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SOME THEOREMS ON APPROXIMATION OF SOLUTIONS OF INTEGRO-DIFFERENCE-DIFFERENTIAL EQUATION WITH NONLOCAL CONDITION IN BANACH SPACE

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Abstract. Results on approximate solutions, closeness, continuous dependence of solutions of a certain integro- differential equation with finite delay with nonlocal condition in Banach Space are discussed. The integral inequality established by B.G. Pachpatte is used to obtain the results.

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1. Introduction

Tychonov's fixed point theorem is applied for the method of successive approximations, and the comparison method, S. Sugiyama [18] explored the existence and uniqueness of solutions of the problem.

$$\frac{dx(\eta)}{d\eta} = \mathcal{F}(\eta, x(\eta), x(\eta-1)), \tag{1}$$

for $0 \leq \eta \leq \eta_1$, with the conditions

$$x(\eta-1) = \phi(\eta) \quad (0 \leq \eta < 1), \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where, x and $\mathcal{F}$ represent n-dimensional vectors (see [18] for details) and Stokes [14] has discussed the same problems as above for nonlinear differential equations. This method can be also be valid for difference-differential equations. Here, it is to be noted that it is essential to obtain the solutions of (1) for $0 \leq \eta \leq \eta_1$, where $\eta_1 > 1$. However, if $\eta_1 < 1$, the problem has been simplified to that of the theory of differential equations, that is, it is enough to consider the of differential equation

$$\frac{dx(\eta)}{d\eta} = \mathcal{F}(\eta, x(\eta), \phi(\eta)),$$

for $0 \leq \eta \leq \eta_1$, along with initial condition $x(0) = x_0$. In [16], S. Sugiyama proved the existence, stability, and boundedness of solutions of the difference-differential problem (1)–(3) by making use of Tychonov's fixed point theorem with additional condition on $\mathcal{F}$.

The aim of this paper is to prove the existence, uniqueness and continuous dependence of solutions of the difference-integrodifferential of the form:

$$x'(\eta) = \mathcal{F}(\eta, x(\eta), \int_0^\eta k(\eta, \omega, x(\omega))d\omega, x(\eta-1)), \tag{4}$$

for $\eta \in \mathbb{R}_+ = [0, \infty)$ under the conditions

$$x(\eta-1) = \phi(\eta) \quad (0 \leq \eta < 1), \tag{5}$$

$$x(0) + g(x) = x^*, \tag{6}$$

where , $\mathcal{F} \in C(J \times X \times X \times X, X)$, $k \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times X, X)$, $g \in C(C(\mathbb{R}_+, X), X)$ and $\phi(\eta)$ is a continuous function for $0 \leq \eta < 1$, $\lim_{\eta \to 1-0} \phi(\eta)$ exists, for which we denote by $\phi(1-0) = c_0$.

The problems of existence, uniqueness and other properties of solutions of various special forms of (4)–(6) by using different techniques, see [2, 6, 11, 12, 8, 18, 17, 16, 15, 19] and the references cited therein. Furthermore, we cite various articles and monographs. [11, 18], [2, p. 342], [5, p.308], [7, p. 18]. Recently, in the exciting paper [12], B. G. Pachpatte has studied the existence, uniqueness and continuous dependence of solutions (1)–(3) with an infinite interval of η, by the well known Banach fixed point theorem and the Gronwall-Bellman integral inequality. Here our approach to extends the results of [12, 13, 14, 18, 17, 20, 21, 22] and the results are obtained by using inequality founded by B. G. Pachpatte. We are motivated by the work of S. Sugiyama [18, 17] and influenced by the work of B. G. Pachpatte [12].

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3, deals with existence through approximation and deals with condition about atmost one solution. In Section 4, we discuss results on continuous dependence of solutions, closeness of two solutions and convergence of solutions. In final Section 5, we summarizes the discussion.

2. Preliminaries and Hypotheses

Some introductions and related hypotheses were utilised in the discussion that occurred.

Let X be the Banach space with norm $\|\cdot\|$. Let S be the space of all continuous functions from J into X and fulfill the condition

$$\|z(\eta)\| = O(\exp(\lambda\eta)), \tag{7}$$

for some positive constant $\lambda > 0$. In this space we define the norm (see [1, 10])

$$\|z\|_S = \sup_{\eta\in J}[\|z(\eta)\| \exp(-\lambda\eta)]. \tag{8}$$

It is much clear to observe that S with the above norm is a Banach space. Note that condition (7) implies the existence of a nonnegative constant N such that $\|z(\eta)\| \leq N \exp(\lambda\eta)$ for $\eta \in \mathbb{R}_+$. Using this fact in (8) we observe that

$$\|z\|_S \leq N. \tag{9}$$

Remark 2.1. *We note that the norm defined in* (8) *was first used by Bielecki* [1] *for proving the existence and uniqueness of global solutions for ordinary differential equations. For the changes associated with Bielecki's method, see* [3].

We required the following Lemma :

Lemma 2.2 ([10],p. 152)**.** *Let* $u(\eta), e(\eta), b(\eta) \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $s \leq \eta$,$a(\eta,\omega), c(\eta,\omega) \in C({\mathbb{R}_+}^2, \mathbb{R}_+)$.*If* $e(\eta)$ *and* $a(\eta,\omega)$ *be nondecreasing* $\eta \in \mathbb{R}_+$ *and*

$$u(\eta) \leq e(\eta) + \int_0^\eta a(\eta,\omega)\Big[b(\omega)u(\omega) + \int_0^\omega c(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega,$$

holds for $\eta \in \mathbb{R}_+$, *then*

$$u(\eta) \leq e(\eta)\exp\Big(\int_0^\eta a(\eta,\omega)\big[b(\omega) + \int_0^\omega c(\omega,\sigma)d\sigma\big]d\omega\Big),$$

for $\eta \in \mathbb{R}_+$.

We list the following hypotheses:

($\mathcal{H}_1$) The function $\mathcal{F}$ in (4) satisfies the condition

$$\|\mathcal{F}(\eta, x, y, z) - \mathcal{F}(\eta, \bar{x}, \bar{y}, \bar{z})\| \leq \mathtt{p}_1(\eta)[\|x-\bar{x}\| + \|y-\bar{y}\| + \|z-\bar{z}\|],$$

for $(\eta, x, y, z), (\eta, \bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}_+ \times X \times X \times X$, where $\mathtt{p}_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ and increasing.

($\mathcal{H}_2$) The function k in (4) satisfies the condition

$$\|k(\eta,\omega,x) - k(\eta,\omega,\bar{x})\| \leq \mathtt{p}_2(\eta,\omega)[\|x-\bar{x}\|],$$

for $(\eta,\omega,x), (\eta,\omega,\bar{x}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times X$, where $\mathtt{p}_2 \in C({\mathbb{R}^2}_+, \mathbb{R}_+)$.

($\mathcal{H}_3$) There exist constants $\mathcal{G}$ such that

$$\|g(x) - g(\bar{x})\| \leq \mathcal{G}\|x-\bar{x}\|, \quad \text{for every} \quad x, \bar{x} \in C(\mathbb{R}_+, X)$$

3. $\epsilon-$ Approximation Solution

3.1. $\epsilon-$ **Approximation Solution:** Let $x_i(\eta)$ $(i = 1, 2)$ be functions which are continuous on $\mathbb{R}_+$, differentiable in $0 < \eta < \infty$ and satisfy the inequalities

$$\|x_i'(\eta) - \mathcal{F}(\eta, x_i(\eta), \int_0^\eta k(\eta,\omega,x_i(\omega))d\omega, x_i(\eta-1))\| \leq \epsilon_i, \tag{10}$$

for given constants $\epsilon_i \geq 0$ $(i = 1, 2)$, where it is supposed that the initial conditions

$$\mathrm{x}_i(\eta - 1) = \phi_i(\eta) \quad (0 \leq \eta < 1) \quad \mathrm{x}_i(0) + g(\mathrm{x}_i) = \mathrm{x}_i{}^* \tag{11}$$

for $i = 1, 2$ are fulfilled and $\phi_i(\eta)$ are continuous functions for which $\lim_{\eta \to 1-0} \phi(\eta)$ exists. Then we call $\mathrm{x}_i(\eta)(i = 1, 2)$ the ϵ_i approximate solutions with respect to the equation (4) with the initial conditions (11). The next theorem tackles to estimate on the difference between the two approximate solutions of equation (4) with (11). We first prove the main conclusion. .

