$\widetilde{C^2} = [M\gamma + N\epsilon]\sqrt{\gamma},$$

Putting value of M and N, we get

$$\widetilde{C^2} = \left(\gamma(1 + \gamma k_1 + \epsilon k_2) + \epsilon \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right) \right) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}.$$

Where,

$$\begin{aligned} k_1 &= \frac{r_1}{r_2} + \frac{r_3}{r_4}, \\ r_1 &= -3\alpha\beta + 12\beta^2, \\ r_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\ r_3 &= 15\alpha^2 \epsilon^2 (-3\alpha\beta + 12\beta^2)^2, \\ r_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times \\ &\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - \\ &\quad 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2)\}], \\ k_2 &= \frac{r_5}{r_6}, \\ r_5 &= (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2, \\ r_6 &= [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\ &\quad 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\ &\quad \epsilon^2(-3\alpha\beta + 12\beta^2)\}]] \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2, \\ r_7 &= 15\epsilon\alpha^2, \\ r_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right), \\ r_9 &= \epsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2, \\ r_{10} &= \left\{ 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right) \right\} \times \\ &\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma). \end{aligned}$$

$$1 + \widetilde{C^2} \neq 0.$$

Clearly, the matrix $\widetilde{H}_{ij}$ is invertible.

$$\tilde{C}^i = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q\left[b^i + \frac{p\in\eta^i}{1+p\gamma}\right]\left[b^j + \frac{p\in\eta^j}{1+p\gamma}\right]}{\tau} \right\} \eta_j,$$

and

$$\tilde{C}^j = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q\left[b^i + \frac{p\in\eta^i}{1+p\gamma}\right]\left[b^j + \frac{p\in\eta^j}{1+p\gamma}\right]}{\tau} \right\} \eta_i,$$

where, $\tilde{C}^i\tilde{C}^j = M^2\eta^i\eta^j + MN(\eta^ib^j + \eta^jb^i) + N^2b^ib^j$. Again by applying Proposition (4.1) we obtain the inverse of $\tilde{H}_{ij}$ as:

$$\begin{aligned} \tilde{H}^{ji} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\epsilon^2}{\tau(1+p\gamma)^2} \right) \eta^i\eta^j + \frac{pq\in}{\tau(1+p\gamma)} (b^i\eta^j + b^j\eta^i) + \frac{q}{\tau} b^ib^j + \\ \frac{M^2\eta^i\eta^j + MN(\eta^ib^j + \eta^jb^i) + N^2b^ib^j}{1 + (M\gamma + N\epsilon)\sqrt{r}}. \end{aligned} \quad (44)$$

Putting all value in equation (44) and we get,

$$\tilde{H}^{ji} = a^{ji} + A_1\eta^i\eta^j + A_2(b^i\eta^j + b^j\eta^i) + A_3b^ib^j + \frac{A_4}{A_5}. \quad (45)$$

Where,

$$\begin{aligned} A_1 &= \frac{h_1}{h_2} + \frac{h_3}{h_4}, \\ h_1 &= -3\alpha\beta + 12\beta^2, \\ h_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\ h_3 &= 15\alpha^2\epsilon^2(-3\alpha\beta + 12\beta^2)^2, \\ h_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) [(2\alpha^2 - 2\alpha\beta - 4\beta^2)(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}], \\ A_2 &= \frac{h_5}{h_6}, \\ h_5 &= 15\alpha^2(-3\alpha\beta + 12\beta^2) \in, \\ h_6 &= (2\alpha^2 - 2\alpha\beta - 4\beta^2) [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}], \\ A_3 &= \frac{h_7}{h_8}, \\ h_7 &= 15\alpha^2, \\ h_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}), \\ A_4 &= (1 + \gamma k_1 + \in k_2)\eta^i\eta^j + (1 + \gamma k_1 + \in k_2)(\frac{r_7}{r_8} + \frac{r_9}{r_{10}})(\eta^ib^j + \eta^jb^i) + \\ &\quad \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right)^2 b^ib^j, \\ A_5 &= 1 + (1 + \gamma k_1 + \in k_2) + \in \left(\frac{r_7}{r_8} + \frac{r_9}{r_{10}} \right) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}, \end{aligned}$$

