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ON CHARACTERIZATIONS ON THE SURFACES OF ROTATION FORMED BY MAGNETIC CURVES IN $\mathbf{Q}^2 \subset E_1^3$

FATMA ALMAZ[1] AND MİHRİBAN ALYAMAÇ KÜLAHCI[2]

[1]DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, FIRAT UNIVERSITY, 23119 ELAZ/TRK˙YE

[2]DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, FIRAT UNIVERSITY, 23119 ELAZ/TRK˙YE

E-MAILS: [1]FB_FAT_ALMAZ@HOTMAIL.COM AND [2]MIHRIBANKULAHCI@GMAIL.COM

Abstract. In this paper, the different magnetic fields on the setting out particle trajectories according to the magnetic flow attached by the Killing magnetic field are given in $\mathbf{Q}^2 \subset E_1^3$. The surfaces generated by the Killing magnetic field of the $y-$magnetic curve are expressed according to asymptotic orthonormal frame field $\{\eta, \beta, y\}$ in $\mathbf{Q}^2 \subset E_1^3$ and, using Clairaut's theorem the conditions being geodesic on these surfaces of rotation are given. Also, the curvatures and the kinetic energy and angular momentum of the surfaces of rotation formed by $y-$magnetic curve are examined.

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1. Introduction

The geometry of rotational surfaces in Euclidean 3-space has been studied widely. The surfaces of rotation in 3D Euclidean space are generated by rotating an arbitrary axis. If the geodesic curvature of the curve is zero it is said to be geodesic. Physically, the geodesic equations are expressed by constant of motion due to energy. Moreover, the equation of action for the energy and the angular momentum is a natural topic using by many applications. Though the mathematicians study very much concerning the submanifolds of the pseudo-Riemannian space forms, there are a few studies on submanifolds of the pseudo-Riemannian lightlike cone. In [1], the authors expressed the magnetic curves in the 2D lightlike cone considering the Killing magnetic field. [2], the authors characterized the magnetic fields on the moving particle trajectories using the magnetic flow associated with the Killing magnetic field in Q^2. [3], the magnetic field on the setting out particle orbit was investigated in Q^2. Clearly, dealing with the Killing magnetic field of α-magnetic curve, the rotational surface generated by α-magnetic is expressed in 3D null cone and the variant kinds of axes of rotation in lightlike cone was given. Furthermore, the specific kinetic energy, the specific angular momentum and conditions being geodesic on rotational surface were expressed with the help of Clairaut's theorem. Studying on the degenerate submanifolds of Lorentzian manifolds generated by degenerate metric supplies to the mathematicians significant appropriateness between null submanifolds and spacetime [4]. In [5], the author classified the magnetic curves (also $N-$magnetic curves that its curvature is constant) in a Galilean $3-$space associated to a Killing vector field. In [6, 7], Barros and Romeo proved the magnetic curves on a Riemannian manifold (L, d) expressed trajectories. Also, each trajectory η was found by calculating the equation $\nabla_{\eta'}\eta' = \Phi(\eta')$, where Φ is the Lorentz force, [7]. In [8], the magnetic flow associated with the Killing magnetic field was examined in orientated Riemannian manifold (L^3, d) by Bozkurt and et al. In [9], the magnetic curves in respect of the Killing magnetic field W were investigated in E_1^3 by the authors. In [11], the first kind and second kind osculating curves were given in the lightlike cone by authors. Furthermore, they characterized osculating curves in terms of their curvature functions. In [15], the cone curves were studied and the some characterizations for the cone curvature function were given, also some samples of cone curves were given in E_1^3. In [16], for the curves in $\mathbf{Q}^2$ and

$\mathbf{Q}^3$, the expressions of formulas were given. In the reference [21] was given some knowledge about F on L being a closed $2-$form on (L, d), where it is related to F with a $(1, 1)-$tensor field. It is written as $d(\Phi(A), B) = F(A, B)$, $\forall A, B \in L$.