Theorem 3.1. *Assume that hypotheses* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold. Let* $\mathrm{x}_i(\eta)(i = 1, 2)$ *be respectively* ϵ_i *-approximate solutions of equation* (1) *on* $\mathbb{R}_+$ *with* (11) *such that*

$$\|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| \leq \delta, \tag{12}$$

where $\delta \geq 0$ *is a constant. Then*

$$\|\mathrm{x}_1(\eta) - \mathrm{x}_2(\eta)\| \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})} \exp\Big(\int_0^\eta \frac{p_1(\omega)}{1-\mathcal{G}}[1 + \int_0^\omega p_2(\omega, \sigma) d\sigma] d\omega\Big) \tag{13}$$

for $0 \leq \eta < 1$ *and*

(14)

$$\|\mathrm{x}_1(\eta) - \mathrm{x}_2(\eta)\| \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})} \exp\Big(\int_0^\eta \frac{(p_1(\omega) + p_1(\omega+1))}{1-\mathcal{G}}[1 + \int_0^\omega p_2(\omega, \sigma) d\sigma] d\omega\Big)$$

for $1 \leq \eta < \infty$ *, where*

$$\mathcal{M}(\eta) = \big((\epsilon_1 + \epsilon_2)\eta + \delta\big) + \int_0^1 p_1(\omega)\|\phi_1(\omega) - \phi_2(\omega)\| d\omega, \tag{15}$$

and $\mathcal{G} < 1$

Proof. Since $\mathrm{x}_i(\eta)(i = 1, 2)$ for $\eta \in \mathbb{R}_+$ are respectively ϵ_i-approximate solutions of equation (4) with (11), we have (10). By taking $\eta = \sigma$ in (10) and integrating both sides from 0 to η, we have

$$\begin{aligned}
\epsilon_i(\eta) &\geq \int_0^\eta \|\mathrm{x}_i'(\sigma) - \mathcal{F}(\sigma, \mathrm{x}_i(\sigma), \int_0^\sigma k(\sigma, \omega, \mathrm{x}_i(\omega)) d\omega, \mathrm{x}_i(\sigma - 1))\| d\sigma \\
&\geq \|\int_0^\eta \{\mathrm{x}_i'(\sigma) - \mathcal{F}(\sigma, \mathrm{x}_i(\sigma), \int_0^\sigma k(\sigma, \omega, \mathrm{x}_i(\omega)) d\omega, \mathrm{x}_i(\sigma - 1))\} d\sigma\| \\
&= \|[\mathrm{x}_i(\eta) - \mathrm{x}_i(0)] - \int_0^\eta \mathcal{F}(\sigma, \mathrm{x}_i(\sigma), \int_0^\sigma k(\sigma, \omega, \mathrm{x}_i(\omega)) d\omega, \mathrm{x}_i(\sigma - 1)) d\sigma\|
\end{aligned}$$

(16)

$$= \|\mathrm{x}_i(\eta) - (\mathrm{x}_i{}^* - g(\mathrm{x}_i)) - \int_0^\eta \mathcal{F}(\sigma, \mathrm{x}_i(\sigma), \int_0^\sigma k(\sigma, \omega, \mathrm{x}_i(\omega)) d\omega, \mathrm{x}_i(\sigma - 1)) d\sigma\|$$

for $i = 1, 2$. From above and using the elementary inequalities $\|v - z\| \leq \|v\| + \|z\|$,$\|v\| - \|z\| \leq \|v - z\|$, We notice that

$$\begin{aligned}
&(\epsilon_1 + \epsilon_2)\eta \\
&\geq \|x_1(\eta) - ({x_1}^* - g(x_1)) - \int_0^\eta \mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1))d\sigma\| \\
&\quad + \|x_2(\eta) - ({x_2}^* - g(x_2)) - \int_0^\eta \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))d\sigma\| \\
&\geq \|x_1(\eta) - ({x_1}^* - g(x_1)) - \int_0^\eta \mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1))d\sigma \\
&\quad - x_2(\eta) + ({x_2}^* - g(x_2)) + \int_0^\eta \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))d\sigma\| \\
&\geq \|(x_1(\eta) - x_2(\eta)) - ({x_1}^* - {x_2}^*) - (g(x_2) - g(x_1)) \\
&\quad - \int_0^\eta [\mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1)) \\
&\quad - \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))]d\sigma\| \\
&\geq \|x_1(\eta) - x_2(\eta)\| - \|{x_1}^* - {x_2}^*\| - \|g(x_2) - g(x_1)\| \\
&\quad - \| \int_0^\eta [\mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1)) \\
&\quad - \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))]\|d\sigma.
\end{aligned} \tag{17}$$

Let $u(\eta) = |x_1(\eta) - x_2(\eta)|, \eta \in \mathbb{R}_+$. From (3.1),We notice that

$$\begin{aligned}
u(\eta) &\leq (\epsilon_1 + \epsilon_2)\eta + \|{x_1}^* - {x_2}^*\| + \|g(x_2) - g(x_1)\| \\
&\quad + \| \int_0^\eta [\mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1)) \\
&\quad - \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))]d\sigma\| \\
&\leq (\epsilon_1 + \epsilon_2)\eta + \|{x_1}^* - {x_2}^*\| + \|g(x_2) - g(x_1)\| \\
&\quad + \int_0^\eta \|[\mathcal{F}(\sigma, x_1(\sigma), \int_0^\sigma k(\sigma, \omega, x_1(\omega))d\omega, x_1(\sigma - 1)) \\
&\quad - \mathcal{F}(\sigma, x_2(\sigma), \int_0^\sigma k(\sigma, \omega, x_2(\omega))d\omega, x_2(\sigma - 1))]\|d\sigma
\end{aligned}$$

$$\begin{aligned}
&\leq (\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| \\
&\quad+\int_0^\eta \|[\mathcal{F}(\sigma,\mathrm{x}_1(\sigma),\int_0^\sigma k(\sigma,\omega,\mathrm{x}_1(\omega))d\omega,\mathrm{x}_1(\sigma-1)) \\
&\qquad-\mathcal{F}(\sigma,\mathrm{x}_2(\sigma),\int_0^\sigma k(\sigma,\omega,\mathrm{x}_2(\omega))d\omega,\mathrm{x}_2(\sigma-1))]\|d\sigma \\
&\leq (\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| \\
&\quad+\int_0^\eta \mathtt{p}_1(\sigma)[\|\mathrm{x}_1(\sigma)-\mathrm{x}_2(\sigma)\|+\int_0^\sigma \|k(\sigma,\omega,\mathrm{x}_1(\omega))-k(\sigma,\omega,\mathrm{x}_2(\omega))\|d\omega \\
&\qquad+\|\mathrm{x}_1(\sigma-1))-\mathrm{x}_2(\sigma-1))\|]d\sigma \\
&\leq (\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| \\
&\quad+\int_0^\eta \mathtt{p}_1(\sigma)[\|\mathrm{x}_1(\sigma)-\mathrm{x}_2(\sigma)\|+\int_0^\sigma \mathtt{p}_2(\sigma,\omega)\|\mathrm{x}_1(\omega)-\mathrm{x}_2(\omega)\|d\omega \\
&\qquad+\|\mathrm{x}_1(\sigma-1))-\mathrm{x}_2(\sigma-1))\|]d\sigma \\
&\leq (\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}u(\eta) \\
&\quad+\int_0^\eta \mathtt{p}_1(\sigma)[u(\sigma)+\int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega+\|\mathrm{x}_1(\sigma-1))-\mathrm{x}_2(\sigma-1))\|]d\sigma. \qquad (18)
\end{aligned}$$

We take into account the next two possibilities..

