Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 22(1 & 2) May & Nov. 2024, pp. 65-85
DOI : 10.47974/JDSGT-2024-005

Generalized second approximation Matsumoto metric

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Abstract

In this paper, we tackle a more general coordinate-free investigation of the second approximation Matsumoto metric. Specifically, we establish various geometric objects corresponding the second approximation Matsumoto metric $\tilde{L}(x,y) = L + \mathfrak{B} + \mathfrak{B}^2/L + \mathfrak{B}^3/L^2$ in terms of the objects of L with a Finsler structure (M,L) and a one form $\mathfrak{B}$. Here we consider L is Finslerian and so we call $\tilde{L}$ the generalized second approximation Matsumoto metric. We describe the metric tensor of $\tilde{L}$ and its non-degeneracy. We limit the one form $\mathfrak{B}$ to a one form associated with a concurent π-vector field in order to determine the geodesic spray, Barthel, and Berwald connections of $\tilde{L}(x,y)$. Next, we compute the Barthel curvature of $\tilde{L}$.

Subject Classification (2020): *53C60, 53B40, 58B20.*

Keywords: *Second approximation matsumoto metric, Geodesic spray, Barthel connection, Berwald connection.*

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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1. Introduction

In 1941, G. Randers [1] introduced a special Finsler space for a one-form B on a differentiable manifold M. The change $\tilde{L} = L + B$, where L is Riemannian, with the goal of building a generalized field theory that would include both gravity and electromagnetism. In 1974, M. Masumoto [2] studied Randers space in a more general setting, by assuming that L is Finslerian. Subsequently, many authors focused more on Finsler metrics of Randers type, leading to an abundance of local studies of Finsler metrics of Randers type in the literature.

In general, there is a lot of study being done in the theory of special Finsler spaces. Additionally, it has numerous uses in many scientific fields, including as biology and physics. Special forms are satisfied by the π-tensor fields (torsions and curvatures) associated with the Cartan connection; these are the foundational spaces of Finsler geometry. Consequently, Finsler geometry has more special spaces than Riemannian geometry. Local Finsler spaces have been examined by a number of academics ([3-12]), among others, utilizing Matsumoto's second approximation Matsumoto metric. However, there aren't many in-depth analyses of these kinds of settings. The authors acknowledge the significant contributions made by A. Tamim, L. Youssef, and other individuals (refer to [13-21]).

In this work, within the pullback formalism of Finsler geoemtry, we investigate a coordinate-free study of generalized second approximation Matsumoto metric with special one π-form. First, by the concept of generalized second approximation Matsumoto metric we mean the deformation of a Finsler metric L (not necessarily Riemannian) by a one form B, that is, $\tilde{L} = L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2}$ and L is Finslerian. Now, in this work, in one hand we consider a Finsler manifold (M, L) that admits a concurrent π-vector field $\overline{p}$, and on the other hand we compute the corresponding π-form $\mathfrak{B}$. Then, we consider the generalized second approximation Matsumoto deformation

$$\tilde{L}(x, y) = L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2}. \tag{1.1}$$

Within the generalized Finsler metric (1.1), we calculate intrinsically some the geometric objects associated with $\tilde{L}$. Namely, the supporting form $\tilde{\ell}$, the angular metric tensors $\tilde{h}$, the Finsler metric $\tilde{g}$, and the Cartan torsion $\tilde{\mathbf{T}}$. Hence, we illustrate the non-degenerate property of the metric $\tilde{g}$, that is, $\tilde{g}$ is non-degenerate if and only if

$$2L^2(3\mathfrak{B} + L)p^2 + L^3 - 3\mathfrak{B}^2L - 8\mathfrak{B}^3 \neq 0.$$

On the other hand, the relationship between the Barthel connections $\tilde{\Gamma}$ and Γ and is obtained, as well as the related canonical sprays corresponding to this change. Moreover, the curvature tensor for the Barthel connection $\tilde{\Gamma}$ is investigated. Also, the corresponding canonical sprays G and $\tilde{G}$ are related by

$$\tilde{G} = G - \frac{L^2(12\mathfrak{B}^5+15\mathfrak{B}^4L+\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\mathcal{C}$$
$$+\frac{2L^4(3\mathfrak{B}+L)}{(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\gamma\overline{p};$$

$\mathcal{C}$ reffers to the Liouville field.

As an example of a Finsler metric (M, L) that admits a concurrent vector field, let $M = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^1 > 0\}$ and L be a conic Finsler metric given by

$$L = x^1 \sqrt{\frac{(y^1)^2 y^3 + (y^2)^3}{y^3}}.$$

Moreover, the components of the corresponding π-form $\mathbf{B}$ are given by $\mathbf{B}_1 = x^1$, $\mathbf{B}_2 = 0$, $\mathbf{B}_3 = 0$, and hence the associated 1- form $\mathfrak{B}$ becomes $\mathfrak{B} = x^1 y^1$. Therefore, we have

$$\tilde{L}(x, y) = L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2}$$

$$= \sqrt{\frac{(y^1)^2 y^2 + (x^1)^2 (y^3)^3}{y^2}} + x^1 y^1 + \frac{x^1 (y^1)^2}{\sqrt{\frac{(y^1)^2 y^2 + (x^1)^2 (y^3)^3}{y^2}}} + \frac{x^1 (y^1)^3}{\frac{(y^1)^2 y^2 + (x^1)^2 (y^3)^3}{y^2}},$$

which defines a special generalized second approximation Matsumoto metric over M.

2. Notations and Preliminaries

Given an n-dimensional smooth manifold M, the tangent bundle with differential $(TTM, d\pi, TM)$ is (TM, π, M). Let $\pi^{-1}(TM)$ be the pullback of the tangent bundle, and $V(TM)$ be the vertical bundle of TM defined by $\ker(d\pi)$. We consider the following short exact sequence [22]

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0;$$

TM denotes the slit tangent bundle, γ represents the natural injection and $\rho := (\pi_{TM}, d\pi)$.

The vertical endomorphism, or almost tangent structure, J of TM can be expressed as $J = \gamma \circ \rho$. The algebra of C^∞ functions on TM is denoted by $C^\infty(TM)$ and $\mathfrak{X}(\pi(M))$ is the $C^\infty(TM)$-module of smooth sections of the bundle $\pi^{-1}(TM)$. The constituents of $\mathfrak{X}(\pi(M))$ are referred to as the π-vector fields, and they will be identified by the barred letters $\overline{X}$.

The Liouville vector field (or, the fundamental π-vector field) $\mathcal{C}$ is given by $\mathcal{C} := \gamma\,\overline{\eta}$, where $\overline{\eta}(u) = (u, u)$, for all $u \in TM$ where $TM := TM \backslash \{0\}$ is the slit tangent bundle.

We have the following some properties of the Klein-Grifone approach to Finsler geometry. We refer to [22-24], for more details.

