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A special class of solutions for three dimensional steady state magnetohydrodynamics equations with nonzero boundary velocity

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Abstract

Following the work of Farwig, Galdi, Simader and Sohr [1, 2], we prove existence of a special class of solutions for three-dimensional steady state magnetohydrodynamic flows with non-zero boundary velocity. We compare our results with previously known results on stationary magnetohydrodynamic equations. We also make some remarks on uniqueness of solutions.

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1. Introduction

We consider steady state viscous magnetohydrodynamic (MHD) flow with nonzero velocity divergence and nonzero boundary velocity, and such a flow is governed by the equations

$$\begin{cases} -\vartheta\Delta u + u.\nabla u - B.\nabla B + \nabla p = f_1 \\ -\lambda\Delta B + u.\nabla B - B.\nabla u \quad = f_2 \\ \text{with nonhomogeneous velocity data} \\ f_1 = div\, F_1, f_2 = div\, F_2 \\ div\, u = k\,, u|_{\partial\Omega} = g \\ \text{and } div\, B = 0\,, B|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where *u, B, p, g, k* denote velocity field, magnetic field, internal pressure, $g = \{g_i\} = 1,2,3$ a vector field, *k* a scalar field respectively, and f_1, f_2 are the external forces, Ω is the bounded domain in which fluid is moving and $\partial\Omega$ denotes its boundary, and ϑ and λ denote kinematic coefficient of viscosity and magnetic diffusivity respectively.

For mathematical convenience we assume $\vartheta = \lambda = 1$. We also assume that *k* and *g* satisfy compatibility condition

$$\int_\Omega k\, dx = \int_{\partial\Omega} N.g\, ds \tag{2}$$

where $N = N(x)$ is the unit normal at $x \in \partial\Omega$. Since *div. B = 0* is one of the Maxwell equations, we don't need compatibility condition for magnetic field *B*, and we assume $B|\partial\Omega = 0$. Here we assume that f_1 and f_2 are of the form $f_1 = div\, F_1$, and $f_2 = div\, F_2$ where F_1 and F_2 are 2 – tensors, so that their divergence are vectors.

Usually, the literature on MHD is studied with $f_2 = 0$ on right hand side of second sub-equation of equation (1). However, while studying MHD equations in plasma physics, if we take into consideration phenomena like Hall effect, thermoelectricity, polarization etc., then these phenomena contribute to the right-hand side and $f_2 \neq 0$, so we can assume that $f_2 \neq 0$ and has the form $f_2 = div.F_2$.

In this paper, we prove existence of a special class of solutions called "very weak solutions" of the system (1) under condition (2). For this, we follow the methods used by Farwig, Galdi and Sohr [1], Galdi, Simader and Sohr [2] and Kim [3] in context of Navier - Stokes equations.

The notion of 'very weak solution' was introduced by Giga [4] in the context of Stokes operator. This notion was later investigated by above mentioned researchers for stationary and non-stationary Navier - Stokes equations who proved existence, uniqueness and regularity of such solutions. We define this concept for our system (1) as follows:

We assume that the data $F_1, F_2, k \in L^q(\Omega)$, *a* Lebesgue space, and *g* belongs to the trace space $W^{1-1/q,\, q}(\partial(\Omega))$, $1 < q < \infty$. Under these conditions, we assume that weak solutions (or distributional solutions) (u, B, p) for system (1) exist, and such existence can be easily proved by extending the result of Galdi,

Simader and Sohr [2] to steady state MHD case, and such solutions belong to the appropriate Sobolev space $W^{1,q}(\Omega)\times W^{1,q}(\Omega)\times$ Lq(Ω).

To define "very weak solutions" of (1) we multiply first two sub-equations of (1) by a test function *w* and applying integrating by parts, we obtain

$$\begin{cases} \langle u, \Delta w\rangle_\Omega = \langle g, curl\ w \times N\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega + \langle BB, \nabla w\rangle_\Omega + \langle F_1, \nabla w\rangle_\Omega \\ \langle B, \Delta w\rangle_\Omega = -\langle uB, \nabla w\rangle_\Omega - \langle kB, w\rangle_\Omega + \langle Bu, \nabla w\rangle_\Omega + \langle F_2, \nabla w\rangle_\Omega \end{cases} \tag{3}$$

We also assume conditions

$$\text{div } u = k\ ,\ N.u|_{\partial\Omega} = N.g\ |_{\partial\Omega} \tag{4}$$

Then (u, B) ∈ L^q (Ω) × L^q(Ω) is called a "very weak solution" of (1) if the conditions (3) and (4) are satisfied. Here *uu, uB* etc. are 2 - tensors u⊗u, u⊗B *etc*., written as $u_i u_j$, $u_i B_j$ etc, *i, j* = 1,2, 3.

Here, we note that a "very weak solutions" need not have any differentiability property. In particular it may not satisfy standard energy inequality which is satisfied by a weak solution.

On the other hand, it can be proved that a very weak solution is unique, contrary to general non- uniqueness of a weak solution.