Case I: For $0 \leq \eta < 1$. From (18) and hypotheses, we observe that

$$\begin{aligned}
u(\eta) &\leq ((\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}u(\eta))+\int_0^\eta \mathtt{p}_1(\sigma)\|\mathrm{x}_1(\sigma-1))-\mathrm{x}_2(\sigma-1))\|d\sigma \\
&\quad+\int_0^\eta \mathtt{p}_1(\sigma)[u(\sigma)+\int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq ((\epsilon_1+\epsilon_2)\eta+\delta+\mathcal{G}u(\eta)) \\
&\quad+\int_0^1 \mathtt{p}_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega+\int_0^\eta \mathtt{p}_1(\omega)[u(\omega)+\int_0^\omega \mathtt{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(1-\mathcal{G})u(\eta) &\leq ((\epsilon_1+\epsilon_2)\eta+\delta \\
&\quad+\int_0^1 \mathtt{p}_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega+\int_0^\eta \mathtt{p}_1(\omega)[u(\omega) \qquad (19) \\
&\quad+\int_0^\omega \mathtt{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
\end{aligned}$$

Let $\mathcal{M}(\eta) = (\epsilon_1+\epsilon_2)\eta + \delta + \int_0^1 \mathrm{p}_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega$. Then in equation (19) becomes

$$(1-\mathcal{G})u(\eta) \leq (\mathcal{M}(\eta)) + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega$$

$$\Rightarrow u(\eta) \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})} + \int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega. \tag{20}$$

Clearly $\frac{\mathcal{M}(\eta)}{(1-\mathcal{G})}$ is nondecreasing in $\eta \in \mathbb{R}_+$. Now an application of Lemma 2.2 to (20) yields

$$u(\eta) \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})}\exp(\int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega). \tag{21}$$

Case II: For $1 \leq \eta < \infty$. From (18) and hypotheses, we observe that

$$\begin{aligned}
&u(\eta)\\
&\leq ((\epsilon_1+\epsilon_2)\eta + \delta + \mathcal{G}u(\eta))\\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)|\|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\||]d\sigma + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma)\\
&\quad + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma\\
&\leq ((\epsilon_1+\epsilon_2)\eta + \delta + \mathcal{G}u(\eta))\\
&\quad + \int_0^1 \mathrm{p}_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega + \int_1^\eta \mathrm{p}_1(\omega)|\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\||]d\omega\\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma\\
&\leq \mathcal{M}(\eta) + \mathcal{G}u(\eta))\\
&\quad + \int_1^\eta \mathrm{p}_1(\omega)|\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\||]d\omega +\\
&\quad \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma\\
&\leq \mathcal{M}(\eta) + \mathcal{G}u(\eta)) + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma + I_1,
\end{aligned}$$

(22)

where $I_1 = \int_1^\eta \mathrm{p}_1(\omega)\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\||]d\omega$ by making change of variable, we obtain

$$I_1 \leq \int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega. \tag{23}$$

Using (23) equation (22) becomes

$$\begin{aligned}
u(\eta) &\leq \mathcal{M}(\eta) + \mathcal{G}u(\eta)) + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega + \\
&\leq \int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega \\
&\leq \mathcal{M}(\eta) + \mathcal{G}u(\eta)) + \int_0^\eta (\mathrm{p}_1(\omega) + \mathrm{p}_1(\omega+1))[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
\end{aligned}$$

This gives

$$u(\eta) \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})} + \int_0^\eta \frac{\mathrm{p}_1(\omega) + \mathrm{p}_1(\omega+1)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega). \tag{24}$$

Using Lemma 2.2, we get

$$u(\eta) \leq \frac{\mathcal{M}(\eta)}{(1-\mathcal{G})} \exp(\int_0^\eta \frac{(\mathrm{p}_1(\omega) + \mathrm{p}_1(\omega+1))}{1-\mathcal{G}}[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega). \tag{25}$$

□

Remark : We note that the estimate obtained in (13)–(14) yields the bound on the difference between the two approximate solutions of equation (4)–(6). If $x_1(\eta)$ is a solution of equation (4)–(6) with $\mathrm{x}_0 = x_1$, then we have $\epsilon_1 = 0$ and from (13)–(14), we see that $\mathrm{x}_2(\eta) \to \mathrm{x}_1(\eta)$ as $\epsilon_2 \to 0$ and $\delta \to 0$. Moreover, if we put (i)$\epsilon_1 = \epsilon_2 = 0$ and $x_1 = \mathrm{x}_2$ in (13)–(14), then the uniqueness of solutions of equation (4)–(6) is established and (ii) $\epsilon_1 = \epsilon_2 = 0$, then we get the bound which shows the dependency of solutions of equation (4)–(6) on given initial values

3.2. **Uniqueness without existence:** Uniqueness without existence is discussed by suitable theorem.

Theorem 3.2. *Assume that hypotheses* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold.Then the problem* (4)–(6) *has at most one solution on* $\mathbb{R}_+$.

Proof. Since $\mathrm{x}_i(\eta)(i = 1,2)$ for $\eta \in \mathbb{R}_+$ are respectively. Let $u(\eta) = \|\mathrm{x}_1(\eta) - \mathrm{x}_2(\eta)\|, \eta \in \mathbb{R}_+$. From (3.1), we observe that

$$\begin{aligned}
u(\eta) \leq \|g(\mathrm{x}_2) - g(\mathrm{x}_1)\| + \| &\int_0^\eta [\mathcal{F}(\sigma, \mathrm{x}_1(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_1(\omega))d\omega, \mathrm{x}_1(\sigma-1)) \\
&- \mathcal{F}(\sigma, \mathrm{x}_2(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_2(\omega))d\omega, \mathrm{x}_2(\sigma-1))]d\sigma\| \\
\leq \|g(\mathrm{x}_2) - g(\mathrm{x}_1)\| + &\int_0^\eta \|[\mathcal{F}(\sigma, \mathrm{x}_1(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_1(\omega))d\omega, \mathrm{x}_1(\sigma-1))
\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{F}(\sigma, \mathrm{x}_2(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_2(\omega))d\omega, \mathrm{x}_2(\sigma-1))]\|d\sigma \\
&\leq \mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| + \int_0^\eta \|[\mathcal{F}(\sigma, \mathrm{x}_1(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_1(\omega))d\omega, \mathrm{x}_1(\sigma-1)) \\
&- \mathcal{F}(\sigma, \mathrm{x}_2(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}_2(\omega))d\omega, \mathrm{x}_2(\sigma-1))]\|d\sigma \\
&\leq \mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| + \int_0^\eta \mathrm{p}_1(\sigma)[\|\mathrm{x}_1(\sigma)-\mathrm{x}_2(\sigma)\| \\
&+ \int_0^\sigma \|k(\sigma,\omega,\mathrm{x}_1(\omega)) - k(\sigma,\omega,\mathrm{x}_2(\omega))\|d\omega + \|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma \\
&\leq \mathcal{G}\|(\mathrm{x}_2)-(\mathrm{x}_1)\| + \int_0^\eta \mathrm{p}_1(\sigma)[\|\mathrm{x}_1(\sigma)-\mathrm{x}_2(\sigma)\| + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)\|\mathrm{x}_1(\omega)-\mathrm{x}_2(\omega)\|d\omega \\
&+ \|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma \\
&\leq \mathcal{G}u(\eta) + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega + \|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma.
\end{aligned}
\tag{26}
$$

We consider the following two cases.