A nonlinear conection on M is a vector 1-form Γ on TM, C^∞ on TM, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

In this case, the horizontal (vertical) projector related with Γ is given by $h := \frac{1}{2}(I + \Gamma)$ ($v := \frac{1}{2}(I - \Gamma)$) respectively. The torsion (curvature) of Γ is given respectively by $t := \frac{1}{2}[J, \Gamma]$ ($\mathfrak{R} := -\frac{1}{2}[h, h]$).

Assume that D be a linear connection on $\pi^{-1}(TM)$, the corresponding connection map K is defined by $K: TTM \longrightarrow \pi^{-1}(TM): X \mapsto D_X\overline{\eta}$, and the horizontal space $H_u(TM)$ to M at u is $H_u(TM) := \{X \in T_u(TM): K(X) = 0\}$. The connection D is said to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall\, u \in TM.$$

For a regular connection D on M, the vector bundle maps $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are isomorphisms. In this case, the map $\beta := (\rho|_{H(TM)})^{-1}$ is called the horizontal map of D.

For a regular connection D on $\pi^{-1}(TM)$ with the horizontal map β and its classical torsion (curvature) tensor T (K), the (h)h-, (h)hv- and (h)v-torsion tensors of the connection D are defined by

$$Q(\overline{W},\overline{Z}) := T(\beta\overline{W},\beta\overline{Z}), \quad T(\overline{W},\overline{Z}) := T(\gamma\overline{W},\beta\overline{Z}), \quad V(\overline{W},\overline{Z}) := T(\gamma\overline{W},\gamma\overline{Z}).$$

Also, the h-, hv-, and v-curvature tensors of D are given by

$$R(\overline{U},\overline{V})\overline{w} := K(\beta\overline{U},\beta\overline{V})\overline{w}, \quad P(\overline{U},\overline{V})\overline{w} := K(\beta\overline{U},\gamma\overline{V})\overline{w},$$

$$S(\overline{U},\overline{V})\overline{w} := K(\gamma\overline{U},\gamma\overline{V})\overline{w}.$$

The (v)v-, (v)h- and (v)hv-torsion of D are:

$$\hat{S}(\overline{W},\overline{Z}) := S(\overline{W},\overline{Z})\overline{\eta}, \quad \hat{P}(\overline{W},\overline{Z}) := P(\overline{W},\overline{Z})\overline{\eta}, \quad \hat{R}(\overline{W},\overline{Z}) := R(\overline{W},\overline{Z})\overline{\eta} \quad .$$

We define the Finsler manifold as follows.

Definition 2.1: A pair (M, L) is knwon as Finsler manifold where M is an n-smooth manifold and L is a map

$$L: TM \longrightarrow \mathbb{R}, -0.1cm$$

with the conditions*:*

1) $L(u) > 0$ for all $u \in TM$ and $L(0) = 0$,
2) L is C^∞ on TM, C^0 on TM,
3) L is homogeneous (h(1)) in the argument y, of degree one*:* $\mathcal{L}_{\mathcal{C}} L = L$,
4) The exterior form $\Omega := dd_J E$ is non-degenerate, where $E := L^2/2$. The metric g obtained by L on $\pi^{-1}(TM)$ is defined as follows

$$g(\rho X, \rho Y) := \Omega(JX, Y), \ \forall X, Y \in \mathfrak{X}(TM). \tag{2.1}$$

Here, L is called a *Finsler structure* and E is the associated *energy function*. In addition, if L fufills the conditions (a)-(d) on a conic subset U of TM, then (M, L) is known a *conic Finsler manifold*.

A semi-spray, G on TM, is a vector field C^∞ on TM, C^1 on TM, provided that $JG = \mathcal{C}$. A semispray G which is h(1) $([\mathcal{C}, G] = G$) is called a spray.

Proposition 2.2 [23,24] For a Finsler manifold (M, L), we associated

1) The canonical spray G: $i_G\, dd_J E = -dE$.

2) The Barthel connection Γ: $\Gamma = [J, G]. -0.3cm$

The theorem below establishes both the existence and uniqueness of the Cartan connection ∇.

Theorem 2.3: [25] *Let* (M, L) *be a Finsler manifold with the attached Finsler metric* g. *There exists a unique regular connection* ∇ *on* $\pi^{-1}(TM)$ *such that*

1) ∇ is metrical *:* $\nabla g = 0$,
2) The torsion Q of ∇ is zero *:* $Q = 0$,
3) The torsion T of ∇ fulfills $g(T(\overline{U}, \overline{V}), \overline{W}) = g(T(\overline{U}, \overline{W}), \overline{V})$.

The connection ∇ indicates the Cartan connection related to (M, L).

The lemma below is useful for subsequent use.

Lemma 2.4: [25] *Let* (M, L) *be a Finsler manifold and* β *be the horizontal map of the Cartan connection* ∇. *Then, the metricity of the Cartan and Berwald connections is characterized by:*

1) $(D^{\circ}_{\gamma\overline{U}}\, g)(\overline{V}, \overline{W}) = 2\mathbf{T}(\overline{U}, \overline{V}, \overline{V}),\ \nabla_{\gamma\overline{U}}\, g = 0.$
2) $(D^{\circ}_{\beta\overline{U}} g)(\overline{V}, \overline{W}) = -2\widehat{\mathbf{P}}(\overline{U}, \overline{V}, \overline{V}),\ \nabla_{\beta\overline{U}}\, g = 0.$

Where $\widehat{\mathbf{P}}$ is the $(v)hv$-torsion of type (0,3) defined by $\widehat{\mathbf{P}}(\overline{U}, \overline{V}, \overline{V}) := g(\hat{P}(\overline{U}, \overline{V}), \overline{V})$ and $\hat{P}$ is the $(v)hv$-torsion tensor of Cartan connection.

For more details about to Finsler geometry of the pullback formalism, we refer for example, to [12, 26-29].

Lemma 2.5: *Let* (M, L) *be a Finsler manifold, then we have*

1) $d_J\, L(\gamma\overline{Y}) = 0, D^{\circ}_{\gamma\overline{Y}}\, L = dL(\gamma\overline{Y}) = d_J\, L(\beta\overline{Y}) = \ell(\overline{Y}).$
2) $d_h\, L(\beta\overline{Y}) = D^{\circ}_{\beta\overline{Y}}\, L = dL(\beta\overline{Y}) = 0.$
3) $(D^{\circ}_{\gamma\overline{Z}}\, \ell)(\overline{Y}) = (\nabla_{\gamma\overline{Z}}\, \ell)(\overline{Y}) = L^{-1}\hbar(\overline{Z}, \overline{Y}).$
4) $dd_J E(\gamma\overline{X}, \beta\overline{Z}) = g(\overline{X}, \overline{Z}),$

where g is the Finser metric associated to L, and ℓ is the normalized supporting element given by $\ell := L^{-1} i_{\overline{\eta}}\, g$.

3. Generalized second approximation Matsumoto metric

We proceed to an intrinsic investigation of the Matsumoto metric in the second approximation in this section. The resulting metric will be referred to as a generalized second approximation Matsumoto metric since we additionally replace a Finslerian metric for the Riemannian one in the second approximation Matsumoto metric.

