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THE SCALAR CURVATURE OF CR SUBMANIFOLD OF MAXIMAL CR DIMENSION WITH A CONFORMAL CHANGE

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Abstract. In this paper, we investigate the codimension reduction theorem for an n-dimensional submanifold of an $(n+p)$-dimensional manifold $(\tilde{M}, \tilde{g})$ with a conformal change $\tilde{g} = exp(-f)g$ where g denotes the Fubini-study metric on $(n+p)$-dimensional complex projective space $P^{\frac{n+p}{2}}(C)$. Moreover, we tend to calculate the scalar curvature of an n-dimensional CR submanifold of maximal CR dimension of $(\tilde{M}, \tilde{g})$ and achieve the sufficient conditions for the existence of a totally geodesic submanifold M' that includes M.

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Keywords: Conformal change; Codimension reduction; Scalar curvature.

1. Introduction

The codimension reduction is a considerable subject which has been attracted by many of authors in a wide variety of views. Allendoerfer considered the Riemannian spaces which are subspaces of Euclidean spaces of one higher dimension [1]. The isomteric immersion of a connected n-dimensional Riemannian manifold into an $(n+p)$-dimensional Riemannian manifold of constant sectional curvature was investigated by Erbacher [15]. Nomizu and Rodriguez's geometric foundations of the

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well-known Reeb theorem was developed by Cecil who represented the codimension reduction theorem for a complex submanifold of the complex projective space [7]. In one attempt, a similar theorem for the real submanifolds was proved by Okumura [23]. In another work, CR-submanifolds of a complex manifold were studied (see for example [4]). An application of the codimension reduction theorem was developed by considering of the condition $h(FX,Y)+h(X,FY)=0$ satisfied by an n-dimensional real submanifold of the CR-dimension $\frac{m-n}{2}$ of an $(n+p)$-dimensional complex space form. In [11] the condition $h(FX,Y)-h(X,FY)=g(FX,Y)\eta$, $\eta\in T^{\perp}(M)$ of an n-dimensional real submanifold of maximal CR dimension of an $(n+p)$-dimensional complex projective space was studied too. The notion F denotes the structure tensor from the complex structure J and h is the second fundamental form of submanifold. Dealing with the real hypersurface as an example of CR-submanifold of CR-dimension $(n-1)$ based on the Lawson's work has been done([8] ,[18]). Similarly, considering the complex hypersurface of real codimension two of complex projective space determines the practical usage of the codimension reduction [9].

The study of locally conformal Kahler manifolds was initiated by Libermann ([13],[19]). The relation between the P. Gauduchon's standard metrics and generalization of the Hopf (g.H) manifolds was accomplished by Vaisman [26]. F.Tricerri and I.Vaisman [26] introduced five classes of Hermitian manifolds and demonstrated that any 2-dimensional Hermitian manifold which belongs to one of these classes, is a g.H manifold. In [16], it was shown that the first Betti number of a compact g.H. manifold is odd. It was also proved that the Hermite-Einstein-Weyl manifolds are g.H. manifolds [24]. Banaru succeeded to classify the sixteen classes of the almost Hermitian manifolds by using the so-called Kirichenko tensors [2]. The sufficient condition under which a compact $l.c.K$ manifold is a complex space form was obtained by Matsumoto [21]. As a fertile field, CR-submanifolds of locally conformal Kahler manifolds have been of great interests (see for example [27]).

In this paper, we aim to develop the above mentioned study of the codimension reduction theorems by considering a conformally changed setting in the ambient space. The computation of the scalar curvature to give some sufficient conditions for a minimal CR submanifold of maximal CR dimension of the ambient space is carried out too.

2. Preliminaries

Assume that $(\tilde{M}, J, \tilde{g})$ is an $(n+p)$-dimensional Hermitian manifold where J and $\tilde{g}$ denote its complex structure and Hermitian metric, respectively. The manifold $\tilde{M}$ equipped with a metric $\tilde{g} = exp(-f)\overline{g}$ is called locally conformally Kahler manifold provided that there exists an open cover $\{U_i\}_{i\in I}$ and a family $\{f_i\}_{i\in I}$ of real-valued functions such that metric $g_i = exp(-f_i)\tilde{g}|U_i$ is Kahlerian.