In section 2, we mention preliminary concepts that are required to establish the main result. In section 3, proof of the main theorem is established and conclusions are specified in section 4.

2. Preliminaries

We now explain certain notations used for function spaces that appear in the proof of existence of solution of (1). We assume Trace theorem and other related results proved/stated in Galdi, Simader and Sohr [2]. For our purpose, we follow the method adopted by [2, 3, 4].

Let Ω ⊂ R^3 be a bounded domain with boundary ∂Ω of class $C^{2,1}$ and let 1< q<∞, q′ = q/(q−1). We denote the usual Lebesgue and Sobolov spaces by $L^q(\Omega)$ and $W^{\alpha,q}$ (Ω), α ≥ 0, respectively, of scalar - valued function in (Ω) [5]. We use the same symbols in the case of vector - valued functions. Also for a given Banach space *X*, we denote the corresponding norm by $\|\ .\ \|_x$. The L^q - $L^{q'}$ duality in (Ω) is defined as follows [5]

$$\langle f, g\rangle_\Omega = \int_\Omega f\ .\ g dx, \qquad f.g = f_1 g_1 + f_2 g_2 + f_3 g_3 \tag{5}$$

where f = (f_1, f_2, f_3) ∈ L^q (Ω*)* with div $f \in L^q(\Omega)$, g = (g_1, g_2, g_3) ∈ $L^{q'}$ (Ω). By C_0^k (Ω), $C^k(\bar{\Omega})$ and $C^k(\partial\Omega)$, k = 1,2, ... , and $k = \infty$, we denote the space of smooth vector -valued functions. In particular, we set

$$\mathcal{K}_0^k(\Omega) = \{\emptyset \in c^k(\bar{\Omega})/\emptyset|_{\partial\Omega} = 0\}$$

and corresponding spaces of divergence free functions are denoted by

$$C_{0,\sigma}^k(\Omega) = \{\varphi \epsilon C_0^k(\Omega); div\ \varphi = 0\}, \mathcal{K}_{0,\sigma}^k(\Omega) = \{\varphi \epsilon \mathcal{K}_0^k(\Omega); div\ \varphi = 0\}$$

Where $div\ \varphi = \nabla.\varphi = D_1\varphi_1 + D_2\varphi_2 + D_3\varphi_3$, $\nabla = (D_1, D_2, D_3)$

$D_j = \frac{\partial}{\partial x_j}, j = 1,2,3, x = (x_1, x_2, x_3) \in \Omega$. We also set

$$W_0^{\alpha,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\ \|_{W_0^{\alpha,q}(\Omega)}} \ \& \ L_\sigma^q(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\ \|_{L^q(\Omega)}}$$

The space $L_\sigma^{q'}(\Omega) = \left[L_\sigma^q(\Omega)\right]'$ is the dual space of $L_\sigma^q(\Omega)$ with the pairing of (5) see [6, 7].

The space $W^{-\alpha,q}(\Omega)$ is the dual space of $W_0^{\alpha,q'}(\Omega)$,$\alpha \geq 0$, it is, the space of distributions in Ω which are continuous in the norm $\|\ .\ \|_{W\alpha,q'(\Omega)}$,$\alpha \geq 0$. Similarly, we define the space $L^q(\partial\Omega)$ and $W^{\alpha,q}(\Omega)$, $\alpha \geq 0$ [5]. The pairing between $L^q(\partial\Omega)$ and its dual $L^{q'}(\partial\Omega) = [L^q(\partial\Omega)]'$ is given by

$$\langle f, g\rangle_{\partial\Omega} = \int_{\partial\Omega} f.g\ dS$$

where $\int_{\partial\Omega}..\ dS$ denotes the surface integral on $\partial\Omega$. In this case, $W^{-\alpha,q}(\partial\Omega)$ is the dual space of $W^{\alpha,q'}(\partial\Omega)$, since the manifold $\partial\Omega$ has no boundary. The space $W^{-\alpha,q}(\partial\Omega)$ consists of boundary distributions which are continuous in the norm of $W^{\alpha,q'}(\partial\Omega)$.

If an integer $\alpha > 1/q$, then $f \in W^{\alpha,q}(\Omega)$ has a well-defined trace $f|_{\partial\Omega} \in W^{\alpha-1/q}(\partial\Omega)$, and we have the following estimate

$$\|f\|_{W^{\propto-\frac{1}{q'}q(\partial\Omega)}} \leq C\ \|f\|_{W^{\propto,q(\Omega)}} \text{ with } C = C(\Omega, \propto, q) > 0 \tag{6}$$