Definition 3.1: [30] For the Berwald connection D° on $\pi^{-1}(TM)$ corresponding to (M, L), a π-vector field $\overline{Z} \in \mathfrak{X}(\pi(M))$ has no dependence on the argument y if and only if $D^{\circ}_{\gamma\overline{U}}\overline{Z} = 0$, $\forall \overline{U} \in \mathfrak{X}(\pi(M))$. Moreover, an independent scalar π-form ω of y equivalent $D^{\circ}_{\gamma\overline{U}}\,\omega = 0\ \forall \overline{U} \in \mathfrak{X}(\pi(M))$.

Definition 3.2: Let (M, L) be a Finsler manifold. Consider the following deformation of the Finsler structure L

$$\tilde{L}(x, y) = L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2}, \tag{3.1}$$

where $\mathfrak{B}$ is a differentiable function defined by $\mathfrak{B}(x, y) := \mathbf{B}(\overline{\eta})$ and $\mathbf{B}$ is a scalar π-form independent of y. Assume that $\tilde{L}$ defines a new Finsler structure on M, then in this case $\tilde{L}$ will be referred as a generalized second approximation Matsumoto metric.

To study the geometric objects associated with $\tilde{L}$, we require the following lemmas.

Lemma 3.3: *Under the change $L \mapsto \tilde{L}$, the vertical counterpart for Berwald connection $D^{\circ}_{\gamma\overline{X}}\overline{Y}$ is invariant. i.e. $\widetilde{D}^{\circ}_{\gamma\overline{X}}\,\overline{Y} = D^{\circ}_{\gamma\overline{X}}\,\overline{Y}$.*

Proof: Under the change $L \mapsto \tilde{L}$, the difference between the horizontal maps $\tilde{\beta}$ and β is a vertical vector field, means that $\tilde{\beta} = \beta + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$. Hence, the proof follows from [25] by the property

$$D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}],$$

together with the facts that $\rho \circ \gamma$ vanishes identically and the vertical distribution is integrable. In more details

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \tilde{\beta}\overline{Y}] = \rho[\gamma\overline{X}, \beta\overline{Y}] + \rho[\gamma\overline{X}, \gamma\overline{\mu}] = \rho[\gamma\overline{X}, \beta\overline{Y}] = D^{\circ}_{\gamma\overline{X}}\overline{Y}.$$

Hence, the result is attained.

Lemma 3.4: *Let (M, L) be a Finsler manifold equipping a scalar π-form $\boldsymbol{B}$ with no dependence on the argument y, and $\overline{p}$ its the associated π-vector field given by $i_{\overline{p}}\, g := \boldsymbol{B}$. Then, the one form $\mathfrak{B}(x, y) := \boldsymbol{B}(\overline{\eta})$ has the following properties*

a) $d_J\mathfrak{B}(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X})$.

b) $d_h\,\mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\,\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p}),\ d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.

c) $D^{\circ}_{\gamma\overline{X}}\,\overline{p} = -2T(\overline{X}, \overline{p})$.

Proof: The proof comes from the facts that $\rho \circ \gamma$ and $K \circ \beta$ vanish identically, $\rho \circ \beta = id_{\mathfrak{X}(\pi(M))}$, $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, $i_{\overline{\eta}}\widehat{\mathbf{P}} = 0$, $D^{\circ}_{\gamma\overline{X}}\mathbf{B} = 0$ and taking into account Definition 3.1 and Lemma 2.4. In more details, we have the following.

(a) For $d_J\mathfrak{B}(\gamma\overline{X})$, we have

$$d_J\mathfrak{B}(\gamma\overline{X}) = (J \circ \gamma\overline{X}) \cdot \mathfrak{B} = \gamma\,(\rho \circ \gamma)\overline{X} \cdot \mathfrak{B} = 0.$$

Moreover, $d_J\mathfrak{B}(\gamma\overline{X})$ can be obtained as follows:

$$d_J\mathfrak{B}(\beta\overline{X}) = J\,(\beta\overline{X} \cdot \mathfrak{B}) = \gamma\,(\rho \circ \beta)\overline{X} \cdot \mathfrak{B} = \gamma\overline{X} \cdot \mathfrak{B}$$

$$= (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{\eta}) + \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\overline{\eta})$$

$$= \mathbf{B}(\overline{X}).$$

(b) One can see that

$$d_h\mathfrak{B}(\beta\overline{X}) = (\beta \circ \rho \circ \beta\overline{X}) \cdot \mathfrak{B} = \beta\overline{X} \cdot \mathfrak{B} = d\mathfrak{B}(\beta\overline{X})$$

$$= \beta\overline{X} \cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\beta\overline{X}}\,g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p},D^{\circ}_{\beta\overline{X}}\overline{\eta})$$

$$= -2\widehat{\mathbf{P}}(\overline{X},\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + 0$$

$$= L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p}).$$

That is, we get that $d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.

(c) We have

$$0 = (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y})$$

$$= D^{\circ}_{\gamma\overline{X}}\mathbf{B}(\overline{Y}) - \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\,\overline{Y})$$

$$= D^{\circ}_{\gamma\overline{X}}\,g(\overline{p},\overline{Y}) - g(\overline{p},D^{\circ}_{\gamma\overline{X}}\,\overline{Y})$$

$$= (D^{\circ}_{\gamma\overline{X}}\,g)(\overline{p},\overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{Y})$$

$$= 2\mathbf{T}(\overline{p},\overline{X},\overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{Y}).$$

Thus, we get that $D^{\circ}_{\gamma\overline{X}}\,\overline{p} = -2T(\overline{X},\overline{p})$.

For the normalized supporting element ℓ and the angular metric tensor $\hbar$, we have:

Proposition 3.5: Under the generalized second approximation Matsumoto metric (3.1), we have

1. The supporting form $\tilde{\ell}$ and ℓ are related by

$$\tilde{\ell}(\overline{X}) = \frac{L^3 - \mathfrak{B}^2L - 2\mathfrak{B}^3}{L^3}\,\ell(\overline{X}) + \frac{L^2 + 2\mathfrak{B}L + 3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{X}). \tag{3.2}$$

2. The angular metric tensors $\tilde{\hbar}$ and $\hbar$ are related by

$$\begin{aligned}\tilde{\hbar}(\overline{X},\overline{Y}) &= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,\hbar(\overline{X},\overline{Y})\\ &+\tfrac{2(3\mathfrak{B}+L)}{L^2}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\tfrac{2\mathfrak{B}^2(3\mathfrak{B}+L)}{L^4}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\ell(\overline{X})\,\ell(\overline{Y})\\ &-\tfrac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned} \tag{3.3}$$

Proof: Under the generalized second approximation Matsumoto metric (3.1), taking Lemma 3.4 into account, we have the following:

1. Due to the facts that $\rho\circ\gamma=0$ and that $\rho\circ\beta=\rho\circ\tilde{\beta}=id_{\mathfrak{X}(\pi(M))}$, it follows that

$$\begin{aligned}&\tilde{\ell}(\overline{X}) = d_J\tilde{L}(\tilde{\beta}\overline{X}) = d_J\tilde{L}(\beta\overline{X})\\ &= \tfrac{\partial\tilde{L}}{\partial L}\,d_JL(\beta\overline{X})+\tfrac{\partial\tilde{L}}{\partial\mathfrak{B}}\,d_J\mathfrak{B}(\beta\overline{X})\\ &= \tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\overline{X})+\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{X}).\end{aligned}$$

2. Applying the previous item, Lemma 3.4, Lemma 2.5, together with Lemma 3.3, and Definition 5.1, one can show that

$$\begin{aligned}&\tilde{\hbar}(\overline{X},\overline{Y}) = \tilde{L}(\tilde{D}^{\circ}_{\gamma\overline{X}}\,\tilde{\ell})(\overline{Y}) = \tilde{L}(D^{\circ}_{\gamma\overline{X}}\,\tilde{\ell})(\overline{Y})\\ &= \tilde{L}\,D^{\circ}_{\gamma\overline{X}}\left\{\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\overline{Y})+\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{Y})\right\}\\ &= \tilde{L}\left\{(D^{\circ}_{\gamma\overline{X}}\,\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3})\,\ell(\overline{Y})+(D^{\circ}_{\gamma\overline{X}}\,\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2})\,\mathbf{B}(\overline{Y})\right\}\\ &\quad+\tilde{L}\left\{\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,(D^{\circ}_{\gamma\overline{X}}\,\ell)(\overline{Y})+\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,(D^{\circ}_{\gamma\overline{X}}\,\mathbf{B})(\overline{Y})\right\}\\ &= (L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2})\left\{\{\tfrac{2\mathfrak{B}^2(3\mathfrak{B}+L)}{L^4}\,\ell(\overline{X})-\tfrac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}\,\mathbf{B}(\overline{X})\right\}\ell(\overline{Y})\\ &\quad+\{-\tfrac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}\,\ell(\overline{X})+\tfrac{2(3\mathfrak{B}+L)}{L^2}\,\mathbf{B}(\overline{X})\}\,\mathbf{B}(\overline{Y})\}\\ &\quad+(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2})\left\{(\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^4})\,\hbar(\overline{X},\overline{Y})+0\right\}.\end{aligned}$$

Hence, the result follows.

4. The metric and Cartan tensors

In this section, we calculate certain geometric objects corresponding to $\tilde{L}(x,y)$. The following statement illustrates the relationship between g and $\tilde{g}$.

Proposition 4.1: The Finsler metric $\tilde{g}$ associated with the generalized second approximation Matsumoto metric (3.1) is given by the following relation:

$$\begin{aligned}\tilde{g}(\overline{X},\overline{Y}) &= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\, g(\overline{X},\overline{Y})\\ &+\tfrac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\, \mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\tfrac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\, \ell(\overline{X})\,\ell(\overline{Y})\\ &+\tfrac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\big\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\big\}.\end{aligned}$$

Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized second approximation Matsumoto metric has the form

$$\begin{aligned}2\widetilde{\mathbf{T}}(\overline{X},\overline{Y},\overline{Z}) &= 2\,\tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\, T(\overline{X},\overline{Y},\overline{Z})\\ &+\tfrac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^7}\,\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y})+\hbar(\overline{Y},\overline{Z})\,\ell(\overline{X})\}\\ &+\tfrac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^6}\big\{\mathbf{B}(\overline{X})\,\hbar(\overline{Y},\overline{Z})+\mathbf{B}(\overline{Y})\,\hbar(\overline{X},\overline{Z})\big\}\\ &+(D^{\circ}_{\gamma\overline{Z}}\tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6})\, g(\overline{X},\overline{Y})\\ &+(D^{\circ}_{\gamma\overline{Z}}\tfrac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4})\, \mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+(D^{\circ}_{\gamma\overline{Z}}\tfrac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6})\, \ell(\overline{X})\,\ell(\overline{Y})\\ &+(D^{\circ}_{\gamma\overline{Z}}\tfrac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5})\big\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\big\}.\end{aligned}$$

where $D^{\circ}_{\gamma\overline{X}}f(L,\mathfrak{B}) = d_Jf(\beta\overline{X}) = \frac{\partial f}{\partial L}\,\ell(\overline{X}) + \frac{\partial f}{\partial \mathfrak{B}}\,\mathbf{B}(\overline{X})$.

Proof: In view of the generalized second approximation Matsumoto metric (3.1), using Proposition 3.5, we have the following:

$$\tilde{\ell}(\overline{X}) = \tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\overline{X}) + \tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{X}).$$

$$\begin{aligned}\tilde{\hbar}(\overline{X},\overline{Y}) &= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,\hbar(\overline{X},\overline{Y})\\ &+\tfrac{2(3\mathfrak{B}+L)}{L^2}(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2})\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\tfrac{2\mathfrak{B}^2(3\mathfrak{B}+L)}{L^4}(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2})\,\ell(\overline{X})\,\ell(\overline{Y})\\ &-\tfrac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2})\,\big\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\big\}.\end{aligned}$$

Hence, using the definition of the angular metric tensor $\tilde{\hbar} := \tilde{g} - \tilde{\ell}\otimes\tilde{\ell}$, is obtained.

In more details,

$$\begin{aligned}
\tilde{g}(\overline{X},\overline{Y}) &= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\ \hbar(\overline{X},\overline{Y})\\
&+\tfrac{2(3\mathfrak{B}+L)}{L^2}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&+\tfrac{2\mathfrak{B}^2(3\mathfrak{B}+L)}{L^4}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\ell(\overline{X})\,\ell(\overline{Y})\\
&-\tfrac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}\left(L+\mathfrak{B}+\tfrac{\mathfrak{B}^2}{L}+\tfrac{\mathfrak{B}^3}{L^2}\right)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}\\
&+\left\{\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\ \ell(\overline{X})+\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\ \mathbf{B}(\overline{X})\right\}\\
&\cdot\left\{\tfrac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\ \ell(\overline{Y})+\tfrac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\ \mathbf{B}(\overline{Y})\right\}\\
&= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\ g(\overline{X},\overline{Y})\\
&+\tfrac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\ \mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&+\tfrac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\ \ell(\overline{X})\,\ell(\overline{Y})\\
&+\tfrac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Consequently, using the expression of the metric $\tilde{g}$, taking into account the fact that $(D^{\circ}_{\overline{Y}\overline{Z}}g)(\overline{X},\overline{Y}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$ (Lemma 2.4), it follows the expression of the Cartan torsion $\tilde{\mathbf{T}}$ of the special generalized second approximation Matsumoto metric. This finishes the proof.

Theorem 4.2: *The metric tensor $\tilde{g}$ of $\tilde{L}$ is non-degenerate if and only if*

$$2L^2(3\mathfrak{B}+L)p^2 + L^3 - 3\mathfrak{B}^2L - 8\mathfrak{B}^3 \neq 0. \tag{4.1}$$

That is, the generalized second approximation Matsumoto metric is a Finsler structure (or, conic Finsler structure) if and only if the condition (4.1) is satisfied.