For a C^∞ manifold $\tilde{M}$ furnished by two conformally related metrics $\tilde{g}$ and $\overline{g}$ we have

$$\tilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\} \tag{1}$$

where $\tilde{\nabla}$ and $\overline{\nabla}$ be two Levi-Civita connections associated with the metrics $\tilde{g}$ and $\overline{g}$, respectively [13]. The 1-form ω is called the Lee form of conformally Kahler manifold $\tilde{M}$, Namely

$$\overline{g}(X, \omega^\sharp) = \omega(X).$$

Let (M, g) be an n-dimensional submanifold of $(\tilde{M}, J, \tilde{g})$ and denote by ∇ the Levi-Civita connection with respect to the induced Riemannian metric g. Then the Gauss and the weingarten formula are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h'(X,Y)$$

$$\tilde{\nabla}_X \xi = -A'_{\xi'} X + \nabla^\perp_X \xi$$

for all X and Y tangent to M. Denoting by $\nabla^\perp$ the connection on the normal bundle $T^\perp(M)$, h' the second fundamental form of submanifold M and $A'_{\xi'}$ the shape operator with respect to the normal vector field ξ', the following so-called relation holds:

$$g(A'_{\xi'}X, Y) = \tilde{g}(h'(X,Y), \xi').$$

Let $\tilde{R}$ and $\overline{R}$ be the curvature tensors with respect to the connections $\tilde{\nabla}$ and $\overline{\nabla}$, respectively. Then the relation between them is given by [13]

$$\begin{aligned} exp(f)\tilde{R}(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) \\ &\quad -\frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\ &\quad +B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\ &\quad -\frac{\|\omega\|^2}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{2}$$

that the symmetric tensor B is defined as follows [13]

$$B = \overline{\nabla}\omega - \frac{1}{2}\omega(X)\otimes\omega(Y). \tag{3}$$

The normal space $T_x^{\perp}(M)$, for an n-dimensional submanifold M, splits into two parts $N_0(x) = \{\xi' \in T_x^{\perp}(M) | A'_{\xi'} = 0\}$ and its orthogonal complement $N_1(x)$ that is called the first normal space of M. The following direct sum decomposition holds

$$T_x^{\perp}(M) = N_0(x) + N_1(x).$$

Then we have

Theorem 2.1. *Let (M, g) be an n-dimensional submanifold of an $(n+p)$-dimensional manifold $\tilde{R}^{n+p}$ equipped with the conformal metric $\tilde{g} = exp(-f)\langle , \rangle$ that $\langle , \rangle$ denotes the Euclidean inner product. If the first normal space $N_1(x)$ is invariant under parallel translation with respect to the normal connection, q is the constant dimension of $N_1(x)$ and the Lee vector field $\omega^{\sharp}$ is tangent to M, then there exists a totally geodesic $(n+q)$-dimensional submanifold $(\tilde{R}^{n+q}, \tilde{g})$ such that $M \subset (\tilde{R}^{n+q}, \tilde{g})$.*

Proof. Let $\tilde{R}$ and $\overline{R}$ denote the curvature tensors of $\tilde{M}$ and Euclidean space E^{n+p}, respectively. From (2) we deduce

$$\begin{aligned} exp(f)\tilde{R}(X,Y,\xi'_a,\xi'_b) &= \overline{R}(X,Y,\xi'_a,\xi'_b) \\ &\quad -\frac{1}{2}\{B(X,\xi'_a)\langle Y,\xi'_b\rangle - B(Y,\xi'_a)\langle X,\xi'_b\rangle \\ &\quad +B(Y,\xi'_b)\langle X,\xi'_a\rangle - B(X,\xi'_b)\langle Y,\xi'_a\rangle\} \\ &\quad -\frac{\|\omega\|^2}{4}\{\langle Y,\xi'_a\rangle\langle X,\xi'_b\rangle - \langle X,\xi'_a\rangle\langle Y,\xi'_b\rangle\} = 0, \end{aligned} \tag{4}$$

and consequently, the Ricci equation becomes

$$\tilde{g}(\tilde{R}^{\perp}(X,Y)\xi'_a,\xi'_b) = g([A'_a, A'_b]X, Y) \tag{5}$$

that A'_a denotes the shape operator with respect to the normal vector fields ξ'_a, $a = 1, ..., p$. Using (5), the orthonormal frame $\xi'_1, ..., \xi'_p$ and a neighborhood $U(x)$ of $x \in M$ can be chosen [10]. Taking into account that $\xi' = exp(\frac{f}{2})\xi$, we obtain