Since, we assume that the boundary $\partial\Omega$ is regular, the outward unit normal vector field

$$N: x \in \partial\Omega \mapsto N(x) = \big(N_1(x), N_2(x), N_3(x)\big), \quad x \in \partial\Omega \tag{7}$$

belongs to $C^{1,1}(\partial\Omega)$. Then it is well known that the normal component $N.g$ is well defined in the sense of distributions on $\partial\Omega$ and $N.g \in W^{-1/q,q}(\partial\Omega)$. Moreover, $N.g$ satisfies the estimate [7]

$$\|N.g\|_{W^{-\frac{1}{q'}q(\partial\Omega)}} \leq C\ \|g\|_{W^{-\frac{1}{q'}q(\partial\Omega)}} \text{ with } C = C(\Omega, q) > 0 \tag{8}$$

Then the normal component $N.f|_{\partial\Omega}$ is well defined as a member of $W^{-1/q,q}(\partial\Omega)$ and satisfies the following inequality [7].

$$\|N.f\|_{W^{-\frac{1}{q'}q(\partial\Omega)}} \leq (\ \|f\|_{L^q(\Omega)} + \|div\ f\|_{L^q(\Omega)} \tag{9}$$

Then the Neumann problem

$$\Delta h = divf, \quad N.(\nabla h - f)|_{\partial\Omega} = 0 \tag{10}$$

has a unique solution $\nabla h \in L^q(\Omega)$ satisfying the estimate

$$||\nabla h||_{L^q(\Omega)} \leq C\ ||f||_{L^q(\Omega)} \tag{11}$$

Let $\mathcal{P}_q: L^q(\Omega) \to L_\sigma^q(\Omega)$ defined by $\mathcal{P}_q f = f - \nabla h$ be Helmholtz projection.

It is well known that function $\mathcal{P}_q$ satisfies the properties

$$\mathcal{P}_q^2 = \mathcal{P}_q, \ \ \langle \mathcal{P}_q f, g\rangle_\Omega = \langle f, \mathcal{P}_{q'} g\rangle_\Omega, \ \ g \in L^{q'}(\Omega)$$

and moreover

$$\left\|\mathcal{P}_q f\right\|_{L^q(\Omega)} \leq C\,.\,||f||_{L^q(\Omega)}$$

We define Stokes operator A_q by $A_q u = -\mathcal{P}_q \Delta u, u \in \mathcal{D}(A_q)$, with its domain $\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W_0^{1,q} \cap W^{2,q}(\Omega)$ and range $\mathcal{R}(A_q) = L^q_\sigma(\Omega)$.

The fractional powers

A_q^α: $\mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ with $\mathcal{D}(A_q) \subset \mathcal{D}(A_q^\alpha) \subset L^q_\sigma(\Omega), 0 \leq \alpha \leq 1$ are also well defined, and they satisfy the relation [5]

$$\langle A_q^\alpha u, v\rangle_\Omega = \left\langle u,\, A_{q'}^\alpha v\right\rangle_\Omega,\; u \in \mathcal{D}(A_q^\alpha),\; v \in \mathcal{D}\left(A_{q'}^\alpha\right) \tag{12}$$

Furthermore, we have the inequality

$$||u||_{W^{2,q}(\Omega)} \leq C\left\|A_q u\right\|_{L^q(\Omega)} \leq C||u||_{W^{2,q}(\Omega)},\; u \in \mathcal{D}(A_q) \tag{13}$$

and $||u||_{W^{1,q}(\Omega)} \leq C_1\left\|A_q^{\frac{1}{2}} u\right\|_{L^q(\Omega)} \leq C_1.||u||_{W^{1,q}(\Omega)},\; u \in \mathcal{D}\left(A_q^{\frac{1}{2}}\right)$

with constants C & $C_1 = C_1(\Omega, q) > 0$. Therefore, $\mathcal{D}\left(A_q^{\frac{1}{2}}\right) = L^q_\sigma(\Omega) \cap W_0^{1,q}(\Omega)$

We also have the following embedding property:

$||u||_{L^q_\sigma(\Omega)} \leq C\,||A_r^\alpha u||_{L^r_\sigma(\Omega)}, u \in \mathcal{D}(A_r^\alpha)$ where $1 < r \leq q < \infty,\; 0 \leq \alpha \leq 1,\; 2\alpha + 3/q = 3/r.$ (14)

Since $L^{q'}_\sigma(\Omega)$ is the dual space of $L^q_\sigma(\Omega)$ with pairing $\langle .,. \rangle_\Omega$, there is well known result that every continuous linear functional

$$F\colon v \mapsto F(v)\,,\; v \in L^q_\sigma(\Omega)$$

can be written in the form

$$F(v) = \left\langle v, \hat{F}\right\rangle_\Omega, \vartheta \in L^q_\sigma(\Omega) \tag{15}$$

with some uniquely determined $\hat{F} \in L^{q'}_\sigma(\Omega)$ For more details about the properties of Stokes operator and trace theorem, we refer the reader to [6, 7, 8, 9, 10]. We now state our main theorem and give its proof.