Proof: Assume that $\tilde{g}$ be the Finsler metric associated with the generalized second approximation Matsumoto metric, defined by (3.1). To prove that $\tilde{g}$ is non-degenerate, let $\tilde{g}(\overline{X},\overline{Y}) = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$. By using Proposition 4.1, we obtain

$$\begin{aligned}
0 &= \tfrac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\ g(\overline{X},\overline{Y})\\
&+\tfrac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\ \mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&+\tfrac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\ \ell(\overline{X})\,\ell(\overline{Y})\\
&+\tfrac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

From which, by substituting $\overline{X} = \overline{p}$, noting that $\ell(\overline{p}) = \frac{\mathfrak{B}}{L}$ and $\mathbf{B}(\overline{p}) = g(\overline{p}, \overline{p}) =: p^2$, one can show that

$$A_1\, \ell(\overline{Y}) + B_1\, \mathbf{B}(\overline{Y}) = 0, \tag{4.2}$$

where

$$A_1 := \frac{(12\mathfrak{B}^5+15\mathfrak{B}^4L+12\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5)(\mathfrak{B}-Lp)(\mathfrak{B}+Lp)}{L^7},$$

$$B_1 := \frac{-14\mathfrak{B}^6-18\mathfrak{B}^5L+15\mathfrak{B}^4L^2(p^2-1)+4\mathfrak{B}^3L^3(5p^2-2)}{L^6} + \frac{18\mathfrak{B}^2L^4p^2+2\mathfrak{B}L^5(6p^2+1)+L^6(3p^2+1)}{L^6}.$$

Similarly, by substituting $\overline{X} = \overline{\eta}$, taking into account the facts that $\ell(\overline{\eta}) = L$ and $\mathbf{B}(\overline{\eta}) = \mathfrak{B}$, we obtain

$$A_2\, \ell(\overline{Y}) + B_2\, \mathbf{B}(\overline{Y}) = 0, \tag{4.3}$$

with

$$A_2 := \frac{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(-2\mathfrak{B}^3-\mathfrak{B}^2L+L^3)}{L^5},$$

$$B_2 := \frac{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(3\mathfrak{B}^2+2\mathfrak{B}L+L^2)}{L^4}.$$

Now, the system of the algebraic equations (4.2) and (4.3) has non-trivial solution for $\ell(\bar{Y})$ *and* $\boldsymbol{B}(\bar{Y})$ if and only if

$$\frac{(\mathfrak{B}^3+\mathfrak{B}^2L+\mathfrak{B}L^2+L^3)^3}{L^{11}}\,(2L^2(3\mathfrak{B}+L)p^2 + L^3 - 3\mathfrak{B}^2L - 8\mathfrak{B}^3) = 0.$$

Hence, as $\tilde{L} \neq 0$ over TM, then we conclude that

$$2L^2(3\mathfrak{B}+L)p^2 + L^3 - 3\mathfrak{B}^2L - 8\mathfrak{B}^3 = 0.$$

Therefore, $\ell(\bar{Y}) = 0$ *and* $\boldsymbol{B}(\bar{Y}) = 0$, if and only if the Finsler structure L and the π-form $\mathfrak{B}$ satisfy the condition

$$2L^2(3\mathfrak{B}+L)p^2 + L^3 - 3\mathfrak{B}^2L - 8\mathfrak{B}^3 \neq 0.$$

From which, together with the first equation of the proof, we conclude that $\bar{Y} = 0$.

This means that the generalized second approximation Matsumoto metric tensor $\tilde{g}$ is non-degenerate if and only if the condition (4.1) is satisfied. Hence, the proof is completed.

Form now on, we consider that the generalized second approximation Matsumoto metric $\tilde{L}$ satisfies the condition (4.1).

5. Geodesic spray and Berwald connection

We limit ourselves to a specific 1-form in order to avoid the difficulties of the formulae and to be able to find the geodesic spray of the generalized second approximation Matsumoto metric. We have the following definition, precisely.

Definition 5.1: [30] Assume that (M, L) is a Finsler manifold, then we mean by a concurrent π-vector field a π-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ fullfing the following conditions

$$\nabla_{\beta\overline{Y}}\,\overline{p} = -\overline{Y} = D^{\circ}_{\beta\overline{Y}}\,\overline{p}, \quad \nabla_{\gamma\overline{Y}}\,\overline{p} = 0 = D^{\circ}_{\gamma\overline{Y}}\,\overline{p}. \tag{5.1}$$

Moreover, if $\mathbf{B}$ is the π-form attached to $\overline{p}$ by using the duality provided by the metric g: $\mathbf{B} = i_{\overline{p}}\, g$, then $\mathbf{B}$ satisfies

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X},\overline{Y}) = (D^{\circ}_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

A particular generalized second approximation Matsumoto metric will be $\tilde{L}(x, y)$ if the π-vector field $\overline{p}(x, y)$ attached to the scalar π-form $\mathbf{B}$ indicates that $\mathbf{B}$ is a concurrent vector field over (M, L).

Nabil et al. addressed an intrinsic investigation of concurrent π-vector fields in Finsler geometry in [30]. They additionally described the concurrent π-vector fields. In other words, we have:

Lemma 5.2: *Consider the Finsler manifold (M, L) equipped with a scalar π-form $\boldsymbol{B}$ which has no dependence on the direction y, and $\overline{p}$ its the associated π-vector field is concurrent π-vector field. Then $d\mathfrak{B}(G) = -L^2$.*

Proof: Applying Lemma 3.4 (b) and taking Definition 5.1 into account the proof is attained.

Theorem 5.3: [30] *A concurrent π-vector field $\overline{p}$ and its attached π-form B have no depndence on the directional variable y.*

Now, we find the relationship between the canonical (geodesic) spray $\tilde{G}$ corresponding to the special generalized second approximation Matsumoto metric $\tilde{L}$, in terms of the geodesic spray G of L. Precisely, we have the following theorem.

Theorem 5.4: *The canonical spray $\tilde{G}$ associated with the special generalized second approximation Matsumoto metric (3.1), is given by*

$$\tilde{G} = G - \frac{L^2(12\mathfrak{B}^5+15\mathfrak{B}^4L+\mathfrak{B}^3L^2+6\mathfrak{B}^2\,L^3-L^5)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\mathcal{C} + \frac{2L^4(3\mathfrak{B}+L)}{(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\gamma\overline{p},$$

where, $\mathcal{C}$ is the Liouville vector field defined by $\mathcal{C} := \gamma\,\overline{\eta}$ and $p^2 := \boldsymbol{B}(\overline{p}) = g(\overline{p}, \overline{p})$.