$$\begin{aligned} A'_a &= \tilde{\nabla}_X\xi'_a = \nabla^E_X\xi'_a - \frac{1}{2}\{\omega(X)\xi'_a + \omega(\xi'_a)X - \langle X,\xi'_a\rangle\omega^{\sharp}\} \\ &= exp(\frac{f}{2})A_a \end{aligned} \tag{6}$$

where A'_a and A_a are the shape operators with respect to the normal vector fields ξ'_a and ξ_a, respectively. Let define $f_a = \tilde{g}(x, \xi'_a)$ where x is the position vector field, in a neighborhood U of $x \in M$. Since $A'_a = 0$ and $\omega^\sharp$ is tangent to M, we compute

$$\langle X, \nabla f_a \rangle = -\frac{1}{2}\langle X, \nabla f \rangle f_a,$$

$$\nabla(\ln f_a + \frac{1}{2}f) = 0,$$

$$f_a = exp(C_a - \frac{1}{2}f),$$

$$exp(\frac{-f}{2})\langle x, \xi_a \rangle = f_a = d_a exp(\frac{-f}{2})$$

Consequently, we obtain

$$\langle x, \xi_a \rangle = d_a \tag{7}$$

for all $a \geq q+1$ and any X tangent to $U(x)$.

Using of the fact that the geodesics of Euclidean spaces are straight lines and combining relations (7) and (6), we conclude $U(x)$ lies in the intersection of $p-q$ hyperplanes. That is, there exists an $(n+q)$-dimensional submanifold $(R^{n+q}, \tilde{g})$ such that $U(x) \subset R^{n+q}$. Further, considering that the submanifold M is connected, the global result will be arrived. Assume that $U(x)$ and $U(y)$ are neighborhoods of $x, y \in M$, respectively, such that the intersection of $U(x)$ and $U(y)$ is nonempty and $U(x) \subset R_1^{n+q}$ and $U(y) \subset R_2^{n+q}$. Then we conclude that $U(x) \cap U(y)$ is subset of $R_1^{n+q} \cap R_2^{n+q}$.

From $R_1^{n+q} \neq R_2^{n+q}$, it follows that there exists an integer r such that $R_1^{n+q} \cap R_2^{n+q} = R^{n+r}$ that implies $dim N_1(t) < q$ for each t contained in the intersection of $U(x)$ and $U(y)$. That is a contradition. Hence $R_1^{n+q} = R_2^{n+q}$ and the proof is complete. □

Theorem 2.2. *Let M be an n-dimensional subamnifold of an $(n+p)$-dimension manifold $\tilde{M}$ endowed with the conformal metric $\tilde{g} = exp(-f)g'$ where g' denotes the Riemannian metric of $(n+p)$-dimensional sphere S^{n+p}. If the first normal space $N_1(x)$ is invariant under parallel translation with respect to the connection in the normal bundle $T^\perp(M)$, q is the constant dimension of N_1 and Lee vector field $\omega^\sharp$ is tangent to M then there exists a totally geodesic $(n+q)$-dimensional subamnifold M' such that $M \subset M'$.*

Proof. Let S^{n+p} denote the unite sphere centered at the origin of E^{n+p+1}. Then mapping $g : R^{n+p+1}\backslash\{0\} \to S^{n+p}$ as a well-known continues function defined by $g(x) = \frac{x}{|x|}$ and the smooth function $f : \tilde{M} \to R$, we can consider M as a submanifold of $(R^{n+p+1}, \overline{g})$ where the composite $h = f \circ g$ is well-defined.
Denoting by ξ' the unit normal of $\tilde{M}$ and $\overline{\nabla}$ the Riemannian connection of $(R^{n+p+1}, \overline{g})$, for any X tangent to M, we conclude

$$\overline{\nabla}_X \xi' = -exp(\frac{h}{2})X. \tag{8}$$

Since the Lee vector field $\bar{\omega}^\sharp$ is tangent to M.
Using the Gauss formula and the notion (8), we obtain