3. Statement of Main Theorem and its Proof

Theorem 3.1: *Let $\Omega \subset R^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$.*

Let $3 \leq q < \infty, 1 < r \leq \frac{q}{2}$ with $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{r}$ and assume $F_1, F_2 \in L^r(\Omega)$, $k \in L^q(\Omega)$, $g \in W^{-1/q,q}(\partial\Omega)$ with $\int_\Omega k\,dx = \int_{\partial\Omega} N\,.\,g\,ds$

There exist a constant $K = K(\Omega, q) > 0$ such that if

$$\|(F_1, F_2)\|_{L^r(\Omega)\times L^r(\Omega)} + \|(k, 0)\|_{L^q \times L^q} + \|(g, 0)\|_{W^{-\frac{1}{q},q} \times W^{-\frac{1}{q},q}} \leq K \tag{16}$$

where $||(F_1, F_2)||$ etc. is the standard product norm. Then there is a unique very weak solution (u, B) to problem (1).

Moreover $(u, B) \in \widehat{W}^{1,q}(\Omega) \times \widehat{W}^{1,q}(\Omega)$ *and* $u|_{\partial\Omega} = g, B|_{\partial\Omega} = 0$.

Furthermore, there is a pressure field $p \in W^{-1,q}(\Omega)$ such that,

$$-\Delta u + u.\nabla u - B.\nabla B + \nabla p = div\, F_1$$
$$-\Delta B + u.\nabla B - B.\nabla u = div\, F_2$$

holds in the sense of distributions. Finally, (u, B, p) satisfy the following inequality

$$||(u,B)||_{L^q \times L^q} + ||(p,0)||_{W^{-1,q}(\Omega)} \tag{17}$$
$$\leq C||(F_1,F_2)||_{L^r(\Omega)\times L^r(\Omega)} + ||(k,0)||_{L^q\times L^q} + ||(g,0)||_{w^{-\frac{1}{q},q}(\partial\Omega)\times w^{-\frac{1}{q},q}(\partial\Omega)}$$

with $C = C(\Omega, q) > 0$.

To prove this theorem, we need the following theorem, which is a linear version of the problem (1):

$$\begin{cases} -\Delta u + \nabla p = f_1 \\ -\Delta B = f_2 \end{cases} \tag{18}$$

with other conditions on *u* and *B* as in (1).

Theorem 3.2: *Let* $\Omega \subset R^3$ *be a bounded domain with boundary* $\partial\Omega$ *of class* $C^{2,1}$. *Let* $3 \leq q < \infty, 1 < r \leq \frac{q}{2}$ *with* $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{r}$ *and assume that* $F_1, F_2 \in L^r(\Omega), k \in L^q(\Omega)$, $g \in W^{-\frac{1}{q},q}(\partial\Omega)$ *with* $\int_\Omega k\, dx = \int_{\partial\Omega} N.g\, ds$.

Then there exist a unique very weak solution $(u, B) \in L^q(\Omega) \times L^q(\Omega)$ *to problem (18). Moreover* $(u, B) \in \widehat{W}^{1,q}(\Omega) \times \widehat{W}^{1,q}(\Omega)$ *and the boundary condition is assumed. Furthermore, there exist* $p \in W^{-1,q}(\Omega)$ *such that* (u, B, p) *satisfy*

$$-\Delta u + \nabla p = div F_1, \qquad -\Delta B = div\, F_2$$

In the sense of distribution. Finally, the following estimates holds.

$$\|(u,B)\|_{L^q\times L^q} + \|P,0\|_{W^{-1,q}(\Omega)} \leq C\, \|(F_1,F_2\|_{L^r(\Omega)\times L^r(\Omega)} + \|(k,0)\|_{L^q\times L^q} + \|(g,0)\|_{w^{-\frac{1}{q},q}(\partial\Omega)\times w^{-\frac{1}{q},q}(\partial\Omega)} \tag{19}$$

Here norm on $\widehat{W}^{1,q}(\Omega)$ is defined as

$$||u||_{\widehat{W}^{1,q}(\Omega)} \equiv ||u||_{L^q(\Omega)} + ||A_r^{-1/2} P_r \Delta u||_{L^r_\sigma(\Omega)}$$

This norm is strictly weaker than $||u||_{W^{1,q}(\Omega)}$. The Banach space $\widehat{W}^{1,q}(\Omega)$ is defined by $\widehat{W}^{1,q}(\Omega) \equiv$ closure of $W^{1,q}(\Omega)$ under the norm $||.||_{\widehat{W}^{1,q}(\Omega)}$ and it contains $W^{1,q}(\Omega)$.

This theorem is the Stokes - MHD version of Theorem 1. It can be proved by extending the proof of Theorem 3 from [2], and to avoid repetition, we omit the proof here.

We now proceed to prove our theorem 3.1

Proof of Theorem 3.1: To establish the proof of existence of solutions (u, B) to system (1), we first assume that $(u, B) \in L^q \times L^q$ is a very weak solution of (1) and prove that it satisfies the integral equations (30) and (31)

below. After this, by using Banach Fixed Point Theorem, we prove that the system (30) – (31) has a solution (u, B) under the smallness assumption (16).