Proof: Due to the special generalized second approximation Matsumoto metric *(3.1)*, taking into account the expression of the exterior π-form $\widetilde{\Omega} := \frac{1}{2}\,dd_J\,\tilde{L}^2$, the fact that the resulting vector from subtracting two sprays is vertical vector field (i.e. $\tilde{G} = G + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$) and using Proposition 2.2, one can show that

$$\begin{aligned} -d\tilde{E}(X) &= i_{\tilde{G}}\,\tilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\,dd_J\tilde{L}^2)(X) \\ &= \tfrac{1}{2}i_G\,dd_J\tilde{L}^2(X) + \tfrac{1}{2}i_{\gamma\overline{\mu}}\,dd_J\tilde{L}^2(X). \end{aligned} \tag{5.2}$$

Therefore, after some computation together the fact thats $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma KX$, together with Lemma 3.4, we have

$$d\tilde{E}(X) = \tfrac{1}{2}d\tilde{L}^2(X) = \tilde{L}\,d\tilde{L}(X)$$

$$= (L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2})\{\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)\}$$

$$= (L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2})\,\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X)$$

$$+(L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2})\,\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X).$$

$$\tfrac{1}{2}\,i_G\,dd_J\tilde{L}^2(X) = \tfrac{1}{2}\{dd_J\tilde{L}^2(\beta\overline{\eta}, X)\}$$

$$= \tfrac{1}{2}\{G\cdot d_J\tilde{L}^2(X) - X\cdot d_J\tilde{L}^2(G) - d_J\tilde{L}^2[G,X]\}$$

$$= \tfrac{1}{2}\{G\cdot(2\tilde{L}\tilde{\ell}(\rho X)) - X\cdot(2\tilde{L}\tilde{\ell}(\overline{\eta})) - 2\tilde{L}\tilde{\ell}(\rho[G,X])\}$$

$$= ((G\cdot\tilde{L})\,\tilde{\ell}(\rho X) + \tilde{L}\,G\cdot\tilde{\ell}(\rho X)) - (X\cdot\tilde{L}^2) - \tilde{L}\,\tilde{\ell}(\rho[G,X]).$$

From which taking into account Lemmas 3.4, and the following facts

$$G\cdot\tilde{L} = d\tilde{L}(G) = \frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(G) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(G)$$

$$= \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(G) = -(L^2 + 2\mathfrak{B}L + 3\mathfrak{B}^2)$$

$$X\cdot\tilde{L} = d\tilde{L}(X) = \frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)$$

$$\tilde{\ell}(\overline{X}) = \frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\overline{X}) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{X}),$$

$$\rho[G,X] = \rho[G, hX + vX] = D^\circ_G\rho X - KX,$$

$$(D^\circ_G\,\mathbf{B})(\overline{X}) = -g(\overline{X},\overline{\eta}) = -L\,\ell(\overline{X}),$$

$$(D^\circ_G\,\ell)(\overline{X}) = (\nabla_G\,\ell)(\overline{X}) = 0,$$

$$d\mathfrak{B}(X) = \mathbf{B}(KX) - L\ell(\rho X),$$

$$dL(X) = dL(\gamma KX) = \ell(KX),$$

the above relation reduces to

$$\tfrac{1}{2}\,i_G\,dd_J\tilde{L}^2(X) =$$

$$= -(L^2 + 2\mathfrak{B}L + 3\mathfrak{B}^2)\,\Big(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\rho X) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\rho X)\Big)$$

$$+(L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2})\,G\cdot\Big(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\rho X) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\rho X)\Big)$$

$$-2(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\left(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X)+\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)\right)$$

$$-(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\left(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\rho[G,X])+\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\rho[G,X])\right).$$

Hence, we obtain

$$\frac{1}{2}\,i_G\,dd_J\tilde{L}^2(X)=$$

$$=\left(\frac{12\mathfrak{B}^5+15\mathfrak{B}^4L+12\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5}{L^3}\right)\ell(\rho X)$$

$$-\left(\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^2}\right)\mathbf{B}(\rho X)$$

$$-(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\left(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X)+\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)\right).$$

On the other hand, using Proposition 4.1, we have

$$\frac{1}{2}\,i_{\gamma\overline{\mu}}\,dd_J\tilde{L}^2(X)=\tilde{g}(\overline{\mu},\rho X)$$

$$=\frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,g(\overline{\mu},\rho X)$$

$$+\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\,\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)$$

$$+\frac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\,\ell(\overline{\mu})\,\ell(\rho X)$$

$$+\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\}.$$

Plugging the last two relations into Equation (5.2), after some calculation, it follows that

$$-(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\left(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X)+\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)\right)$$

$$=\left(\frac{12\mathfrak{B}^5+15\mathfrak{B}^4L+12\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5}{L^3}\right)\ell(\rho X)$$

$$-\left(\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^2}\right)\mathbf{B}(\rho X)$$

$$-(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\left(\frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,dL(X)+\frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,d\mathfrak{B}(X)\right)$$

$$+\frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,g(\overline{\mu},\rho X)$$

$$+\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\,\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)$$

$$+\frac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\,\ell(\overline{\mu})\,\ell(\rho X)$$

$$+\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\}.$$

In view of the non-degenerate property of the Finsler metric g, one can show that

$$\begin{aligned}
&\frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,\overline{\mu} = \\
&= \{\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^4} \\
&-\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^6}\,\mathbf{B}(\overline{\mu}) \\
&-\frac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^7}\,\ell(\overline{\mu})\}\overline{\eta} \\
&+\{\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^2} \\
&-\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\,\mathbf{B}(\overline{\mu}) \\
&-\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\,\ell(\overline{\mu})\}\overline{p}.
\end{aligned} \tag{5.3}$$

where $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ are geometric quantities determined by the following two equations

$$\begin{aligned}
A_1\,\ell(\overline{\mu}) + B_1\,\mathbf{B}(\overline{\mu}) &= C_1. \\
A_2\,\ell(\overline{\mu}) + B_2\,\mathbf{B}(\overline{\mu}) &= C_2,
\end{aligned} \tag{5.4}$$

where

$$A_1 := \frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(-2\mathfrak{B}^3-\mathfrak{B}^2L+L^3)}{L^6}.$$

$$B_1 := \frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(3\mathfrak{B}^2+2\mathfrak{B}L+L^2)}{L^5}.$$

$$C_1 := \frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(3\mathfrak{B}^2+2\mathfrak{B}L+L^2)}{L^3}.$$

$$A_2 := \frac{(12\mathfrak{B}^5+15\mathfrak{B}^4L+12\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5)(\mathfrak{B}^2-L^2p^2)}{L^7}.$$

$$\begin{aligned}
B_2 :=\ & \frac{-14\mathfrak{B}^6-18\mathfrak{B}^5L+18\mathfrak{B}^2L^4p^2+15\mathfrak{B}^4L^2(p^2-1)}{L^6} \\
&+\frac{L^6(1+3p^2)+4\mathfrak{B}^3L^3(-2+5p^2)+2\mathfrak{B}L^5(1+6p^2)}{L^6}
\end{aligned}$$

$$\begin{aligned}
C_2 :=\ & \frac{-12\mathfrak{B}^6-15\mathfrak{B}^5L+18\mathfrak{B}^2L^4p^2+3L^6p^2+3\mathfrak{B}^4L^2(-4+5p^2)}{L^4} \\
&+\frac{2\mathfrak{B}^3L^3(-3+10p^2)+\mathfrak{B}L^5(1+12p^2)}{L^4}.
\end{aligned}$$

$$p^2 := \mathbf{B}(\overline{p}).$$

Making use of the condition (4.1), the system (5.4) has the following solution

$$\ell(\overline{\mu}) = \frac{L^3(3\mathfrak{B}^2+2\mathfrak{B}L+L^2)(2\mathfrak{B}^3+\mathfrak{B}^2L-L^3)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(8\mathfrak{B}^3+3\mathfrak{B}^2L-2L^2p^2(3\mathfrak{B}+L)-L^3)},$$

$$\mathbf{B}(\overline{\mu}) = L^2\left(\frac{(2\mathfrak{B}^3+\mathfrak{B}^2L-L^3)^2}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(8\mathfrak{B}^3+3\mathfrak{B}^2L-2L^2p^2(3\mathfrak{B}+L)-L^3)}+1\right).$$