$$\overline{\nabla}_X \xi'_a = \nabla_X \xi'_a + \overline{g}(\overline{\nabla}_X \xi'_a, \xi')\xi' = \nabla_X \xi'_a \tag{9}$$

where ∇ is the induced connection from $\overline{\nabla}$ and ξ'_a, $a = 1, ..., p$ were admited in the theorem 2.1. Consequently, it follows

$$\tilde{N_1}(x) = N_1(x) + span\{\xi'(x)\}$$

and therefore, $\tilde{N_1}$ is invariant under parallel translation with respect to the normal connection. Thus by theorem 2.1 there exists a totally geodesic subspace R^{n+q+1} such that $M \subset R^{n+q+1}$, namely

$$R^{n+q+1} = T_x(M) + N_1(x) + span\{\xi'(x)\},$$

which shows that R^{n+q+1} passes through the origin of $(R^{n+p+1}, \overline{g})$, which completes the proof. □

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional manifold $\tilde{M}$ with an admited conformal metric $\tilde{g}$, that is, $\tilde{g}$ satisfies $\tilde{g} = exp(-f)g$ where g denotes the Fubini-Study metric defined on $(n+p)$-dimensional complex projective space $P^{\frac{n+p}{2}}(C)$ with the complex structure J. For any $x \in T(M)$ and $\xi'_a \in T^\perp(M)$, the skew-symmetric $(1,1)$-tensor F is defined by

$$JX = FX + \Sigma_{a=1}^{p} u'^a(x)\xi'_a$$

and we have the tangential part and the normal part of $J\xi'_a$ as follows

$$J\xi'_a = -U'_a + \Sigma_{a=1}^{p} P_{ab}\xi'_a$$

where $U'_a \in T(M)$.

Let us define $H_0(x) = JN_0(x) \cap JN_0(x)$ for the subspace $N_0(x)$ of $T_x^{\perp}(M)$. Then we conclude that $H_0(x)$ is the maximal J-invariant subspace of $N_0(x)$.

Moreovere, considering $H_1(x)$ as the orthogonal complement of $H_0(x)$ in $T^{\perp}(M)$ implies that $N_1(x)$ is a subspace of H_1 and in conclusion we have

Proposition 2.3. *If M is a complex subamnifold of a conformal Kahler manifold $(\tilde{M}, \tilde{g})$ such that the relation $\tilde{g} = exp(f)g$ holds and the Lee vector field $\omega^{\sharp}$ is tangent to M, then $H_1(x) = N_1(x)$.*

Proof. Let denote by $\tilde{\nabla}$ and $\overline{\nabla}$ the Riemannian connections with respect to the metrics $\tilde{g}$ and g, respectively. Using (2) and applying the Weingarten formula, we obtain

$$\tilde{\nabla}_X J\xi_a' = -JA'_{\xi_a'}X + JD'_X\xi_a'. \tag{10}$$

On the other hand, for any X tangent to M, the Weingarten formula yeilds

$$\tilde{\nabla}_X(J\xi_a') = -A'_J\xi_a' + D'_X(J\xi_a'). \tag{11}$$

Comparing the tangential part and the normal part of the equations (10) and (11), it follows

$$A'_{J\xi_a'}X = JA'_{\xi_a'}X.$$

Namely, if $\xi_a' \in N_0(x)$ then $\xi_a' \in H_0(x)$. This completes the proof. □

Proposition 2.4. [10] *Let $H(x)$ be a J-invariant subspace of $H_0(x)$ and $H_2(x)$ be the orthogonal complement of $H(x)$ in $T_x^{\perp}(M)$. Then $T_x(M) \oplus H_2(x)$ is a J-invariant subspace of $T_x(\tilde{M})$.*

Theorem 2.5. *Let M be an n-dimensional real submanifold of a real $(n+p)$-dimensional manifold $\tilde{M}$ with the conformal metric $\tilde{g} = exp(-f)g$ that g denotes the Fubini-Study metric on $(n+p)$-dimensional complex projective space $P^{\frac{n+p}{2}}(C)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x^{\perp}(M)$ is invariant under parallel translation with respect to the normal connection, q is the constant dimension of H_2 and the Lee vector field $\omega^{\sharp}$ is tangent to M, then there exists a real $(n+q)$-dimensional totally geodesic submanifold M' of $\tilde{M}$ such that $M \subset M'$.*