To prove this, we begin to prove some estimates with the help of elementary identities such as

$$u.\nabla u = div(uu) - ku$$
$$B.\nabla B = div(BB) \text{ since } div.B = 0$$
$$u.\nabla B = div(uB) - kB$$
$$B.\nabla u = div\,(Bu) \text{ since } div.B = 0$$

For first term $u.\nabla u$, we follow proof of Theorem 4 [2]. Thus, we get

$$|\langle u.\nabla u, w\rangle|_\Omega \leq |\langle div(uu) - ku, w\rangle_\Omega| \leq |\langle div\,(uu), w\rangle_\Omega + \langle ku, w\rangle_\Omega|$$
$$\leq \left|\langle uu, \nabla w\rangle_\Omega + \left\langle ku, A_r^{-\frac{1}{2}} A_r^{\frac{1}{2}}, w\right\rangle_\Omega\right|$$
$$\leq C_1\left(||uu||_{L^r(\Omega)}||\nabla w||_{L^{r'}(\Omega)} + ||ku||_{L^r(\Omega)}\,||A_r^{\frac{1}{2}}, w\,||_{L^{r'}(\Omega)}\right)$$

Since $$\left\|A_r^{\frac{1}{2}}, w\right\|_{L^{r'}} \leq C\ \|\nabla w\|_{L^{r'}(\Omega)}$$

$$|\langle u.\nabla u, w\rangle|_\Omega \leq C_2\left(||u||_{L^q(\Omega)}^2 + ||k||_{L^q(\Omega)}||u||_{L^q(\Omega)}\right)||\nabla w||_{L^{r'}(\Omega)}\ \forall\ w \in \mathcal{K}_{0,\sigma}^2(\Omega) \tag{20}$$

For the second term $B.\nabla B$ since $div.B = 0$, we get

$$|\langle B.\nabla B, w\rangle|_\Omega \leq |\langle divBB, w\rangle_\Omega| \leq |\langle BB, \nabla w\rangle_v|$$
$$|\langle B.\nabla B, w\rangle|_\Omega \leq C_1||B||_{L^q(\Omega)}^2||\nabla w||_{L^{r'}(\Omega)} \tag{21}$$

Third term (in second equation) $u.\nabla B$ is estimated as

$$|\langle u.\nabla B, w\rangle|_\Omega \leq |\langle div(uB) - kB, w\rangle_\Omega|$$
$$\leq C_1\left(\|uB\|_{L^r(\Omega)}\|\nabla w\|_{L^{r'}(\Omega)} + \|kB\|_{L^r(\Omega)}\|\nabla w\|_{L^{r'}(\Omega)}\right) \tag{22}$$

Thus,

$$|\langle u.\nabla B, w\rangle|_\Omega \leq C_2\left(\|u\|_{L^q(\Omega)}\|B\|_{L^q(\Omega)} + \|k\|_{L^q(\Omega)}\|B\|_{L^q(\Omega)}\right)\|\nabla w\|_{L^{r'}(\Omega)}$$

And for the last term $B.\nabla u$, again using $div.B = 0$, we get

$$|\langle B.\nabla u, w\rangle|_\Omega \leq |\langle div(Bu), w\rangle_\Omega| \leq |\langle Bu, \nabla w\rangle_\Omega|$$
$$|\langle B.\nabla B, w\rangle|_\Omega \leq C_1\,||Bu||_{L^r(\Omega)}||\nabla w||_{L^{r'}(\Omega)}$$
$$|\langle B.\nabla B, w\rangle|_v \leq C_2\,||B||_{L^q(\Omega)}\,||u||_{L^q(\Omega)}||\nabla w||_{L^{r'}(\Omega)} \tag{23}$$

We now write

$\langle u.\nabla u, w\rangle - \langle B.\nabla B, w\rangle = \langle div(uu) - div\ BB - ku, w\rangle = -\langle uu - BB, \nabla w\rangle - \langle ku, w\rangle$.

And similarly,

$\langle u.\nabla B - B.\nabla u, w\rangle = \langle div\,(uB) - div\,(Bu) - kB, w\rangle = -\langle uB - Bu, \nabla w\rangle - \langle kB, w\rangle$.

Using functional analytic argument given in [7] p.61-62, we have the existence of 2-tensors $W_1(u,B)$ and $W_2(u,B)$ in $L^r(\Omega)^2$ such that

$$\langle u.\nabla u - B.\nabla B, w\rangle = \langle div.\, W_1(u,B), w\rangle = -\langle W_1(u,B), \nabla w\rangle \tag{24}$$

and $$\langle u.\nabla B - B.\nabla u, w\rangle = \langle div.\, W_2(u,B), w\rangle = -\langle W_2(u,B), \nabla w\rangle \tag{25}$$