To understand the comparison with Newton's theory or the transition to quantum theory, those physical terms as energy and momentum can be replaced by the specific quantities obtained by partitioning out the mass. Therefore, specific energy of the particle is constant due to references [22, 23]. Also, it is perpendicular to the velocity of the particle its the energy, the speed and specific energy E must be constant.

In this paper, we expressed the specific energy and specific angular momentum on surfaces of rotation using by $y-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$ and with the help of Clairaut's theorem we obtained along a geodesic, the speed is constant .

2. Preliminaries

Let E_1^3 be the pseudo-Euclidean space by

$$d(A, B) = \langle A, B\rangle = a_1b_1 + a_2b_2 - a_3b_3,$$

for $\forall\ A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3) \in E_1^3$. Let L be a submanifold of E_1^3 and if the pseudo-Riemannian metric d of E_1^3 reduces metric $\widetilde{d}$(respectively, a Riemannian metric) on L, then L is called as a time-like (respectively, space-like, degenerate) of E_1^3, [11].

The null cone is expressed as follows

$$\mathbf{Q}^2 = \left\{\eta \in E_1^3 : d(\eta, \eta) = 0\right\},$$

and a vector $D \neq 0$ in E_1^3 is said to be space-like, time-like, null, if $\langle D, D\rangle > 0$, $\langle D, D\rangle < 0$, $\langle D, D\rangle = 0$, respectively [15]. A frame field $\{\eta, \beta, y\}$ on E_1^3 is said to be as asymptotic orthonormal frame field given as

$$\langle \eta, \eta\rangle = \langle y, y\rangle = \langle \eta, \beta\rangle = \langle y, \eta\rangle = 0, \langle \eta, y\rangle = \langle \beta, \beta\rangle = 1,$$

and that curve $\eta : I \to \mathbf{Q}^2 \subset E_1^3$ is a regular curve in $\mathbf{Q}^2, \forall\ \vartheta \in I$, taking $\eta'(\vartheta) = \beta(\vartheta)$. Then, the cone Frenet formulas of $\eta(\vartheta)$ are given as

$$\begin{aligned} \eta'(\vartheta) &= \beta(\vartheta) \\ \beta'(\vartheta) &= \kappa(\vartheta)\eta(\vartheta) - y(\vartheta) \\ y'(\vartheta) &= -\kappa(\vartheta)\beta(\vartheta), \end{aligned} \tag{2.1}$$

where the function $\kappa(\vartheta)$ is the cone curvature function of $\eta(\vartheta)$, [15].

Let $\eta : I \to \mathbf{Q}^2 \subset E_1^3$ be a space-like curve that it has arc length parameter s in $\mathbf{Q}^2$. Hence, $\eta(s) = (\eta_1, \eta_2, \eta_3)$ is given as

$$\eta(s) = \frac{g_s^{-1}}{2}(g^2 - 1, 2g, g^2 + 1),$$

for $g(s) \neq$constant, $g_s = g'$, [16].

The Lorentzian cross−product $\times : E_1^3 \times E_1^3 \to E_1^3$ is written as

$$A \times B = \begin{bmatrix} i & j & -k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix},$$

where $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3) \in E_1^3$. Thus, it is antisymmetric and $A \times B$ is orthogonal on A and B, [11].

The Lorentz force Φ of F on $\mathbf{Q}^2$ is written as follows

$$d(\Phi(A), B) = F(A, B), \forall \in A, B \in \mathbf{Q}^2,$$

[11, 21]. The curves η on $\mathbf{Q}^2$are the β−magnetic trajectories of F that supplies the equation

$$\nabla_{\eta'}\eta' = \Phi(\eta').$$

Also, for $A, B, C \in \mathbf{Q}^2$ the mixed product is given as follows

$$d(A \times B, C) = dv_d(A, B, C),$$

where denotes a volume on $\mathbf{Q}^2$ is pointed out as dv_d, [11, 21].