Proof. If $\xi_a' \in H(x)$ then we conclude that $A'_a = 0$ and $U'_a = 0$. Making use of the Hopf fibration $\pi : S^{n+p} \to P^{\frac{n+p}{2}}(C)$ and denoting by f the smooth function

$\tilde{M} \to R$, we can define a Riemannian metric g' on $\tilde{M}'$ by

$$g'(X,Y) = \pi^*\tilde{g}(X,Y) + exp(h)g^S(X_{V'}, X_{V'})$$

where we set $h = f \circ \pi$, g^S denotes the Riemannian metric on S^{n+p} and $X_{V'}$ belongs to the vertical subspace of $T_p(\tilde{M}')$. Let ξ'_a are the orthonormal normal fields to M in $\tilde{M}$ and ξ'^*_a $a = 1, ..., p$ be the horizontal lifts of ξ'_a. From the Weingarten formula for ξ^*_a, we get

$$\nabla'_{X^*}\xi'^*_a = -A''_a X^* + D'_{X^*}\xi'^*_a \tag{12}$$

where A'' is the shape operator with respect to the normal vector field ξ'^*_a. Therefore, we get

$$exp(-\frac{h}{2})(-A''_a X^* + D^*_X\xi'_a) = exp(-\frac{f}{2})(-A'_a X + D_X\xi'_a)^* - u'^a(X)V'. \tag{13}$$

Now we caculate $\nabla'_{V'}\xi'^*_a$ as follows

$$\nabla'_{V'}\xi'^*_a = exp(\frac{f}{2})(U'^*_a + P_{ab}\xi'^*_b). \tag{14}$$

On the other hand, the Weingarten formula yeilds

$$\nabla'_{V'}\xi'^*_a = -A''_a V' + D'_{V'}\xi'^*_a. \tag{15}$$

Consequently, from (14) and (15) we arrive at

$$A''_a V' = exp(\frac{f}{2})U'^*_a. \tag{16}$$

Now assume that

$$N'_0(x') = \{\xi' \in T^{\perp}_{x'}(\pi^{-1}(M)) | A''_{\xi'_a} = 0\}. \tag{17}$$

, Using (13) and (15), implies that

$$N'_0(x') = span\{\xi'^*_a | A'_a = 0, U'_a = 0\}$$

for $x' \in \pi^{-1}(M)$ that holds $\pi(x') = x$.
Consequently,

$$H(x)^* = \{\xi'^* | \xi \in H(x)\}$$

is a subspace of $N'_0(x')$ and its orthogonal complement

$$H_2(x)^* = \{\xi^*_a | \xi'_a \in H_2(x)\}$$

is a subspace of $T^{\perp}_{x'}(\pi^{-1}(M)$ that $N'_1(x') \subset H_2(x)^*$. As $H_2(x)$ and $H(x)$ are invariant under parallel translation , making use of (13) and (14), we conclude that

$H(x)^*$ is invariant under parallel translation with respect to the normal conection D'. Then $\pi^{-1}(M)$ satisfies the conditions of theorem 2.2 and using 2.3, we obtain the result. □

3. The scalar curvature of CR submanifold of maximal CR dimension

Let (M, g) be an n-dimensional CR submanifold of maximal CR dimension of $(n+p)$-dimensional complex projective space $(P^{\frac{n+p}{2}}(C), \overline{g})$.

Such submanifolds are equipped with an almost contact metric structure (F, U, u, g) induced from the almost complex structure $(P^{\frac{n+p}{2}}(C), J, \overline{g})$.

Applying a conformal change $\tilde{g} = exp(-f)\overline{g}$, we can regard (M', g') as an n-dimensional submanifold of $(\tilde{M}, \tilde{g})$ with a conformal almost contact metric structure $(F', \tilde{U}, \tilde{u}, g')$ defined by

$$F' = F, \quad \tilde{U} = exp(\frac{f}{2})U, \quad \tilde{u} = exp(\frac{-f}{2})u. \tag{18}$$

