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SOME PROPERTIES OF BASES INTERSECTION GRAPH

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Abstract. The objective of this paper is to introduce and provide an overview for graphical representation of a vector space $\mathbb{V}$ over a field $\mathbb{F}$. The fundamental characteristics of the graph, such as connectivity, diameter, girth, and clique number have also be examined.

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1. Introduction

In the year 1988, I. Beck[3] initiated the study of zero divisor graph(ZDG). Subsequently, Anderson and Livingston[2] provided a brief overview of the properties of zero divisor graph(ZDG) over commutative ring with a unit element. Since then, numerous researchers have investigated the zero divisor graph on different algebraic structures including semi-group, group, ring, field, and vector space(VS) etc. In recent times, S. Akabari and M. Habibi[1] have focused on studying the zero divisor graph(ZDG) on ideals of the ring, while A. Das explored the ZDG on various

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algebraic structures, such as the Non-Zero Component(NZC) Union Graph of a VS having finite dimension [4].The study of algebraic structures and their properties with the aid of graphs has gained significant interest in recent years. For instance, NZC Union Graph of a Finite Dimensional VS[5], Subspace Inclusion Graph of VS [6], and Non-Zero Component Graph(NZC) of a Finite Fields[7] are among the structures that have been characterized using graphs. Muneshwar Et al. [8] introduced the inclusion graph of an open subset over topological space and studied its properties. In a subsequent work [9] they introduced and proved several results on the concept of topological space intersection graph , including the connectivity and determination of it's diameter and girth. They also showed that for a discrete topological space (X, τ) with $|X| \geq 3$, the chromatic and clique number of graph are equal.Additional, they determined domination number, independence number, degree of vertices etc. They also introduced the union graph of a topological space [10] and proved several results, including the connectivity and determination of its diameter and girth. They further showed that for a discrete topological space (X_i, τ) with $|X| \geq 4$, the chromatic and clique number of the union graph are equal to the dimension of VS i.e n. They also determined different variants of domination numbers, independence numbers, and degrees of intersection graphs.

In this paper, we aim to define a Bases Intersection Graph of a vector space(VS) over a finite field.

2. Definition and Preliminaries:

This section provides some fundamental definitions related to bases intersection graph. A graph $G = (V, E)$ is an ordered pair where E is the set of edges and V is the set of vertices, which forms a binary relation on V. Any two vertices u and v of V if there exists an edge between them, then they are said to be adjacent. A subgraph $H = (U, F)$ of $G = (U', K)$ is a graph where $\phi \neq U \subseteq U'$ and $F \subseteq K$. If V is a finite set, then G is called a finite graph; otherwise, it is called an infinite graph. A graph G is complete if all vertices of G are adjacent pairwise. A clique is complete subgraph of the graph G, and the one which is maximal with respect to inclusion is called maximal clique. The largest size of a clique in G is the clique number of G, denoted by $\omega(G)$. Consequently $\chi(G)$ represents the chromatic number of G, which is nothing but the least number of colors required for labeling the vertices of

G in such a way that different colors was given to vertices which are adjacent. "A triangulated graph is graph in which every cycle of length four or more has a chord, which is an edge connecting two non adjacent vertices of the cycle.The diameter of a graph is the maximum distance between any two vertices in the graph, where the distance between two vertices is defined as the length of the shortest path between them. In other words, diameter is the longest shortest path in the graph if it is exist otherwise it is defined as ∞.The girth of a graph is defined as the length of its smallest cycle. If the graph has no cycles, it is considered acyclic and its girth is defined as ∞.For readers who are not already familiar with the foundational concepts and terminology related to the topic discussed in this article, it is strongly recommended that they seek assistance from external sources such as [11] and [12]. These sources can provide a more comprehensive explanation of the basic principles and terminology that the article assumes the reader has prior knowledge.

3. Bases Intersection graph:

Definition 3.1. *Let $\mathbb{V}$ be any finite dimensional vector space over a field F. We define a Bases Intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V}) = (\mathcal{V}, E)$, where $\mathcal{V}$ = the collection of all bases of $\mathbb{V}$ over F and for $B_1, B_2 \in \mathcal{V}, B_1 \sim B_2$ or $(B_1, B_2) \in E$ if and only if $B_1 \cap B_2 \neq \phi$.*

Throughout this paper, it is understood that the underline field is F and $\mathbb{V}$ is a finite dimensional vector space.
Now we study some basic properties of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.