Moreover W_1 and W_2 satisfy the inequalities (Comparing (20) - (23) with (24) -(25)),

$$\begin{cases} \|W_1(u,B)\|_{L^r} \leq C_1\left(\|u\|_{L^q}^2 + \|k\|_{L^q}.\|u\|_{L^q} + \|B\|_{L^q}^2\right) \\ \text{and } \|W_2(u,B)\|_{L^r} \leq C_2(\|u\|_{L^q}.\|B\|_{L^q} + \|k\|_{L^q}.\|B\|_{L^q}) \end{cases} \tag{26}$$

Using (3), (24), (25) and previous equations, we get

$$\begin{aligned} \langle u, \Delta w\rangle_\Omega &= \langle g,\ curl\, w \times N\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_v - \langle ku, w\rangle_\Omega + \langle BB, \nabla w\rangle_\Omega + \langle F_1, \nabla w\rangle_\Omega \\ &= \langle g,\ curl\, w \times N\rangle_{\partial\Omega} + \langle F_1 - W_1(u,B), \nabla w\rangle_\Omega \end{aligned} \tag{27}$$

$$\text{Also,}\quad \langle B, \Delta w\rangle_\Omega = -\langle uB, \nabla w\rangle_\Omega - \langle kB, w\rangle_\Omega + \langle Bu, \nabla w\rangle_\Omega + \langle F_2, \nabla w\rangle_\Omega = \langle F_2 - W_2(u,B), \nabla w\rangle_\Omega \tag{28}$$

Comparison of (27) and (28) with (3) shows that (u,B) is a very weak solution of the linear system

$$\begin{cases} -\Delta u + p = div\left(F_1 - W_1(u,B)\right) \\ div.\, u = k, u|_{\partial\Omega} = g \end{cases} \tag{29}$$

and $-\Delta B = div\left(F_2 - W_2(u,B)\right)$

Thus, we can apply Theorem 3 and Lemma 3 in [2] to conclude integral representation of u and B as follows:

$$u = \nabla h - \hat{F}_1 - \hat{G} - \widehat{W}_1(u,B) \tag{30}$$

$$B = -\hat{F}_2 - \widehat{W}_2(u,B) \tag{31}$$

where $\nabla h \in L^q(\Omega)$ is the solution to the Neumann Problem

$$\Delta\, h = k, \quad N.\nabla\, h|_{\partial\,\Omega} = N.g \tag{32}$$

$$\langle \hat{F}_1, v\rangle = \left\langle F_1, \nabla A_{q'}^{-1} v\right\rangle,\ v = A_{q'} w,\ w \in \mathcal{K}^2_{0,\sigma(\Omega)},\ w = A_{q'}^{-1} v \tag{33}$$

$$\langle \hat{F}_2, v\rangle = \left\langle F_2, \nabla A_{q'}^{-1} v\right\rangle \tag{34}$$

$$\langle \hat{G}, v\rangle = \left\langle g - \nabla h|_{\partial\Omega}, N.\nabla A_{q'}^{-1} v\right\rangle \tag{35}$$

and $\widehat{W}_1(u,B)$ and $\widehat{W}_2(u,B)$ are determined by (using (24) and (25)).

$$\langle \widehat{W}_1(u,B), v\rangle = -\left\langle uu - BB, \nabla A_{q'}^{-1} v\right\rangle - \left\langle ku,\ A_{q'}^{-1} v\right\rangle \tag{36}$$

$$\langle \widehat{W}_2(u,B), v\rangle = -\left\langle uB - Bu,\ \nabla A_{q'}^{-1} v\right\rangle \tag{37}$$

Here, we note that

$$div.\,(u - \nabla h) = div.\, u - div.\nabla h = k - \Delta h = k - k = 0$$

and the space $\mathcal{K}^2_{0,\sigma}(\Omega) = \{w \in C^2(\{\overline{\Omega}\}); w|_{\partial v} = 0, div.w = 0\}$ is the ``test" space.

We shall now treat (30) - (31), a non-linear system in (u, B) and determine a solution by applying Banach Fixed Point Theorem. For this purpose, we write $\hat{u} = u - \nabla h$ and $\hat{B} = B$ so that $\hat{u}$ and $\hat{B}$ are divergence free. Thus, we write (30) - (31) in the form

$$\hat{u} = \mathcal{F}_1(\hat{u}, \hat{B}), \quad \mathcal{F}_1(\hat{u}, \hat{B}) = -\hat{F}_1 - \hat{G} - \widehat{W}_1(\hat{u} + \nabla h, \hat{B}) \tag{38}$$

and $$\hat{B} = \mathcal{F}_2(\hat{u}, \hat{B}), \quad \mathcal{F}_2(\hat{u}, \hat{B}) = -\hat{F}_2 - \widehat{W}_2(\hat{u} + \nabla h, \hat{B}) \tag{39}$$

Combining them, we write

$$(\hat{u}, \hat{B}) = \left(\mathcal{F}_1(\hat{u}, \hat{B}), \mathcal{F}_2(\hat{u}, \hat{B})\right), \mathcal{F}_1 \text{ and } \mathcal{F}_2 \text{ as above} \tag{40}$$
$$= \mathcal{F}(\hat{u}, \hat{B})$$

We wish to show that the product operator $\mathcal{F}$ satisfies condition of Fixed-Point Theorem.