Let W be a Killing vector in $\mathbf{Q}^2$ and for the equation $F_W = \imath_W vol_d$, here the inner product is determined by $\imath$. Therefore, the Lorentz force of F_W is written as

$$\Phi(A) = W \times A,$$

for $\forall A \in \mathbf{Q}^2$. Hence, the Lorentz equation is taken down as

$$\nabla_{\eta'}\eta' = \Phi(\eta') = W \times \eta',$$

taking into account the Killing vector field $W = s_1\partial_\eta + s_2\partial_\beta + s_3\partial_y$; $s_1, s_2, s_3 \in \mathbb{R}_0$ and the solutions are obtained as

$$\eta'' = W \times \eta'$$

that is the magnetic trajectories $\eta : I \to \mathbf{Q}^2 \subset E_1^3$ indicated by W, [21].

Definition 2.1. *Let γ be a curve written as*

$$\gamma(s) = \left(x\left(u(s), v(s)\right), y\left(u(s), v(s)\right), z\left(u(s), v(s)\right)\right),$$

which is an arc length parametrized geodesic on a rotational surface. The Lagrangian:

$$L = \dot{w}^2 + \rho^2\dot{v}^2, \tag{2.2}$$

the Euler-Lagrange equations are given as

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}; \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} \text{ or } \ddot{u} = \rho\rho'\dot{v}^2, \frac{d}{ds}\left(\rho\dot{v}^2\right) = 0, \tag{2.3}$$

that is a constant of the motion, [11, 20].

Theorem 2.2. (*Clairaut's Theorem*)*Let γ be a geodesic on a surface of revolution S, let ρ be the distance function of a point of S from the axis of rotation, and let θ be the angle between γ and the meridians of S. The $\rho\sin\theta$ is constant along γ. Conversely, if $\rho\sin\theta$ is constant along some curve γ in the surface, and if no part of γ is part of some parallel of S, then γ is a geodesic,* [20].

Definition 2.3. *A one-parameter group of diffeomorphism of a manifold L is a map $\Phi : L \times \mathbb{R} \to L$, $\Phi_t(\vartheta) = \Phi(\vartheta, t)$, where*

(1) $\Phi_t : L \to L$ is a diffeomorphism,
(2) $\Phi_0 = id$.
(3) $\Phi_{s+t} = \Phi_s o \Phi_t$.

This group is attached generated by a vector field W expressed as $\frac{d}{dt}\Phi_t(\vartheta) = W(\vartheta)$, and the group of diffeomorphisms is said to be as the flow of W.

If a one-parameter group of isometries is originated by W, then W is said to be as a Killing vector field, [12].

3. The characterizations of $y-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$

In this part, $y-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$ are defined and some characterizations according to $y-$magnetic curves are given.

Definition 3.1. *Let $\eta : I \to \mathbf{Q}^2 \subset E_1^3$ be a space-like curve in $\mathbf{Q}^2$ and F_W be a magnetic field on $\mathbf{Q}^2 \subset E_1^3$. The curve η is said to be $y-$magnetic curve if its W_y vector field satisfies the Lorentz force equations*

$$\nabla_\beta y = \Phi^y(y) = W_y \times y.$$

Theorem 3.2. *Let $\eta(\vartheta)$ be a space-like $y-$magnetic curve in $\mathbf{Q}^2 \subset E_1^3$ given as the asymptotic orthonormal frame $\{\eta, \beta, y\}$. The Lorentz force in the Frenet frame is expressed by*

$$\Phi^y = \begin{bmatrix} 0 & w_2 & 0 \\ \kappa & 0 & -w_2 \\ 0 & -\kappa & 0 \end{bmatrix}, \tag{3.1}$$

where w_2 is a function defined by $w_2 = d(\Phi^y(\eta), \beta), \kappa = -w_2$.