Theorem 3.2. *If $\mathbb{V}$ is a vector space over a field F and W is a subspace of $\mathbb{V}$ then $\mathcal{I}_{\mathcal{B}}(W)$ is a subgraph of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.*

Proof. Let $\mathbb{V}$ is a vector space over a field F and W is a subspace of $\mathbb{V}$. Let B_1 and B_2 are bases of W such that $B_1 \cap B_2 \neq \phi$, therefore by definition of $\mathcal{I}_{\mathcal{B}}(W)$, $B_1 \sim B_2$. Now as B_1 and B_2 are bases of W, they contained linearly independent vectors of W and hence $\mathbb{V}(\because W \subseteq \mathbb{V})$. Therefore by bases extension theorem, these linearly independent set B_1 and B_2 can be extended to form a bases $B_1'$and B_2' of $\mathbb{V}$ such that $B_1' \cap B_2' \neq \phi$, and hence $B_1' \sim B_2'$. This proves that $\mathcal{I}_{\mathcal{B}}(W)$ is a subgraph of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$. □

Corollary 3.3. *If $\mathbb{V}$ is a vector space over a field F and $dim(\mathbb{V}) = 1$, then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is an null graph.*

Proof. We have $dim(\mathbb{V}) = 1$ then every basis of $\mathbb{V}$ contains only one nonzero vector. Thus any two distinct bases B_i, B_j of a vector space $\mathbb{V}$ dose not have any vector in common, i.e $B_i \cap B_j = \phi$, for all $i \neq j$ and $B_i, B_j \in B$. This shows that $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is null graph. □

Corollary 3.4. *If $\mathbb{V}$ is a vector space over a field F then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not complete graph.*

Proof. Let $\mathbb{V}$ is a vector space over a field F and $dim(\mathbb{V}) = n$. If $dim(\mathbb{V}) = 1$, then it is obvious from Corollary 3.3 . Let $dim(\mathbb{V}) = n > 1$, and let $B_1 = \{e_1, e_2, e_3, \ldots, e_n\}$ be any basis of $\mathbb{V}$. thus $B_2 = \{e_1 + e_2, e_2 + e_3, \ldots, e_n + e_1\}$ is also a basis of $\mathbb{V}$, such that $B_1 \cap B_2 = \phi$, and hence $B_1 \not\sim B_2$. This shows that $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not complete graph. □

4. Diameter and girth of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$:

In this section, we will check the connectivity, diameter and girth of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.

Lemma 4.1. *Let $\mathbb{V}$ be a finite dimensional vector space over a field F and $dim(\mathbb{V}) \geq 2$, then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is connected and $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) \geq 2$.*

Proof. Let $dim(\mathbb{V}) = n \geq 2$, and let $B_1 = \{u_1, u_2, \ldots, u_n\}, B_2 = \{v_1, v_2, \ldots, v_n\}$ are bases of $\mathbb{V}$. Then we have two cases.
Case 1 : If $B_1 \cap B_2 \neq \phi$, then by definition of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$, we have, $B_1 \sim B_2$ be a required path from B_1 to B_2 and $dist(B_1, B_2) = 1$.
Case 2 : If $B_1 \cap B_2 = \phi$, then there exists $u_i \in B_1$, and $v_j \in B_2$, for some i, j, such that, $u_i \neq \alpha v_j$, and hence u_i and v_j are linearly independent vectors of $\mathbb{V}$. Therefore by bases extension theorem, we can construct a basis B_3 as, $B_3 = \{u_i, v_j, w_3, w_4, w_5, \ldots, w_n\}$ such that, $B_1 \cap B_3 = u_i \neq \phi$ and $B_2 \cap B_3 = v_j \neq \phi$ and hence $B_1 \sim B_3$ and $B_2 \sim B_3$. Thus $B_1 \sim B_3 \sim B_2$ be a required path between B_1 and B_2 and $dist(B_1, B_2) = 2$.
Hence $\mathcal{I}_{\mathcal{B}}(V)$ is connected and $diam(\mathcal{I}_{\mathcal{B}}(V)) \geq 2$. □

Theorem 4.2. *If $\mathbb{V}$ is a vector space over a field F and $dim(\mathbb{V}) \geq 2$, then $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = 2$.*

Proof. Let B_1 and B_2 be any two bases then by the lemma 4.1, either $B_1 \sim B_2$ or there exist $B_3 \in \mathcal{B}$ such that, $B_1 \sim B_2 \sim B_3$. Therefore $dist(B_1, B_2) \leq 2$ and hence $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = 2$. □

Theorem 4.3. *Let $\mathbb{V}$ be an n-dimensional vector space over a field F, then*

$$girth(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \begin{cases} 3 & \text{if } n > 1. \\ \infty & \text{if } n = 1. \end{cases}$$

Proof. Let $\mathbb{V}$ be an n-dimensional vector space over a field F.

Case-I: If $n = 1$ then by Corollary 3.2, the graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is null graph and hence $girth(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \infty$.