Case I: $0 \leq \eta < 1$. From (26) and hypotheses, we observe that

$$
\begin{aligned}
u(\eta) &\leq \mathcal{G}u(\eta) + \int_0^\eta \mathrm{p}_1(\sigma)\|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) \\
&\quad + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq \mathcal{G}u(\eta) + \int_0^1 \mathrm{p}_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega)+ \\
&\leq \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
(1-\mathcal{G})u(\eta) &\leq \int_0^1 \mathrm{p}_1(\omega)\|\phi(\omega)-\phi(\omega)\|d\omega + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega \\
&\leq 0 + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
\end{aligned}
\tag{27}
$$

$$
\Rightarrow u(\eta) \leq 0 + \int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega.
$$

Clearly $e(\eta)=0$. Now an application of Lemma 2.2 to (27) in equation, yields

$$u(\eta) \leq 0\exp(\int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega)$$

$$\Rightarrow u(\eta) \leq 0$$

$$\Rightarrow \|\mathrm{x}_1(\eta)-\mathrm{x}_2(\eta)\| \leq 0. \tag{28}$$

Case II: For $1 \leq \eta < \infty$. From (26) and hypotheses, we observe that

$$\begin{aligned}
u(\eta) &\leq \mathcal{G}u(\eta) + \int_0^\eta \mathrm{p}_1(\sigma)\|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) \\
&\quad + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq \mathcal{G}u(\eta) + \int_0^1 \mathrm{p}_1(\omega)\|\phi(\omega)-\phi(\omega)\|d\omega + \int_1^\eta \mathrm{p}_1(\omega)\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega \\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq 0 + \mathcal{G}u(\eta) + \int_1^\eta \mathrm{p}_1(\omega)|\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) \\
&\quad + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma
\end{aligned}$$

(29)

$$\leq 0 + \mathcal{G}u(\eta) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma + I_1,$$

where $I_1 = \int_1^\eta \mathrm{p}_1(\omega)\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega$ by making change of variable, we obtained

$$I_1 \leq \int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega \tag{30}$$

Using (30) equation (29) becomes

$$\begin{aligned}
u(\eta) &\leq 0 + \mathcal{G}u(\eta) + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega \\
&\quad + \int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega \\
&\leq 0 + \mathcal{G}u(\eta) + \int_0^\eta (\mathrm{p}_1(\omega)+\mathrm{p}_1(\omega+1))[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega
\end{aligned}$$

$$\Rightarrow u(\eta) \leq 0 + \int_0^\eta \frac{\mathrm{p}_1(\omega)+\mathrm{p}_1(\omega+1)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega). \tag{31}$$

Using Lemma 2.2, we get

$$u(\eta) \leq 0 \exp(\int_0^\eta \frac{(\mathrm{p}_1(\omega)+\mathrm{p}_1(\omega+1))}{1-\mathcal{G}}[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega)$$

$$\Rightarrow \|\mathrm{x}_1(\eta)-\mathrm{x}_2(\eta)\| \leq 0. \tag{32}$$

So, we have $\mathrm{x}_1(\eta)=\mathrm{x}_2(\eta)$ for $\eta \in \mathbb{R}_+$. Thus there is atmost one solution to (4)-(6) on $\mathbb{R}_+$. □

4. Continuous dependence

In this section we study the continuous dependence of solutions to (4) on the given initial data, and on the function $\mathcal{F}$. Also we show the continuous dependence of solutions of equations of the form (4) on certain parameters. First, we shall give the following theorem concerning the continuous dependence of solutions to (4) on the given initial data.

Continuous dependence of solutions is discussed by suitable theorem.

Theorem 4.1. *Suppose that the hypotheses* $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ *hold and let* $\mathrm{x}_1(\eta)$, $\mathrm{x}_2(\eta)$ *be the solutions of* (4) *with the initial conditions*

$$\mathrm{x}_1(\eta-1)=\phi_1(\eta) \quad (0 \leq \eta < 1), \quad \mathrm{x}_1(0)+g(\mathrm{x}_1)=\mathrm{x}_1{}^*, \tag{33}$$

$$\mathrm{x}_2(\eta-1)=\phi_2(\eta) \quad (0 \leq \eta < 1), \quad \mathrm{x}_2(0)+g(\mathrm{x}_2)=\mathrm{x}_2{}^*, \tag{34}$$

respectively, where $\mathrm{x}_1{}^*, \mathrm{x}_2{}^*$ *are elements of* X. *Then*

$$\|\mathrm{x}_1(\eta)-\mathrm{x}_2(\eta)\| \leq \frac{c(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{p_1(\omega)}{(1-\mathcal{G})}\Big[1+\int_0^\omega p_2(\omega,\sigma)d\sigma\Big]d\omega\Big) \tag{35}$$

for $0 \leq \eta < 1$ *and*

$$\|\mathrm{x}_1(\eta)-\mathrm{x}_2(\eta)\| \leq \frac{c(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{(p_1(\omega)+p_1(\omega+1))}{(1-\mathcal{G})}\Big[1+\int_0^\omega p_2(\omega,\sigma)d\sigma\Big]d\omega\Big) \tag{36}$$

for $1 \leq \eta < \infty$, *where*

$$c(\eta)=\|\mathrm{x}_1{}^*-\mathrm{x}_2{}^*\|+\int_0^1 p_1(\omega)\|\phi_1(\omega)-\phi_2(\omega)\|d\omega. \tag{37}$$

and $\mathcal{G}<1$

Proof. Let $u(\eta) = \|\mathrm{x}_1(\eta) - \mathrm{x}_2(\eta)\|$ for $\eta \in \mathbb{R}_+$. We consider the following two cases.

Case 1: For $0 \leq \eta < 1$. From (3.1) and hypotheses, we observe that

$$\begin{aligned} u(\eta) \leq \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| + \mathcal{G}u(\eta) &+ \int_0^\eta \mathrm{p}_1(\sigma)[\|\mathrm{x}_1(\sigma-1) - \mathrm{x}_2(\sigma-1)\|]d\sigma \\ &+ \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\ \leq \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| + \mathcal{G}u(\eta) &+ \int_0^1 \mathrm{p}_1(\omega)\|\|\phi_1(\omega) - \phi_2(\omega)\|d\omega \\ &+ \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} (1-\mathcal{G})u(\eta) \leq \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| &+ \int_0^1 \mathrm{p}_1(\omega)\|\|\phi_1(\omega) - \phi_2(\omega)\|d\omega \\ &+ \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega. \end{aligned} \tag{38}$$

Let $c(\eta) = \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| + \int_0^1 \mathrm{p}_1(\omega)\|\|\phi_1(\omega) - \phi_2(\omega)\|d\omega$, in equation (38) becomes

$$(1-\mathcal{G})u(\eta) \leq (c(\eta)) + \int_0^\eta \mathrm{p}_1(\omega)[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega$$

$$u(\eta) \leq \frac{c(\eta)}{(1-\mathcal{G})} + \int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega. \tag{39}$$

Clearly $\frac{c(\eta)}{(1-\mathcal{G})}$ is nondecreasing in $\eta \in \mathbb{R}_+$. Now an application of Lemma 2.2 to (39), yields

$$u(\eta) \leq \frac{c(\eta)}{(1-\mathcal{G})} \exp(\int_0^\eta \frac{\mathrm{p}_1(\omega)}{1-\mathcal{G}}[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega). \tag{40}$$

Case II: For $1 \leq \eta < \infty$. From (3.1) and hypotheses, we observe that

$$\begin{aligned} u(\eta) \leq \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| + \mathcal{G}u(\eta) &+ \int_0^\eta \mathrm{p}_1(\sigma)[\|\mathrm{x}_1(\sigma-1)) - \mathrm{x}_2(\sigma-1))\|]d\sigma \\ &+ \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\ \leq \|\mathrm{x}_1{}^* - \mathrm{x}_2{}^*\| + \mathcal{G}u(\eta) &+ \int_0^1 \mathrm{p}_1(\omega)\|\phi_1(\omega) - \phi_2(\omega)\|d\omega \\ &+ \int_1^\eta \mathrm{p}_1(\omega)[\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega \\ &+ \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \end{aligned}$$

$$\leq c(\eta) + \mathcal{G}u(\eta) + \int_1^\eta \mathtt{p}_1(\omega)|\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\||]d\omega$$

$$+ \int_0^\eta \mathtt{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma$$

$$\leq c(\eta) + \mathcal{G}u(\eta) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma + I_1, \tag{41}$$

where $I_1 = \int_1^\eta \mathtt{p}_1(\omega)|\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\||]d\omega$ by making change of variable, we obtain

$$I_1 \leq \int_0^\eta \mathtt{p}_1(\omega+1)u(\omega)d\omega. \tag{42}$$

Using (42) equation (41) becomes

$$u(\eta) \leq c(\eta) + \mathcal{G}u(\eta)) + \int_0^\eta \mathtt{p}_1(\omega)|[u(\omega) + \int_0^\omega \mathtt{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega +$$

$$\int_0^\eta \mathtt{p}_1(\omega+1)u(\omega)d\omega$$

$$\leq c(\eta) + \mathcal{G}u(\eta)) + \int_0^\eta (\mathtt{p}_1(\omega) + \mathtt{p}_1(\omega+1))[u(\omega) + \int_0^\omega \mathtt{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega$$

$$\Rightarrow u(\eta) \leq \frac{c(\eta)}{(1-\mathcal{G})} + \int_0^\eta \frac{\mathtt{p}_1(\omega) + \mathtt{p}_1(\omega+1)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathtt{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega).$$