We use the following estimates:

$$\left\|\hat{F}_1\right\|_{L^q_\sigma} \leq C \, \|F_1\|_{L^r} \tag{41}$$

$$\left\|\hat{F}_2\right\|_{L^q_\sigma} \leq C \, \|F_2\|_{L^r} \tag{42}$$

and $$\left\|\hat{G}\right\|_{L^q_\sigma} \leq C \left(\|g\|_{w^{-\frac{1}{q}q}(\partial\Omega)} + \|k\|_{L^q(\Omega)}\right) \tag{43}$$

We also note that

$$\langle \widehat{W}_1(u, B), \vartheta \rangle = -\langle W_1(u, B), \nabla w \rangle = -\langle W_1(u, B), \nabla A_q'^{-1} \vartheta \rangle$$

and $$\langle \widehat{W}_2(u, B), \vartheta \rangle = -\langle W_2(u, B), \nabla w \rangle = -\langle W_2(u, B), \nabla A_q'^{-1} \vartheta \rangle$$

This resembles $\left\langle F, \nabla A_q'^{-1} v \right\rangle = \langle \hat{F}, v \rangle$ (see 3.10 in [2]) and we conclude that (41) and (42) remain valid with $\hat{F}_1, \hat{F}_2, F_1, F_2$ replaced by $\widehat{W}_1, \widehat{W}_2, W_1, W_2$ respectively.

We also use the inequality

$$\|\nabla h\|_{L^q} \leq C_1 \left(\|g\|_{w^{-\frac{1}{q}q}(\partial\Omega)} + \|k\|_{L^q(\Omega)}\right) \tag{44}$$

Using above inequalities along with inequalities (26) we get

$$|| \, \widehat{W}_1(u, B) \, ||_{L^q_\sigma} \leq C_1 || \, W_1(u, B) \, ||_{L^r} \leq C_1 \left(\|u\|^2_{L^q} + \|k\|_{L^q} . \|u\|_{L^q} + \|B\|^2_{L^q}\right)$$

Using $u = \hat{u} + \nabla h$ and (44), this becomes

$$\left\|\widehat{W}_1(u, B)\right\|_{L^q_\sigma} \leq C_1 \left[\begin{array}{l} \left[\left(\|\hat{u}\|_{L^q_\sigma} + \|g\|_{w^{-\frac{1}{q}q}(\partial\Omega)} + \|k\|_{L^q(\Omega)}\right)^2 + \|B\|^2_{L^q_\sigma}\right] \\ + \|k\|_{L^q} \left(\|\hat{u}\|_{L^q_\sigma} + \|g\|_{w^{-\frac{1}{q}q}(\partial\Omega)} + \|k\|_{L^q(\Omega)}\right) \end{array}\right] \tag{45}$$

and $||\widehat{W}_2(u,B)|| \leq C_2\left(||u||_{L^q}^2 + ||B||_{L^q}^2 + ||k||_{L^q}.||B||_{L^q}\right)$

$$||\widehat{W}_2(u,B)|| \leq C_2\left[\left(\|\hat{u}\|_{L^q_\sigma} + \|g\|_{w^{-\frac{1}{q},q}(\partial\Omega)} + \|k\|_{L^q(\Omega)}\right)^2 + \|B\|_{L^q_\sigma}^2 + C_2\,\|k\|_{L^q}.\|B\|_{L^q}\right] \tag{46}$$

where we have used $||u||.||B|| \leq \frac{1}{2}\,(||u||^2 + ||B||^2)$.

We wish to combine these inequalities using product norm $||(u,B)|| = (||u||^2 + ||B||^2)^{\frac{1}{2}}$

We also use the fact that

$$||u|| + ||B|| \leq C\,||u,B|| \text{ and } ||(u,B)|| \leq (||u|| + ||B||)$$

(i.e. $||(u,B)||$ and $||u|| + ||B||$ are equivalent).

And estimate $||\mathcal{F}(\hat{u},\hat{B})||$ where $\mathcal{F}(\hat{u},\hat{B})$ is defined by (40) through (38) and (39):

First of all, we use the fact that for $a,b>0$, $a^2+b^2 \leq (a+b)^2$, to bring $||\widehat{B}||_{L^q}^2$ inside the first bracket in (45) and (46) and obtain

$$\begin{aligned}\left|\left|\widehat{W}_1(u,B)\right|\right|_{L^q_\sigma} \leq\ & C_1\left[\left(||\widehat{u}||+||\widehat{B}||+||g||+||k||\right)^2\right. \\ & \left.+||k||(||\hat{u}||+||g||+||k||)\right]\end{aligned}$$

$$\left|\left|\widehat{W}_1(u,B)\right|\right|_{L^q_\sigma} \leq C_1\begin{bmatrix}\left(||\hat{u},\hat{B}||+||g,0||+||k,0||\right)^2 \\ +||k,0||\left(||\hat{u},\hat{B}||+||g,0||+||k,0||\right)\end{bmatrix} \tag{47}$$