Case -II: If $n > 1$, then consider $B_1 = \{u_1, u_2, u_3, \ldots, u_n\}$ be a basis of $\mathbb{V}$, then clearly $u_1, u_2, u_3, \ldots, u_n$ are linearly independent vectors of $\mathbb{V}$. In particular, the set $\{u_1\}$ is linearly independent, so by basis extension theorem, we can extend this up to a basis $B_2 = \{u_1, v_2, v_3, \ldots, v_n\}$ of V. Then clearly $B_1 \cap B_2 = \{u_1\} \neq \phi$, and hence $B_1 \sim B_2$. Choose some u_i in B_1 such that $u_i \neq \alpha v_k, \forall k = 2, 3, \ldots, n$, and let it be u_1 and v_2 in particular, then clearly u_1 and v_2 are linearly independent vectors in $\mathbb{V}$. Then by bases extension theorem, we can form a basis B_3 as, $B_3 = \{u_1, v_2, w_3, w_4, \ldots, w_n\}$ such that$B_1 \cap B_2 \neq \phi$, $B_1 \cap B_3 \neq \phi$, $B_2 \cap B_3 \neq \phi$, and hence $B_1 \sim B_2 \sim B_3 \sim B_1$ be the required cycle of length 3.

□

Theorem 4.4. *Let $dim(\mathbb{V}) > 1$, then $I_{\mathcal{B}}(\mathbb{V})$ is triangulated graph.*

Proof. Proof of this theorem follows obviously from Theorem 4.2. □

5. Clique Number of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$

In this section, we will find clique and chromatic number of $\mathcal{I}_{\mathcal{B}}(V)$.

Theorem 5.1. *Let $\mathbb{V}$ be a n-dimensional vector space over a finite field F of order p . Let K be a collection of all bases of a vector space $\mathbb{V}$ having one vector in common then K be the maximal clique of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ having clique number*

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1})}{(n-1)!}$$

Proof. Let $\mathbb{V}$ be a finite dimensional vector space over a field F and B be a collection of all bases of a vector space $\mathbb{V}$. Let K is a collection of all bases of a vector space $\mathbb{V}$ having one vector in common.

$$i.e. K = \{B_i | \alpha \in B_i \forall i\}\, i.e. \cap B_i = \{\alpha\} \tag{1}$$

i.e. $K = \{B_i | B_i \cap B_j = \{\alpha\} \neq \phi, \forall i \neq j\}$ then $B_i \cap B_j = \{\alpha\} \neq \phi$ and hence $B_i \sim B_j, \forall B_i, B_j \in K$. Thus K is a clique in $\mathcal{I}_{\mathcal{B}}(V)$. We know that, if V be a finite dimensional vector space over a field F of order p then the number of unordered basis of V is

$$\frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{n!}$$

.

We have to find the clique number, as clique is the collection of those bases having one vector in common.

$\therefore$ Number of possibilities of selecting one vector in common is $\frac{p^n-1}{n}$

$\therefore$ The total number of basis having one vector in common is given by

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{\frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{n!}}{\frac{p^n-1}{n}} \tag{2}$$

$$\therefore \omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{(n-1)!} \tag{3}$$

Which is the required clique number of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.

Now as $K = \{B_i | \alpha \in B_i \forall i\}\, i.e. \cap B_i = \{\alpha\}$ We will show this K is a maximal clique. If possible suppose $K' = K \cup \{B\}$ is a clique where $B \notin K$ i.e. $\alpha \notin B$. As $B \notin K'$ and K' is a clique. $\Rightarrow B \cap B_i \neq \phi \forall i$ therefore there exist some $\gamma \neq$ such that $\gamma \in B \cap B_i \forall i \Rightarrow \gamma \in B$ and $\gamma \in B_i, \forall i \Rightarrow \alpha, \gamma \in B_i \forall i. \Rightarrow K$ contains two vectors in common. Number of possibilities of selecting two vectors in common is $\frac{(p^n-1)(p^n-p)}{n(n-1)}$

$\therefore$ The total number of basis having one vector in common is given by

$$\frac{\frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{n!}}{\frac{(p^n-1)(p^n-p)}{n(n-1)}}$$

which is the number less than number given in equation (3), the contradiction, hence K' is not a clique. □

Corollary 5.2. *Let $\mathbb{V}$ be an n-dimensional vector space over a finite field of order P with $n > 1$ and $p > 2$ then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not a planar graph.*

Proof. Let $\mathbb{V}$ be a finite dimensional vector space over a field F of order p then

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{(n-1)!}$$

Then for $n > 1$ and $p > 2$, $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ has a clique of size at least 6," i.e. there exist a subgraph which is isomorphic to K_5 or $K_{3,3}$. Thus it is not planar. □

6. Conclusion

The article introduces a bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ of a finite dimensional vector space $\mathbb{V}$ and studied relationship between $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ and $\mathbb{V}$ as a vector space. Discussion contains the fundamental properties of graph such as connectedness, subgraph, completeness etc. clique number, planarity property etc. Further discussion contains girth and diameter of graph and found it 2 and 3 respectively for VS of $dim(\mathbb{V}) \geq 2$. In addition clique number is found

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{(n-1)!}$$

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