Using Lemma 2.2, we get

$$u(\eta) \leq \frac{c(\eta)}{(1-\mathcal{G})} \exp(\int_0^\eta \frac{(\mathtt{p}_1(\omega) + \mathtt{p}_1(\omega+1))}{1-\mathcal{G}}[1 + \int_0^\omega \mathtt{p}_2(\omega,\sigma)d\sigma]d\omega).$$

□

4.1. **Closeness of Solution:** The initial value problem (4)-(6) along difference-integrodifferential is considered and discussed. Suitable theorems with different cases also discussed.

$$y'(\eta) = F(\eta, y(\eta), \int_0^\eta k(\eta,\omega,y(\omega))d\omega, y(\eta-1)), \tag{43}$$

for $\eta \in \mathbb{R}_+ = [0,\infty)$ under the conditions

$$y(\eta-1) = \psi(\eta) \quad (0 \leq \eta < 1), \tag{44}$$

$$y(0) + l(y) = y^*, \tag{45}$$

where , $F \in C(\mathbb{R}_+ \times X \times X \times X, X)$, $k \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times X, X)$, $l \in C(C(\mathbb{R}_+, X), X)$ and $\psi(\eta)$ is a continuous function for $0 \leq \eta < 1$, $\lim_{\eta\to 1-0}\psi(\eta)$ exists, for which we denote by $\psi(1-0) = c_0$.

The next theorem deals with the nearness of solutions of initial value problem (4)-(6) and solutions of initial value problem (43)-(45) .

Theorem 4.2. *Suppose that the functions $\mathcal{F}$, k,g in* (4)-(6) *satisfy the hypotheses* $(\mathcal{H}_1)$ - $(\mathcal{H}_3)$ *hold and there exist nonnegative constants* $\overline{\epsilon_1}, \overline{\epsilon_2}, \overline{\delta}$*such that*

$$\|\mathcal{F}(\eta, \mathrm{u}, \mathrm{v}, z) - F(\eta, \mathrm{u}, \mathrm{v}, z)\| \leq \overline{\epsilon_1} \tag{46}$$

$$\|g(u) - l(u)\| \leq \overline{\epsilon_2} \tag{47}$$

$$\|\mathrm{x}^* - y^*\| \leq \overline{\delta} \tag{48}$$

where $\mathrm{x}^*, g, \mathcal{F}$ *and* y^*, l, F *are as in* (4)-(6) *and* (43)-(45) *respectively. Let* $\mathrm{x}(\eta)$ *and* $y(\eta)$ *be respectively, solutions of* (4)-(6) *and* (43)-(45) *on* $\mathbb{R}_+$. *Then*

$$\|\mathrm{x}(\eta) - y(\eta)\| \leq \frac{d(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{\mathfrak{p}_1(\omega)}{1-\mathcal{G}}[1 + \int_0^\omega \mathfrak{p}_2(\omega,\sigma)d\sigma]d\omega\Big)$$

for $0 \leq \eta < 1$ *and*

$$\|\mathrm{x}(\eta) - y(\eta)\| \leq \frac{d(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{(\mathfrak{p}_1(\omega) + \mathfrak{p}_1(\omega+1))}{1-\mathcal{G}}[1 + \int_0^\omega \mathfrak{p}_2(\omega,\sigma)d\sigma]d\omega\Big)$$

for $1 \leq \eta < \infty$, *where*

$$d(\eta) = [\overline{\delta} + \overline{\epsilon_1}\eta + \overline{\epsilon_2}] + \int_0^1 \mathfrak{p}_1(\omega)\|\phi(\omega) - \psi(\omega)\|d\omega. \tag{49}$$

and $\mathcal{G} < 1$.

Proof. Let $u(\eta) = \|\mathrm{x}_1(\eta) - \mathrm{x}_2(\eta)\|, \eta \in \mathbb{R}_+$. From the hypotheses, it follows that

$$\begin{aligned}
u(\eta) \leq\ & \|\mathrm{x}^* - y^*\| + \|g(\mathrm{x}) - l(y)\| + \overline{\epsilon_1}\eta + \\
& \| \int_0^\eta [\mathcal{F}(\sigma, \mathrm{x}(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}(\omega))d\omega, \mathrm{x}(\sigma-1)) \\
& \quad - F(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1))]d\sigma\| \\
& \leq \overline{\delta} + \|g(\mathrm{x}) - g(y) + g(y) - l(y)\| + \\
& \int_0^\eta \|[\mathcal{F}(\sigma, \mathrm{x}(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}(\omega))d\omega, \mathrm{x}(\sigma-1)) \\
& \quad - F(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1))]\|d\sigma \\
& \leq \overline{\delta} + \|g(\mathrm{x}) - g(y)\| + \|g(y) - l(y)\| + \int_0^\eta \|[\mathcal{F}(\sigma, \mathrm{x}(\sigma), \int_0^\sigma k(\sigma,\omega,\mathrm{x}(\omega))d\omega, \mathrm{x}(\sigma-1)) \\
& \quad - F(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1))]\|d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\le \overline{\delta} + \mathcal{G}\|x - y\| + \overline{\epsilon_2} + \\
&\int_0^\eta \|[\mathcal{F}(\sigma, x(\sigma), \int_0^\sigma k(\sigma,\omega,x(\omega))d\omega, x(\sigma-1)) \\
&\qquad - \mathcal{F}(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1))]\|d\sigma \\
&\;+ \\
&\int_0^\eta \|[\mathcal{F}(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1)) \\
&\qquad - F(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1))]\|d\sigma \\
&\le \overline{\delta} + \mathcal{G}\|x - y\| + \overline{\epsilon_2} + \overline{\epsilon_1}\eta \\
&\quad + \int_0^\eta \mathtt{p}_1(\sigma)[\|x(\sigma) - y(\sigma)\| + \int_0^\sigma \|k(\sigma,\omega,x(\omega)) - k(\sigma,\omega,y(\omega))\|d\omega \\
&\qquad + \|x(\sigma-1)) - y(\sigma-1))\|]d\sigma \\
&\le [\overline{\delta} + \mathcal{G}\|x - y\| + \overline{\epsilon_2} + \overline{\epsilon_1}\eta] \\
&\quad + \int_0^\eta \mathtt{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega + \|x_1(\sigma-1)) - x_2(\sigma-1))\|]d\sigma
\end{aligned}
$$

We consider the following two cases:

Case 1: For $0 \le \eta < 1$.

$$
\begin{aligned}
(1-\mathcal{G})u(\eta) &\le [\overline{\delta} + \overline{\epsilon_2} + \overline{\epsilon_1}\eta] + \int_0^\eta \mathtt{p}_1(\sigma)\|x(\sigma-1)) - y(\sigma-1))\|]d\sigma \\
&\qquad + \int_0^\eta \mathtt{p}_1(\sigma)\Big[u(\sigma) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\le [\overline{\delta} + \overline{\epsilon_2} + \overline{\epsilon_1}\eta] \\
&\qquad + \int_0^1 \mathtt{p}_1(\sigma)\|\phi(\omega) - \psi(\omega)\|]d\omega + \\
&\int_0^\eta \mathtt{p}_1(\sigma)\Big[u(\sigma) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\le d(\eta) + \int_0^\eta \mathtt{p}_1(\sigma)\Big[u(\sigma) + \int_0^\sigma \mathtt{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma. \qquad (50)
\end{aligned}
$$