$$\begin{aligned}
k_1 &= \frac{r_1}{r_2} + \frac{r_3}{r_4}, \\
r_1 &= -3\alpha\beta + 12\beta^2, \\
r_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\
r_3 &= 15\alpha^2 \epsilon^2 (-3\alpha\beta + 12\beta^2)^2, \\
r_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) [(2\alpha^2 - 2\alpha\beta - 4\beta^2)(2\alpha^4 - \\
&\quad 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega(2\alpha^4 - 2\alpha^3\beta - \\
&\quad 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2)\}], \\
k_2 &= \frac{r_5}{r_6}, \\
r_5 &= (-45\alpha\beta + 180\beta^2)\alpha^6(2\alpha^2 - 2\alpha\beta - 4\beta^2)^2, \\
r_6 &= [(2\alpha^2 - 2\alpha\beta - 4\beta^2)[(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\
&\quad 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\
&\quad \epsilon^2(-3\alpha\beta + 12\beta^2))\}](2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2, \\
r_7 &= 15\epsilon\alpha^2, \\
r_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}), \\
r_9 &= \epsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2, \\
r_{10} &= \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma})\} \\
&\quad (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma). \\
\det\left(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}b_i b_j + \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}\eta_i \eta_j\right) = \\
&\quad [1 + (\gamma(1 + \gamma k_1 + \epsilon k_2)) + \epsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}})]\sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}]r[1 + \omega + \\
&\quad \frac{\epsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}](1 + 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + \\
&\quad 12\beta^2)\gamma)\det(a_{ij}). \tag{46}
\end{aligned}$$

But $\widetilde{g}_{ij} = \frac{1}{2\alpha^6}(6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6)\widetilde{H}_{ij}$, with $\widetilde{H}_{ij}$ from last step. Thus

$$\widetilde{g}^{ji} = \frac{2\alpha^6 \widetilde{H}^{ji}}{6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6}. \tag{47}$$

Therefore, from equation (45), (46) and we obtained claims (i) and (ii). Where, $\widetilde{g}_{ij}$ and $\widetilde{g}^{ij}$ represents the R-complex Finsler space with Cubic(α, β)-metric.

We observed the terms of γ, ϵ , and δ from above Theorem 4.1, immediately we state:

Proposition 4.3: In a non-Hermitian R-complex Finsler space with Cubic(α, β)-metric following properties holds good:

$$\gamma + \bar{\gamma} = l_i \eta^i + l_j \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}k} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \tag{48}$$

$$\epsilon + \bar{\epsilon} = b_j \eta^j + \bar{b}_j \bar{\eta}^j = 2\beta, \quad \delta = \epsilon, \quad (49)$$

where,

$$\begin{aligned} l_i &= a_{ij} \eta^j, \quad \eta_i = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) a_{ij} \eta^i + \frac{3(\alpha+\beta)^5}{\alpha^4} b_i, \\ \gamma &= a_{jk} \eta^j \eta^k = l_k \eta^k, \quad \epsilon = b_j \eta^j, \quad b^k = a^{jk} b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk} \eta^j b^k = l_k b^k, \\ &= a^{jl} l_i = \eta^j. \end{aligned}$$

Example 1: We set α as

$$\alpha^2(z, \eta) := \frac{(1+\epsilon|z|^2) \sum_{k=1}^n \text{Re}(\eta^k)^2 - \epsilon \text{Re}\langle z, \eta \rangle^2}{(1+\epsilon|z|^2)^2}.$$