Consequently, using (5.3) taking into account the assumption $\tilde{G} = G + \gamma\overline{\mu}$, it follows that the canonical sprays G and $\tilde{G}$, are related by

$$\tilde{G} = G - \frac{L^2(12\mathfrak{B}^5+15\mathfrak{B}^4L+\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\mathcal{C}$$

$$+\frac{2L^4(3\mathfrak{B}+L)}{(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\gamma\overline{p}.$$

Hence, the proof is completed.

Theorem 5.5: *The Barthel connection $\tilde{\Gamma}$ associated with the special generalized second approximation Matsumoto metric (3.1), is given by*

$$\tilde{\Gamma} = \Gamma - \lambda_1 J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p},$$

where

$$\lambda_1 := \frac{L^2(12\mathfrak{B}^5+15\mathfrak{B}^4L+\mathfrak{B}^3L^2+6\mathfrak{B}^2L^3-L^5)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)},$$

$$\lambda_2 := \frac{2L^4(3\mathfrak{B}+L)}{(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}.$$

Consequently, the horizontal map $\tilde{\beta}$ associated with the special generalized second approximation Matsumoto metric has the form

$$\tilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\}.$$

Proof: By using Theorem 5.4 and the formula [31]:

$$[fX, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,$$

and using the given assumption for λ_1 and λ_2, one can show that

$$\tilde{\Gamma} = [J, \tilde{G}] = [J, G - \lambda_1\,\gamma\overline{\eta} + \lambda_2\gamma\overline{p}] = [J, G] + [\lambda_1\,\gamma\overline{\eta} - \lambda_2\gamma\overline{p}, J]$$

$$= [J, G] + \lambda_1[\gamma\overline{\eta}, J] + d\lambda_1 \wedge i_{\gamma\overline{\eta}} J - d_J\lambda_1 \otimes \gamma\overline{\eta}$$

$$-\lambda_2[\gamma\overline{p}, J] - d\lambda_2 \wedge i_{\gamma\overline{p}} J + d_J\lambda_2 \otimes \gamma\overline{p}.$$

On the other hand, we obtain

$$d_J p^2 = 0$$

$$i_{\gamma\overline{\eta}} J = 0 = i_{\gamma\overline{p}} J, \qquad (\text{as } J \circ \gamma = 0),$$

whereas

$$[\gamma\overline{p}, J]X = [\gamma\overline{p}, JX] - J[\gamma\overline{p}, X]$$

$$= \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - \nabla_{JX}\,\overline{p}\} - \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - T(\overline{p}, \rho X)\} = 0.$$

$$[\gamma\overline{\eta}, J]X = -JX.$$

Therefore,

$$\tilde{\Gamma} = \Gamma - \lambda_1 J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p}.$$

Consequently, using the fact that $\Gamma = 2\beta \circ \rho - I$, the horizontal map $\tilde{\beta}$ associated with the special generalized Matsumoto metric has the form

$$\tilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\{\lambda_1\, \gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\, \gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\, \gamma\overline{p}\}.$$

This completes the proof.

Theorem 5.6: *The Barthel curvature tensor $\widetilde{\mathfrak{R}}$ associated with the special generalized second approximation Matsumoto metric (3.1) is determined by*

$$\widetilde{\mathfrak{R}} = \mathfrak{R} - [h, \mathbb{L}] - N_{\mathbb{L}},$$

where $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ is the Nijenhuis torsion of a vector 1-form $\mathbb{L}$ defined by

$$\mathbb{L} := -\frac{1}{2}\{\lambda_1 J + d_J\lambda_1 \otimes \gamma\overline{\eta} - d_J\lambda_2 \otimes \gamma\overline{p}\}. \tag{5.5}$$

Proof: According to Theorem 5.5, we see that the horizontal projection $\tilde{h}$ and vertical projection $\tilde{v}$ associated with the special generalized second approximation Matsumoto metric has the form

$$\tilde{h} = h + \mathbb{L}, \quad \tilde{v} = v - \mathbb{L},$$

where $\mathbb{L}$ is defined by (5.5). Hence, the Nijenhuis torsion [3] of a vector 1-form $\mathbb{L}$ is given by

$$N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}](X, Y) = [\mathbb{L}X, \mathbb{L}Y] + \mathbb{L}^2[X, Y] - \mathbb{L}[\mathbb{L}X, Y] - \mathbb{L}[X, \mathbb{L}Y].$$

Now, the proof is obtained by the fact that $\widetilde{\mathfrak{R}} = -\frac{1}{2}[\tilde{h}, \tilde{h}]$, and taking into account the proprties of the Frölicher-Nijenhuis bracket.

The Berwald vertical counterpart is given by Lemma 3.3 and the Berwald horizontal counterpart is given by the following result.

Proposition 5.7: *For the special generalized second approximation Matsumoto metric (3.1), the Berwald horizontal counterpart is given by*

$$\widetilde{D^{\circ}}_{\tilde{\beta}\overline{X}}\,\overline{Y} = D^{\circ}{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\, D^{\circ}_{\gamma\overline{X}}\,\overline{Y} + d_J\lambda_1(\beta\overline{X})\, D^{\circ}_{\gamma\overline{\eta}}\,\overline{Y}$$

$$-d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\, D^{\circ}_{\gamma\overline{p}}\,\overline{Y}\}$$

$$+\frac{1}{2}\{dd_J\lambda_1(\gamma\overline{Y}, \beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y}, \beta\overline{X}))\,\overline{p}\}.$$

Proof: The proof comes from the fact that $v := \gamma \circ K$, $h := \beta \circ \rho$, $\gamma D^{\circ}{}_{hX}\,\overline{Y} := v[hX, JY]$ and $D^{\circ}{}_{\gamma\overline{X}}\,\rho Y := \rho[\gamma\overline{X}, \beta\overline{Y}]$ ([25, Proposition 4.4]), taking into account Theorem 5.6, and the facts that the map $\gamma: \pi^{-1}(TM) \to VTM$ is an

isomorphism, the Berwlad (v)v-curvature $\tilde{S}^{\circ}=0$, $[JX,JY]=J[X,JY]+J[JX,Y]$, $vJ=J$ and $Jv=0$.