Now an application of Lemma 2.2 by taking $e(\eta) = \frac{d(\eta)}{1-\mathcal{G}}$, in equation (50) becomes

.

$$
u(\eta) \le \frac{d(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{\mathtt{p}_1(\omega)}{(1-\mathcal{G})}\Big[1 + \int_0^\omega \mathtt{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big). \qquad (51)
$$

Case 2: For $1 \leq \eta < \infty$. By following a similar arguments as in Case 1 of the proof of Theorem 4.1 and from the hypotheses, it follows that

$$
\begin{aligned}
(1-\mathcal{G})u(\eta) &\leq [\overline{\delta} + \overline{\epsilon_2} + \overline{\epsilon_1}\eta] + \int_0^\eta \mathrm{p}_1(\sigma)\|\mathrm{x}(\sigma-1)) - y(\sigma-1))\|]d\sigma \\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)\Big[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\leq [\overline{\delta} + \overline{\epsilon_2} + \overline{\epsilon_1}\eta] + \int_0^1 \mathrm{p}_1(\sigma)\|\mathrm{x}(\sigma-1)) - y(\sigma-1))\|]d\sigma \\
&\quad + \int_1^\eta \mathrm{p}_1(\sigma)\|\mathrm{x}(\sigma-1)) - y(\sigma-1))\|]d\sigma \\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)\Big[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\leq (\overline{\delta} + \overline{\epsilon_2} + \overline{\epsilon_1}\eta) + \int_0^1 \mathrm{p}_1(\omega)\|\phi(\omega) - \psi(\omega)\|d\omega \\
&\quad + \int_1^\eta \mathrm{p}_1(\omega)|[\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega \\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq d(\eta) + \int_1^\eta \mathrm{p}_1(\omega)|[\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega \\
&\quad + \int_0^\eta \mathrm{p}_1(\sigma)[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma \\
&\leq d(\eta) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega]d\sigma + I_1, \qquad (52)
\end{aligned}
$$

where $I_1 = \int_1^\eta \mathrm{p}_1(\omega)[\|\mathrm{x}_1(\omega-1)) - \mathrm{x}_2(\omega-1))\|]d\omega$, by making change of variable, we obtain

$$
I_1 \leq \int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega. \tag{53}
$$

Using (53) in equation (52) becomes

$$
\begin{aligned}
(1-\mathcal{G})u(\eta) \leq d(\eta) + \int_0^\eta \mathrm{p}_1(\omega)|[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega + \\
\int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega
\end{aligned}
$$

$$
u(\eta) \leq \frac{d(\eta)}{(1-\mathcal{G})} + \int_0^\eta \frac{\mathrm{p}_1(\omega) + \mathrm{p}_1(\omega+1)}{1-\mathcal{G}}[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega) \tag{54}
$$

Using Lemma 2.2,and put $u(\eta) = \|\mathrm{x}(\eta) - y(\eta)\|$ in (54) ,we get

$$\|\mathrm{x}(\eta) - y(\eta)\| \leq \frac{d(\eta)}{1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{\mathrm{p}_1(\omega) + \mathrm{p}_1(\omega+1)}{1-\mathcal{G}}[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma]d\omega\Big)$$

From this inequality and (51), it follows that solutions of (4)–(6) depends continuously on the functions involved therein. □

Remark 4.3. *The result given in Theorem 4.2 related to solutions of IVP* (4)–(6) *and of IVP* (43)-(45) *in the sense that if $\mathcal{F}$ is close to F, x^* is close to y^*, ϕ is close to ψ and g is close to l, then the solutions of IVPs* (4)–(6) *and* (43)-(45) *are also close together.*

4.2. **Convergence of Solutions:** We consider the IVP (4)–(6) together with

$$y'(\eta) = \overline{\mathcal{F}}_n(\eta, y(\eta), \int_0^\eta \mathcal{K}(\eta,\omega,y(\omega))d\omega, y(\eta-1)), \tag{55}$$

$$y(\eta-1) = \psi_n(\eta), \quad y(0) + l_n(y) = c_n{}^*, \tag{56}$$

for $n = 1, 2, \cdots$, where $\overline{\mathcal{F}}_n \in C(\mathbb{R}_+ \times X \times X \times X, X)$, $l_n \in C(C(\mathbb{R}_+, X), X)$, $\mathcal{K} \in C(\mathbb{R}_+^2 \times X, X)$, $\mathbb{R})$ and for each $n = 1, 2, \cdots$, $\psi_k(\eta)$ is a continuous function for which the limit $\lim\limits_{\eta \to 1-0} \psi_n(\eta)$ exists.

As a direct result of the theorem 4.2, we have the following corollary.

Corollary 4.3.1. *Suppose that the hypotheses $(\mathcal{H}_1) - (\mathcal{H}_3)$ hold and there exist nonnegative constants $\overline{\epsilon_n}, \overline{\epsilon_n{}^*}$, $\overline{\delta}_n$ $(n = 1, 2, \cdots)$ such that*

$$\|\mathcal{F}(\eta, u, v, w) - \overline{\mathcal{F}}_n(\eta, u, v, w)\| \leq \overline{\epsilon_n}, \tag{57}$$

$$\|g(u) - l_n(u)\| \leq \overline{\epsilon_n{}^*}, \quad \|\mathrm{x}^* - c_n{}^*\| \leq \overline{\delta}_n, \tag{58}$$

with $\overline{\epsilon_n} \to 0$ and $\overline{\epsilon_n{}^*}$, $\overline{\delta}_n \to 0$ as $n \to \infty$, where x^*, g, ϕ, $\mathcal{F}$ and $c_n{}^*$, l_n, ψ_n, $\overline{\mathcal{F}}_n$ are as in (4)–(6) *and* (55)–(56). *If $y_n(\eta)$ $(n = 1, 2, \cdots)$ and $\mathrm{x}(\eta)$ are respectively the solutions of* (55)–(56) *and* (4)–(6) *on $\mathbb{R}_+$, then $y_n(\eta) \to \mathrm{x}(\eta)$ as $n \to \infty$ on $\mathbb{R}_+$.*

Proof. For $n = 1, 2, \cdots$, the conditions of of Theorem 4.2 fulfilled. As an application of Theorem 4.2 and Lemma 2.2 yields

$$\|y_n(\eta) - \mathrm{x}(\eta)\| \leq \frac{\overline{c_n}}{(1-\mathcal{G}} \exp\Big(\int_0^\eta \frac{\mathrm{p}_1(\omega)}{(1-\mathcal{G})}\Big[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big) \tag{59}$$

for $0 \leq \eta < 1$ and

$$\|y_n(\eta) - x(\eta)\| \leq \frac{\overline{c_n}}{(1-\mathcal{G})} \exp\Big(\int_0^\eta \frac{p_1(\omega) + p_1(\omega+1)}{1-\mathcal{G}}[1 + \int_0^\omega p_2(\omega,\sigma)d\sigma]d\omega\Big)$$

for $1 \leq \eta < \infty$, where

$$\overline{c_n} = \max\Big\{[\overline{\delta_n} + \overline{\epsilon_n}^* + \overline{\epsilon_n}\eta] + \int_0^1 p_1(\omega)\|\phi(\omega) - \psi_n(\omega)\|d\omega\Big\}.$$

The necessary results derived from Theorem 4.2. It follows that (4)–(6) depends continuously on the functions involved therein. □

Remark 4.4. *The result obtained in above Corollary provide sufficient conditions that ensure solutions of IVPs* (55)–(56) *will converge to the solution of IVP* (43)-(45) *.*

4.3. **Dependency on parameters:** Next, we consider difference-differential equations

$$x'(\eta) = \overline{\mathcal{F}}(\eta, x(\eta), \int_0^\eta k(\eta,\omega,x(\omega))d\omega, x(\eta-1), \mu_1), \tag{60}$$

$$x'(\eta) = \overline{\mathcal{F}}(\eta, x(\eta), \int_0^\eta k(\eta,\omega,x(\omega))d\omega, x(\eta-1), \mu_2), \tag{61}$$

for $\eta \in \mathbb{R}_+$, where $\overline{\mathcal{F}} \in C(\mathbb{R}_+ \times X \times X \times X \times \mathbb{R}, X)$, and with the initial conditions given by (5)–(6).