Where $|z|^2 := \sum_{k=1}^n z^k \bar{z}^k$, $\langle z, \eta \rangle := \sum_{k=1}^n z^k \bar{\eta}^k$, defined over the disk

$$\Delta_r^\eta = \{z \in C^\eta, |z| < r, r := \sqrt{\frac{1}{|\epsilon|}}\}$$

If $\epsilon < 0$, on C^η if $\epsilon = 0$ and on the complex projective space $P^\eta(C)$ is $\epsilon > 0$. Through computing, we get $a_{ij} = \frac{1}{1+\epsilon|z|^2} \left(\delta_{ij} - \epsilon \frac{z^i \bar{z}^j}{1+\epsilon|z|^2} \right)$ and $a_{i\bar{j}} = 0$ and so, $\alpha^2(z, \eta) = \frac{1}{2} (a_{ij} \eta^i \eta^j + a_{i\bar{j}} \bar{\eta}^i \bar{\eta}^j)$. Now, taking $\beta(z, \eta) := \text{Re} \frac{\langle z, \eta \rangle}{1+\epsilon|z|^2}$, where $b_i := \frac{\bar{z}^i}{1+\epsilon|z|^2}$. Here are a few examples of R-complex non-Hermitian Cubic metrics:

$$F_\epsilon := \frac{(1+\epsilon|z|^2)^2 \left(\sqrt{\frac{(1+\epsilon|z|^2) \sum_{k=1}^n \text{Re}(\eta^k)^2 - \epsilon \text{Re}\langle z, \eta \rangle^2}{1+\epsilon|z|^2}} + \text{Re} \frac{\langle z, \eta \rangle}{1+\epsilon|z|^2} \right)^3}{(1+\epsilon|z|^2) \sum_{k=1}^n \text{Re}(\eta^k)^2 - \epsilon \text{Re}\langle z, \eta \rangle^2}.$$

Example 2: We set α as

$$\alpha^2(z, \eta) := \frac{|\eta|^2 + \epsilon(|z|^2 |\eta|^2 - |\langle z, \eta \rangle|^2)}{(1+\epsilon|z|^2)^2},$$

defined over the disk

$$\Delta_r^\eta = \{z \in C^\eta, |z| < r, r := \sqrt{\frac{1}{|\epsilon|}}\}$$

If $\epsilon < 0$, on C^η if $\epsilon = 0$ and on the complex projective space $P^\eta(C)$ is $\epsilon > 0$, where $|\langle z, \eta \rangle|^2 := \langle z, \eta \rangle \overline{\langle z, \eta \rangle}$. Through computing, we get $a_{ij} = 0$ and $a_{i\bar{j}} = \frac{1}{1+\epsilon|z|^2} \left(\delta_{i\bar{j}} - \epsilon \frac{\bar{z}^i z^j}{1+\epsilon|z|^2} \right)$ and so, $\alpha^2(z, \eta) = a_{i\bar{j}}(z) \eta^i \bar{\eta}^j$. Setting $\beta(z, \eta)$ as in Example 1., here are a few examples of R-complex Hermitian Cubic metrics:

$$F_\epsilon := \frac{(1+\epsilon|z|^2)^2 \left(\sqrt{\frac{|\eta|^2 + \epsilon(|z|^2 |\eta|^2 - |\langle z, \eta \rangle|^2)}{1+\epsilon|z|^2}} + \text{Re} \frac{\langle z, \eta \rangle}{1+\epsilon|z|^2} \right)^3}{|\eta|^2 + \epsilon(|z|^2 |\eta|^2 - |\langle z, \eta \rangle|^2)}.$$

5. Conclusion

The concept of R-complex Finsler space is a generalised Finsler space which is based on the homogeneity of the fundamental function for any real scalars λ with respect to the fibre variables and recently various authors studied its importance in various cases. In this paper we determined the fundamental metric tensors $\tilde{g}_{ij}$ and $\tilde{g}_{i\bar{j}}$ of R-complex Finsler space with Cubic (α, β) metric and also find their determinants and its inverse metric tensor. Finally, we studied the property of non-Hermitian space in theorem 4.2 and proposition 4.3 for the R-complex Finsler space equipped with Cubic (α, β) -metric.

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