Hence, for any $X \in T(M')$, we have put

$$JX = F'X + \tilde{u}(X)\xi'$$

where $(1,1)$ tensor field F and the one form $\tilde{u}$ satisfy

$$J\xi' = \tilde{U} \tag{19}$$
$$F^2X = -X + \tilde{u}(X)\tilde{U}$$

Differentiating (19) covariantly and comparing the tangential part and the normal part, it follows

$$\begin{aligned}(\nabla'_X F)Y &= \tilde{u}(Y)A'X - g'(A'Y, X)\tilde{U} \\ &\quad -\frac{1}{2}\omega(FY)X + \frac{1}{2}\overline{g}(X, FY)\omega^\sharp \\ &\quad +\frac{1}{2}\omega(Y)FX - \frac{1}{2}\overline{g}(X, Y)F\omega^\sharp\end{aligned} \tag{20}$$

$$\nabla'_X\tilde{U} = FA'X - \frac{1}{2}\omega(\tilde{U})X + \frac{1}{2}exp(f)\tilde{u}(X)\omega^\sharp \tag{21}$$

since the $\omega^\sharp$ is to be considered as a tangent vector field.

Now, let $\xi', \xi'_1, ..., \xi'_q, \xi'_{1^*}, ..., \xi'_{q^*}$ be a canonical local orthonormal basis of $T^\perp(\tilde{M})$ and A' , A'_a and A'_{a^*} denote the shape operators with respect to the normal vector fields ξ', ξ'_a and ξ'_{a^*} respectively.

Then we compute the Ricci tensor $\tilde{Ric}$ and the scalar curvature $\tilde{\rho}$ of M as follows

$$\begin{aligned}(22)\quad \tilde{Ric}(X,Y) &= (n+2)exp(f)g'(X,Y) - 3exp(f)\tilde{u}(X)\tilde{u}(Y)\\ &+(traceA')g'(A'X,Y) - g'(A'^2X,Y)\\ &+\sum_{a=1}^{q}\{(traceA'_a)g'(A_aX,Y) + (trace(A'_{a^*})g'(A'_{a^*}X,Y)\\ &-g'(A'^2_aX,Y) - g'(A'^2_{a^*}X,Y)\}\\ &-\frac{1}{2}(\{(2-n)\overline{g}(\nabla_X\omega^\sharp,Y) + \frac{n-2}{2}\omega(X)\omega(Y)\\ &-(divf - \frac{\|\omega\|^2}{2})\overline{g}(X,Y)\} - (n-1)\frac{\|\omega\|^2}{4}\overline{g}(X,Y),\end{aligned}$$

$$\begin{aligned}(23)\quad \tilde{\rho} &= exp(f)(n+3)(n+1) + (traceA')^2 - traceA'^2\\ &+\sum_{a=1}^{q}\{(traceA'_a)^2 + (traceA'_{a^*})^2 - traceA'^2_a - traceA'^2_{a^*}\}.\end{aligned}$$

Let us consider the integral formula

$$(24)\qquad \int_M\{(\tilde{Ric}(X,X) + \frac{1}{2}|L(X)g'|^2 - |\nabla'X|^2 - (divX)^2\} * 1 = 0,$$

now with using the realtion (21), we compute

$$(25)\qquad div(\tilde{U}) = traceFA' + \frac{n-1}{2}\omega(\tilde{U}) = \frac{n-1}{2}\omega(\tilde{U}),$$

$$\begin{aligned}(26)\quad \sum_{a=1}^{q}|\nabla'\tilde{U}|^2 &= traceA'^2 - g'(A'^2\tilde{U},\tilde{U}) + exp(f)g(A'F\omega^\sharp,U)\\ &+\frac{n-2}{4}\omega^2(\tilde{U}) + \frac{1}{4}exp(f)|\omega^\sharp|^2,\end{aligned}$$

$$\begin{aligned}(27)\quad \sum_{a=1}^{q}|L_{\tilde{U}}g'|^2 &= 2traceA'^2 + 2trace(FA')^2 - g'(A'^2U,U)\\ &+\frac{2n-3}{2}\omega^2(U) + 2exp(f)g'(A'F\omega^\sharp,\tilde{U}) + \frac{1}{2}exp(f)|\omega^\sharp|^2,\end{aligned}$$

substituing $X = \tilde{U}$ into the relation (22), we obtain

$$\begin{aligned}(28)\qquad \tilde{Ric}(\tilde{U},\tilde{U}) &= (n-1)exp(f) - g'(A'^2\tilde{U},\tilde{U}) \\ &\quad -\sum_{a=1}^{q}\{g'(A'^2_a\tilde{U},\tilde{U}) - g'(A'^2_{a^*}\tilde{U},\tilde{U})\} \\ &\quad +\{\frac{n-2}{2}\overline{g}(\nabla_{\tilde{U}}\omega^\sharp,\tilde{U}) - \frac{n-2}{4}\omega^2(U) \\ &\quad -exp(f)divf - \frac{n+2}{4}exp(f)|\omega^\sharp|^2\}.\end{aligned}$$