In more details.

$$\gamma\widehat{D^{\circ}}_{hX}\,\rho Y=\tilde{v}[\overline{h}X,JY]=(v-\mathbb{L})[hX+\mathbb{L}X,JY]$$

$$=v[hX,JY]+v[\mathbb{L}X,JY]-\mathbb{L}[hX,JY]-\mathbb{L}[\mathbb{L}X,JY]$$

$$=\gamma D^{\circ}{}_{hX}\overline{Y}-\frac{\gamma}{2}\{\lambda_1\,K[JX,JY]+d_J\lambda_1(X)\,K[\gamma\overline{\eta},JY]-d_J\lambda_2(X)\,K[\gamma\overline{p},JY]\}$$

$$+\frac{\gamma}{2}\{(JY\cdot\lambda_1)\,\rho X+(JY\cdot d_J\lambda_1(X))\,\overline{\eta}-(JY\cdot d_J\lambda_2(X))\,\overline{p}\}$$

$$+\frac{\gamma}{2}\{\lambda_1\,\rho([hX,JY])+d_J\lambda_1([hX,JY])\,\overline{\eta}-d_J\lambda_2([hX,JY])\,\overline{p}\}$$

$$=\gamma D^{\circ}{}_{hX}\rho Y-\frac{\gamma}{2}\{\lambda_1\,D^{\circ}_{JX}\,\rho Y+d_J\lambda_1(X)\,D^{\circ}_{\gamma\overline{\eta}}\,\rho Y$$

$$-d_J\lambda_1(X)\,\rho Y-d_J\lambda_1(Y)\,\rho X-d_J\lambda_2(X)\,D^{\circ}_{\gamma\overline{p}}\,\rho Y\}$$

$$+\frac{\gamma}{2}\{dd_J\lambda_1(JY,X)\,\overline{\eta}-dd_J\lambda_2(JY,X))\,\overline{p}\}.$$

Consequently,

$$\widehat{D^{\circ}}_{\beta\overline{X}}\,\overline{Y}=D^{\circ}{}_{\beta\overline{X}}\overline{Y}-\frac{1}{2}\{\lambda_1\,D^{\circ}_{\gamma\overline{X}}\,\overline{Y}+d_J\lambda_1(\beta\overline{X})\,D^{\circ}_{\gamma\overline{\eta}}\,\overline{Y}$$

$$-d_J\lambda_1(\beta\overline{X})\,\overline{Y}-d_J\lambda_1(\beta\overline{Y})\,\overline{X}-d_J\lambda_2(\beta\overline{X})\,D^{\circ}_{\gamma\overline{p}}\,\overline{Y}\}$$

$$+\frac{1}{2}\{dd_J\lambda_1(\gamma\overline{Y},\beta\overline{X})\,\overline{\eta}-dd_J\lambda_2(\gamma\overline{Y},\beta\overline{X}))\,\overline{p}\}.$$

This completes the proof.

We now present an example of a Finsler metric providing a concurrent π-vector field, and compute the corresponding π-form. Note that Tachibana [12] investigated the presence of a concurrent vector field on Finsler manifolds. The so-called semi-concurrent vector fields are a generalization of concurrent vector fields; for further information, see [19].

Example 1: Consider the manifold $M=\{(x^1,x^2,x^3)\in\mathbb{R}^3: x^1>0\}$ and the conic Finsler structure L is given by

$$L=\sqrt{\frac{(y^1)^2y^2+(x^1)^2(y^3)^3}{y^2}}.$$

The components of the metric tensor g_{ij} are

$$g_{11}=1,\quad g_{22}=\frac{(x^1)^2(y^3)^3}{(y^2)^3},\quad g_{23}=-\frac{3}{2}\frac{(x^1)^2(y^3)^2}{(y^2)^2},\qquad g_{33}=\frac{3(x^1)^2y^3}{y^2}.$$

The components of the Cartan tensor C_{ijk} are

$$C_{222}=-\frac{3}{2}\frac{(x^1)^2(y^3)^3}{(y^2)^4},C_{223}=\frac{3}{2}\frac{(x^1)^2(y^3)^2}{(y^2)^3},C_{233}=-\frac{3}{2}\frac{(x^1)^2y^3}{(y^2)^2},\ C_{333}=\frac{3}{2}\frac{(x^1)^2}{y^2}.$$

The components of the inverse metric tensor g^{ij} are calculated as follows:

$$g^{11} = 1, \quad g^{22} = \frac{4(y^2)^3}{(x^1)^2(y^3)^3}, \quad g^{23} = \frac{2(y^2)^2}{(x^1)^2(y^3)^2} \quad g^{33} = \frac{4}{3}\frac{y^2}{(x^1)^2 y^3}.$$

Assume the vector field with the components

$$p^1(x) = x^1, \qquad p^2(x) = 0, \qquad p^3(x) = 0.$$

By making use of the Finsler package [31], we list the required coefficients of Cartan connection:

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{13} = \Gamma^2_{11} = \Gamma^2_{13} = \Gamma^3_{11} = \Gamma^3_{21} = 0, \quad \Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^3_{13} = \frac{1}{x^1}.$$

Now, we have $p^i C_{ijk} = 0$. Moreover;

$$p^1_{|1} = \delta_1 p^1 + p^1\Gamma^1_{11} + p^2\Gamma^1_{12} + p^3\Gamma^1_{13} = 1,$$

similarly, $p^2_{|2} = 1$, $p^3_{|3} = 1$ as well as the other components of $p^i_{|j}$ vanish. Therefore, one can confirm that this metric admits a concurrent π-vector field $\overline{p} = p^i\overline{\partial_i}$, where $\overline{\partial_i}$ are the basis of fibers of $\pi^{-1}(TM)$, $p^1(x) = x^1$, $p^2(x) = 0$, $p^3(x) = 0$.

The components of the corresponding π-form $\mathbf{B}$ are given by $\mathbf{B}^1 = x^1$, $\mathbf{B}^2 = 0$, $\mathbf{B}^3 = 0$, and hence the associated one form $\mathfrak{B}$ becomes $\mathfrak{B}(x, y) = x^1 y^1$.

Consequently, we have

$$\tilde{L}(x,y) = L + \mathfrak{B} + \frac{\mathfrak{B}^2}{L} + \frac{\mathfrak{B}^3}{L^2}$$

$$= \sqrt{\frac{(y^1)^2y^2 + (x^1)^2(y^3)^3}{y^2}} + x^1y^1 + \frac{x^1(y^1)^2}{\sqrt{\frac{(y^1)^2y^2 + (x^1)^2(y^3)^3}{y^2}}} + \frac{x^1(y^1)^3}{\frac{(y^1)^2y^2 + (x^1)^2(y^3)^3}{y^2}},$$

which defines a special generalized second approximation Matsumoto metric over M.

Remark 5.8: It should be noted that Finsler geometry and physics both make extensive use of the second approximation Matsumoto metric. We will look into some geometric applications of this metric intrinsically as a continuation of the present investigation

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Received April, 2024