The following theorem states the continuous dependence of solutions to (60) and (61) with initial conditions given by (5)–(6) on parameters.

Theorem 4.5. *Assume that hypotheses* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold and there exists an increasing function* $L_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\|\overline{\mathcal{F}}(\eta, x, y, z, \mu_1) - \overline{\mathcal{F}}(\eta, \bar{x}, \bar{y}, \bar{z}, \mu_2)\| \leq L_1\Big[\|x - \bar{x}\| + \|y - \bar{y}\| + \|z - \bar{z}\| + |\mu_1 - \mu_2|\Big].$$

Let $x_1(\eta)$ *and* $x_2(\eta)$ *be solutions of* (60) *with* (5)–(6) *and* (61) *with* (5)–(6) *respectively. Then*

$$\|x(\eta) - y(\eta)\| \leq \Big(\frac{L_1|\mu_1 - \mu_2|\eta}{(1-\mathcal{G})}\Big) \exp\Big(\int_0^\eta \frac{L_1}{(1-\mathcal{G})}\Big[1 + \int_0^\omega p_2(\omega,\sigma)d\sigma\Big]d\omega\Big),$$

for $0 \leq \eta < 1$ *and*

$$\|x(\eta) - y(\eta)\| \leq \Big(\frac{L_1|\mu_1 - \mu_2|\eta}{(1-\mathcal{G})}\Big) \exp\Big(\int_0^\eta \frac{2L_1}{(1-\mathcal{G})}\Big[1 + \int_0^\omega p_2(\omega,\sigma)d\sigma\Big]d\omega\Big),$$

for $1 \leq \eta < \infty$. *and* $\mathcal{G} < 1$

Proof. Let $u(\eta) = \|x_1(\eta) - x_2(\eta)\|$ for $\eta \in \mathbb{R}_+$. We have

$$\begin{aligned}
u(\eta) &\leq \|g(x) - g(y)\| \\
&\quad + \| \int_0^\eta [\overline{\mathcal{F}}(\sigma, x(\sigma), \int_0^\sigma k(\sigma,\omega,x(\omega))d\omega, x(\sigma-1), \mu_1) \\
&\quad - \overline{\mathcal{F}}(\sigma, y(\sigma), \int_0^\sigma k(\sigma,\omega,y(\omega))d\omega, y(\sigma-1), \mu_2)]d\sigma\| \\
&\leq \mathcal{G}\|x - y\| \\
&\quad + \int_0^\eta [L_1\Big[\|x(\sigma) - y(\sigma)\| + \int_0^\sigma p_2(\sigma,\omega)\|x(\omega) - y(\omega)\|d\omega \\
&\quad + \|x(\sigma-1)) - y(\sigma-1))\| + |\mu_1 - \mu_2|\Big]d\sigma \\
&\leq \mathcal{G}u(\eta) \\
&\quad + \int_0^\eta [L_1\Big[u(\sigma) + \int_0^\sigma p_2(\sigma,\omega)u(\omega)d\omega + \|x(\sigma-1)) - y(\sigma-1))\| + |\mu_1 - \mu_2|\Big]d\sigma \\
&\leq \mathcal{G}u(\eta) \\
&\quad + \int_0^\eta [L_1\|x(\sigma-1)) - y(\sigma-1))\|d\sigma + L_1|\mu_1 - \mu_2|\eta \\
&\quad + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma p_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma.
\end{aligned}$$

Think about each of the following two cases..

Case 1: For $0 \leq \eta < 1$ From the hypotheses, we have

$$\begin{aligned}
(1-\mathcal{G})u(\eta) &\leq \int_0^1 L_1\|\phi(\sigma) - \phi(\sigma)\|d\sigma + L_1|\mu_1 - \mu_2| \\
&\quad + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma p_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\leq L_1|\mu_1 - \mu_2|\eta + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma p_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma.
\end{aligned}$$

$$\Rightarrow u(\eta) \leq \Big(\frac{L_1|\mu_1 - \mu_2|\eta}{(1-\mathcal{G})}\Big) + \Big(\int_0^\eta \frac{L_1}{(1-\mathcal{G})}\Big[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega\Big). \tag{62}$$

Using Lemma 2.2 in equation (62) becomes

$$u(\eta) \leq \Big(\frac{L_1|\mu_1 - \mu_2|\eta}{(1-\mathcal{G})}\Big)\exp\Big(\int_0^\eta \frac{L_1}{(1-\mathcal{G})}\Big[1 + \int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big).$$

So ,we get

$$\|x(\eta)-y(\eta)\| \leq \Big(\frac{L_1|\mu_1-\mu_2|\eta}{(1-\mathcal{G})}\Big)\exp\Big(\int_0^\eta \frac{L_1}{(1-\mathcal{G})}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big),$$

Case 2: For $1 \leq \eta < \infty$.

$$\begin{aligned}
u(\eta) &\leq \mathcal{G}u(\eta) + \int_0^\eta [L_1\|x(\sigma-1)) - y(\sigma-1))\|d\sigma \\
&\quad + L_1|\mu_1-\mu_2|\eta + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma \\
&\leq \mathcal{G}u(\eta) + L_1|\mu_1-\mu_2|\eta \\
&\quad + \int_0^1 [L_1\|\phi(\sigma)-\phi(\sigma)\|d\sigma + \int_1^\eta \|x(\omega-1)-y(\omega-1)\|d\omega \\
&\quad + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma.
\end{aligned}$$

$$\Rightarrow u(\eta) \leq \mathcal{G}u(\eta) + 0 + I_1 + L_1|\mu_1-\mu_2|\eta + \int_0^\eta L_1[u(\sigma) + \int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma.$$

where $I_1 = \int_1^\eta L_1\|x(\omega-1)-y(\omega-1)\|d\omega$. By changing the variable, we get

$$I_1 = L_1\int_0^\eta u(\omega)d\omega. \tag{63}$$

Using this in above inequality, we get

$$u(\eta) \leq \mathcal{G}u(\eta) + L_1|\mu_1-\mu_2|\eta + \int_0^\eta L_1[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega + \int_0^\eta L_1 u(\omega)d\omega.$$

$$(1-\mathcal{G})u(\eta) \leq L_1|\mu_1-\mu_2|\eta + \int_0^\eta 2L_1[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega$$

$$\Rightarrow u(\eta) \leq \frac{L_1|\mu_1-\mu_2|\eta}{(1-\mathcal{G})} + \int_0^\eta \frac{1}{(1-\mathcal{G})}2L_1[u(\omega) + \int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega. \tag{64}$$

Using Lemma 2.2, in equation (64) becomes

$$u(\eta) \leq \Big(\frac{L_1|\mu_1-\mu_2|\eta}{(1-\mathcal{G})}\Big)\exp\Big(\int_0^\eta \frac{2L_1}{(1-\mathcal{G})}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big),$$

$$\Rightarrow \|x(\eta)-y(\eta)\| \leq \Big(\frac{L_1|\mu_1-\mu_2|\eta}{(1-\mathcal{G})}\Big)\exp\Big(\int_0^\eta \frac{2L_1}{(1-\mathcal{G})}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big),$$

From above, it follows that the solutions (60) with (5)–(6) and (61) with (5)–(6) depend continuously on the parameters μ_1, μ_2. □

Remark 4.6. *the result touching the property of a solution called "dependence of solutions on parameters" . Here parameters are scalars. Notice that initial conditions are detached from parameters . The dependence on parameters are an important aspect in various physical problems.*