$(since\ ||u|| \leq ||(u,B)||\ and\ ||B|| \leq ||(u,B)||)$ using properties of product norm. Similarly,

$$||\widehat{W}_2(u,B)||_{L^q_\sigma} \leq C_2\left[\left(||\hat{u},\hat{B}||+||g,0||+||k,0||\right)^2 + ||k,0||\ ||\hat{u},\hat{B}||\right] \tag{48}$$

Thus, using (41) - (43) and (47), (48), we get finally,

$$\begin{aligned}||\mathcal{F}(\hat{u},\hat{B})|| =\ & ||\mathcal{F}_1(\hat{u},\hat{B}),\mathcal{F}_2(\hat{u},\hat{B})|| \leq ||\,\mathcal{F}_1(\hat{u},\hat{B})|| + ||\mathcal{F}_2(\hat{u},\hat{B})|| \\ & \leq ||\hat{F}_1|| + ||\hat{G}||\ + ||\widehat{W}_1(\hat{u}+\nabla h,\hat{B})|| + ||\hat{F}_2|| + ||\widehat{W}_2(\hat{u}+\nabla h,\hat{B})||\end{aligned}$$

$$||\mathcal{F}(\hat{u},\hat{B})|| \leq C\begin{bmatrix}\|F_1,F_2\| + \|g,0\| + \|k,0\| + \left(\left\|\hat{u},\hat{B}\right\| + \|g,0\| + \|k,0\|\right)^2 \\ +\|k,0\|\left(\left\|\hat{u},\hat{B}\right\| + \|g,0\| + \|k,0\|\right)\end{bmatrix} \tag{49}$$

where C is adjusted as per requirements.

We now observe that the form of inequality (49) is exactly same as inequality (4.15) in the proof of Theorem 4 [2], with $\hat{u}$ replaced by $(\hat{u},\hat{B})$, g by $(g,0)$, k by $(k,0)$ and F by (F_1,F_2). Thus, the remaining argument to prove that the mapping

$\mathcal{F}$ is a contraction mapping follows as in the proof of Theorem 4 [2]. We describe only major steps to avoid repetition, we set

$$a \equiv C, \beta \equiv ||g, 0|| \ + \ +||k, 0||, \gamma \equiv C\left(||F_1, F_{2|}|_{L^r} + \beta\right) + \beta, \delta \equiv C||k, 0||$$

Then, (49) implies

$$||\mathcal{F}(\hat{u}, \hat{B})||_{L^q_\sigma} + \beta \leq a\left(||\hat{u}, \hat{B}\,|| + \beta\right)^2 + \delta\left(||\hat{u}, \hat{B}|| + \beta\right) + \gamma \tag{50}$$

We consider the closed ball of $L^q_{\sigma(\Omega)} \times L^q_{\sigma(\Omega)}$:

$$\mathcal{B} = \left\{\left(\hat{u}, \hat{B}\right) \in \ L^q_\sigma \times L^q_\sigma / ||\hat{u}, \hat{B}||_{L^q_\sigma \times L^q_\sigma} + \beta \leq Y_1\right\} \tag{51}$$

where Y_1 is the smallest positive root of the equation

$$Y = aY^2 + \delta Y + \ \gamma \tag{52}$$

Then, proceeding as in the proof of Theorem 4 [2], and assuming

$$4a\gamma + 2\delta < \ 1,$$

We get unique fixed point $\left(\hat{u}, \hat{B}\right)$ of $\mathcal{F}\left(\hat{u}, \hat{B}\right)$ satisfying (38)-(40). Consequently, (u, B) with $u = \hat{u} + \nabla h$, $B = \hat{B}$, is a solution of (30)-(31), and hence a very weak solution to the system (1). Thus, existence has been proved. The smallness condition (53) gets satisfied if we take

$$K = \frac{1}{2C\,(2C+4)} \text{ in equation (16)} \tag{54}$$

The estimate (17) then follows as in the proof of Theorem 4 [2].

The proof of Theorem 1 is thus completed.

4. Conclusion

To the best of our knowledge regarding viscous conducting fluid with nonzero velocity at the boundary has not been treated so far in the literature. However such results are available for Navier -Stokes equations. By using these results as in the work of Farwig and Galdi, we have proved the existence of class of solutions with non - zero velocity at the boundary.

Keeping in view length of the paper, Proof of uniqueness of solution (u, B) to the system (1) is omitted here. We can obtain this proof by proceeding similarly along the lines of uniqueness proof as given in Theorem 4 [2].

In Reference [11], the authors have proved stability of solutions of steady state MHD equations in the whole space R^3, with homogeneous data i.e. $div.u = 0$ and $u = 0$ on $\partial\Omega$. The results in the present paper prove existence of very weak solutions with non-homogeneous data. In future work, we intend to prove stability of these solutions under suitable conditions.

Conflicts of interest:

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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