From (25) , (26) ,(27) and (28) it follows

$$\begin{aligned}(29\int_{M'}\{\frac{1}{2}|L(X)g'|^2 &+ \rho' - exp(f)(n+2)(n-1) + \frac{(n-2)}{4}\overline{g}(\nabla_{\tilde{U}}\omega^\sharp,\tilde{U}) \\ &+ \sum_{a=1}^{q}\{traceA'^2_a + traceA'^2_{a^*} - g'(A'^2_a\tilde{U},\tilde{U}) - g'(A'^2_{a^*}\tilde{U},\tilde{U})\} \\ &+ \frac{(n+1)}{4}exp(f) - \omega(FA'\tilde{U}) - \frac{(n-1)^2}{4}\omega^2(\tilde{U})\} = 0.\end{aligned}$$

Further, let $e'_1, ..., e'_n$ be a chosen orthonormal frame in such a way that $e_n = \tilde{U}$. Then the notion $\sum_{i=1}^{n-1}\{g'(A'_a e'_i, A'_a e'_i)\}$ is positive which implies

$$(30)\qquad L_{\tilde{U}}g' = 0,$$

and for any $i = 1, ...n-1$ we have

$$(31)\qquad A'_a e'_i = A'_{a^*}e'_i = 0,$$

and conseqently for any $X \perp \tilde{U}$ we get

$$(32)\qquad A'_a X = A'_{a^*}X = 0,$$

provided that

$$\begin{aligned}(33)\qquad \rho' &\geq exp(f)(n+2)(n-1) - \frac{n-2}{4}\overline{g}(\nabla_{\tilde{U}}\omega^\sharp,\tilde{U}) \\ &\quad -\frac{n+1}{4}exp(f) + \omega(FA'\tilde{U}) + \frac{(n-1)^2}{4}\omega^2(\tilde{U}).\end{aligned}$$

From(30), it follows

$$(34)\qquad g'((FA' - A'F)X, Y) + \frac{1}{2}\tilde{u}(X)\omega(Y). + \frac{1}{2}\tilde{u}(Y)\omega(X) = 0.$$

Since M' is minimal and making use of (32), we obtain

$$s_a(\tilde{U}) = s_{a^*}(\tilde{U}) = 0.$$

Using (13) and from the Weingarten formula for ξ'_a and ξ_{a^*}, we deduce

$$s_a(X) = s_{a^*}(X) = 0. \tag{35}$$

This together with (32) implies

$$A'_a = A'_{a^*} = 0.$$

Then, we have proved

Lemma 3.1. *Let (M', g') be an n-dimensional compact minimal submanifold which satisfies the condition* (18)*. If the condition* (33) *holds, then*

$$g'((FA' - A'F)X, Y) + \frac{1}{2}\tilde{u}(X)\omega(Y) + \frac{1}{2}\tilde{u}(Y)\omega(X) = 0$$

and

$$A'_a = A'_{a^*} = 0.$$

Theorem 3.2. *Let (M', g') be an n-dimensional compact minimal submanifold which satisfies the condition* (18)*. If the condition* (33) *holds, then there exists a real $(n+1)$-dimensional totally geodesic submanifold M' of $\tilde{M}$ so that $M \subset \tilde{M}$.*

Proof. Let us consider N_0 defined in 2.5. Using 3.1, for any $\eta \in N_0(x)$, we have $A_\eta = 0$ where we have put $\eta = q^0\xi' + \{q^a\xi'_a + \xi'_{a^*}\}$. Consequently $q^0 = 0$ which implies that $H_0 = span\{\xi'_1, .., \xi'_q, \xi'_{1^*}, ..., \xi'_{q^*}\}$ and then H_1 satisfies the condition of 2.5. This completes the proof. □

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