A small change of Theorem 4.2 is given in the following theorem

Theorem 4.7. *Suppose that*

$$\|\mathcal{F}(\eta, x, y, z) - \overline{\mathcal{F}}(\eta, \bar{x}, \bar{y}, \bar{z})\| \leq p(\eta)\Big[\|x - \bar{x}\| + \|y - \bar{y}\| + \|z - \bar{x}\|\Big],$$

$$\|g(u) - l(v)\| \leq \mathcal{G}_1\|u\text{-}v)\|,$$

$$\|x^* - y^*\| \leq \overline{\delta},$$

where an increasing function $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ and hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_3) - (\mathcal{H}_5)$. Let $x(\eta)$ and $y(\eta)$ be respectively, solutions of (4)–(6) *and* (43)-(45) *on $\mathbb{R}_+$. Then*

$$\|x(\eta) - y(\eta)\| \leq \Big(\frac{\overline{c}}{1 - \mathcal{G}_1}\Big) \exp\Big(\int_0^{\eta} \frac{p(\omega)}{1 - \mathcal{G}_1}\Big[1 + \int_0^{\omega} p_2(\omega, \sigma)d\sigma\Big]d\omega\Big),$$

for $0 \leq \eta < 1$ and

$$\|x(\eta) - y(\eta)\| \leq \Big(\frac{\overline{c}}{(1 - \mathcal{G}_1)}\Big) \exp\Big(\int_0^{\eta} \frac{(p(\omega) + p(\omega + 1))}{(1 - \mathcal{G}_1)}\Big[1 + \int_0^{\omega} p_2(\omega, \sigma)d\sigma\Big]d\omega\Big),$$

for $1 \leq \eta < \infty$, where

$$\overline{c} = \overline{\delta} + \int_0^1 p(\omega)\|\phi(\omega) - \psi(\omega)\|d\omega.$$

and $\mathcal{G}_1 < 1$

Proof. Let $u(\eta) = \|x(\eta) - y(\eta)\|$ for $\eta \in \mathbb{R}_+$. We have

$$\begin{aligned}
\|x(\eta) - y(\eta)\| \leq \overline{\delta} + \|g(x) - g(y)\| + \| \int_0^{\eta} [\mathcal{F}(\sigma, x(\sigma), \int_0^{\sigma} k(\sigma, \omega, x(\omega))d\omega, x(\sigma - 1))& \\
- F(\sigma, y(\sigma), \int_0^{\sigma} k(\sigma, \omega, y(\omega))d\omega, y(\sigma - 1)]d\sigma\|& \\
\leq \overline{\delta} + \mathcal{G}_1\|x - y\| + \int_0^{\eta} p(\sigma)\Big[\|x(\sigma) - y(\sigma)\|& \\
+ \int_0^{\sigma} p_2(\sigma, \omega)\|x(\omega) - y(\omega)\|d\omega + \|x(\sigma - 1)) - y(\sigma - 1))\|\Big]d\sigma& \\
\leq \overline{\delta} + \mathcal{G}_1 u(\eta)&
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\eta \mathrm{p}(\sigma)\Big[u(\sigma)+\int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega+\|\mathrm{x}(\sigma-1))-y(\sigma-1))\|\Big]d\sigma\\
&\le \overline{\delta}+\mathcal{G}_1 u(\eta)+\int_0^\eta \mathrm{p}(\omega)\|\mathrm{x}(\omega-1))-y(\omega-1))\|d\omega\\
&+\int_0^\eta \mathrm{p}(\sigma)\Big[u(\sigma)+\int_0^\sigma \mathrm{p}_2(\sigma,\omega)u(\omega)d\omega\Big]d\sigma\\
&\le \overline{\delta}+\mathcal{G}_1 u(\eta)+\int_0^\eta \mathrm{p}(\omega)\|\mathrm{x}(\omega-1))-y(\omega-1))\|d\omega\\
&+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega.
\end{aligned}
$$

Consider the following two cases.

Case 1: For $0 \le \eta < 1$. From the hypotheses, we have

$$
\begin{aligned}
(1-\mathcal{G}_1)u(\eta) &\le \overline{\delta}+\int_0^1 \mathrm{p}(\omega)\|\mathrm{x}(\omega-1))-y(\omega-1))\|d\omega\\
&\quad+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega\\
&\le \Big[\overline{\delta}+\int_0^1 \mathrm{p}(\omega)\|\phi(\omega)-\psi(\omega)\|d\omega\Big]\\
&\quad+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega\\
&\le \overline{c}+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega
\end{aligned}
$$

$$
\Rightarrow u(\eta)\le \frac{\overline{c}}{1-\mathcal{G}_1}+\int_0^\eta \frac{\mathrm{p}(\omega)}{1-\mathcal{G}_1}\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega. \tag{65}
$$

Using Lemma 2.2, in equation (65) becomes

$$
u(\eta)\le\Big(\frac{\overline{c}}{1-\mathcal{G}_1}\Big)\exp\Big(\int_0^\eta \frac{\mathrm{p}(\omega)}{1-\mathcal{G}_1}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big), \tag{66}
$$

$$
\Rightarrow \|\mathrm{x}(\eta)-y(\eta)\|\le\Big(\frac{\overline{c}}{1-\mathcal{G}_1}\Big)\exp\Big(\int_0^\eta \frac{\mathrm{p}(\omega)}{1-\mathcal{G}_1}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big),
$$

Case 2: For $1 \le \eta < \infty$.

$$
\begin{aligned}
(1-\mathcal{G}_1)u(\eta) &\le \overline{\delta}+\int_0^\eta \mathrm{p}(\omega)\|\mathrm{x}(\omega-1))-y(\omega-1))\|d\omega\\
&\quad+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega\\
&\le \overline{c}+\int_1^\eta \mathrm{p}(\omega)\|\mathrm{x}(\omega-1))-y(\omega-1))\|d\omega
\end{aligned}
$$

$$+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega$$
$$\leq \overline{c}+I_1+\int_0^\eta \mathrm{p}(\omega)\Big[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma\Big]d\omega.$$

where $I_1=\int_1^\eta \mathrm{p}(\omega)\|x(\omega-1))-y(\omega-1))\|d\omega$. By making the change of variable,we obtain

$$I_1=\int_0^\eta \mathrm{p}(\omega+1)u(\omega)d\omega.$$

Using this equation above inequality becomes

$$(1-\mathcal{G})u(\eta)\leq \overline{c}+\int_0^\eta \mathrm{p}_1(\omega)[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega+\int_0^\eta \mathrm{p}_1(\omega+1)u(\omega)d\omega$$
$$\leq \overline{c}+\int_0^\eta (\mathrm{p}_1(\omega)+\mathrm{p}_1(\omega+1))[u(\omega)+\int_0^\omega \mathrm{p}_2(\omega,\sigma)u(\sigma)d\sigma]d\omega. \tag{67}$$

Using Lemma 2.2 in equation (67) becomes

$$u(\eta)\leq\Big(\frac{\overline{c}}{(1-\mathcal{G}_1)}\Big)\exp\Big(\int_0^\eta \frac{(\mathrm{p}(\omega)+\mathrm{p}(\omega+1))}{(1-\mathcal{G}_1)}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big).$$
$$\Rightarrow \|x(\eta)-y(\eta)\|\leq\Big(\frac{\overline{c}}{(1-\mathcal{G}_1)}\Big)\exp\Big(\int_0^\eta \frac{(\mathrm{p}(\omega)+\mathrm{p}(\omega+1))}{(1-\mathcal{G}_1)}\Big[1+\int_0^\omega \mathrm{p}_2(\omega,\sigma)d\sigma\Big]d\omega\Big)$$

□

5. Conclusion:

We have studied the first result regarding $\epsilon-$approximate solutions as fundamental result and secondly, we established uniqueness of solution without existence part. In the succession, we have discussed some qualitative properties such as continuous dependence, closeness and convergence of solutions of a certain integro- differential equation with finite delay with nonlocal condition in Banach Space with the contribution of integral inequality founded by B. G. Pachpatte.

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