Proof. Let η be a space-like $y-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ and let $\{\eta, \beta, y\}$ be asymptotic orthonormal frame. By using the definition of the $y-$magnetic curve and equation (2.1), we can write

$$\nabla_\beta y = \Phi^y(y) = -\kappa \overrightarrow{\beta},$$

since $\Phi^y(\eta) \in Span\{\eta, \beta, y\}$, we can write

$$\Phi^y(\eta) = c_1 \overrightarrow{\eta} + c_2 \overrightarrow{\beta} + c_3 \overrightarrow{y}.$$

Then, by using the following equalities,

$$c_1 = d(\Phi^y(\eta), y) = -d(\Phi^y(y), \eta) = -d(-\kappa\beta, \eta) = 0; c_2 = d(\Phi^y(\eta), \beta) = w_2;$$

$$c_3 = d(\Phi^y(\eta), \eta) = 0,$$

we obtain

$$\Phi^y(\eta) = w_2 \overrightarrow{\beta}.$$

Similarly, we can get that

$$\Phi^y(\beta) = \kappa \overrightarrow{\eta} - w_2 \overrightarrow{y}.$$

□

Theorem 3.3. *Let η be a space-like curve in the $\mathbf{Q}^2 \subset E_1^3$ and let the curve η be a $y-$magnetic trajectory of a magnetic field W_y if and only if the vector field W_y is written as*

$$W_y = \mp\kappa(\overrightarrow{\eta} + \overrightarrow{y}), \tag{3.2}$$

where $\kappa = d(\Phi^y(\alpha), \eta)$.

Proof. Let η be a space-like $y-$magnetic curve trajectory of a magnetic field W_y. Taking up theorem 2 and definition 2, we get

$$W_y = \mp\kappa(\overrightarrow{\eta} + \overrightarrow{y}).$$

On the contrary, that the equation (3.2) holds is supposed. Hence, $\Phi^y(y) = W_y \times y$ is obtained and δ is a $y-$magnetic curve. □

Theorem 3.4. *Let η be a $y-$magnetic trajectory as to the killing vector field $W_y = \mp\kappa(\overrightarrow{\eta} + \overrightarrow{y})$ in $\mathbf{Q}^2 \subset E_1^3$. Then the $y-$magnetic trajectory of curve η is given by*

$$\eta_y(\vartheta) = \eta(0) + 2\kappa^2 \left(c_2\vartheta + 2c_3 \cosh\vartheta\right)\eta''(0)$$

$$+ \left(c\vartheta^2 + c_2\left(1 - \vartheta - 2\cosh\vartheta\right)\langle W_y, \eta''(0)\rangle\right) W_y + 2c_3(\vartheta - \sinh\vartheta)W_y \times \eta''(0), \tag{3.3}$$

where $c, c_3, c_2 \in \mathbb{R}_0^+$.

Proof. We try to prove the theorem as to the W_y. Since W_y and $\eta''(0)$ are linearly independent, we can write $\langle W_y, \eta''(0)\rangle \neq 0$. We consider $\langle W_y, W_y\rangle > 0$ and we write

$$Z = 2\kappa^2\eta''(0) - \langle W_y, \eta''(0)\rangle W_y.$$

Hence, $W_y, W_y \times \eta''(0)$ and Z are linearly independent and supply

$$\langle W_y, Z\rangle = 0, \langle W_y \times \eta''(0), Z >= 0, \langle W_y \times \eta''(0), W_y\rangle = 0,$$

so, we can write

$$\eta(\vartheta) = \eta(0) + \lambda_y(\vartheta)W_y + \mu_y(\vartheta)W_y \times \eta''(0) + \rho_y(\vartheta)Z,$$

where $\lambda_y(\vartheta), \mu_y(\vartheta), \rho_y(\vartheta)$ are functions and for $\vartheta = 0$, we have

$$\lambda_y(0) = 0, \mu_y(0) = 0, \rho_y(0) = 0, \mu_y'(0) = 0, \rho_y'(0) = c_2, \lambda_y'(0) = c_2\langle W_y, \eta''(0)\rangle, \tag{3.4}$$

the equation $\eta'''(\vartheta) = W_y \times \eta''(\vartheta)$ is given by

$$\lambda_y'''(\vartheta)W_y + \mu_y'''(\vartheta)W_y \times \eta''(0) + \rho_y'''(\vartheta)Z = -\mu_y''(\vartheta)Z - \rho_y''(\vartheta)W_y \times \eta''(0),$$

which is equivalent to

$$\lambda_y'''(\vartheta) = 0, \rho_y'''(\vartheta) = -\mu_y''(\vartheta), \mu_y'''(\vartheta) = -\rho_y''(\vartheta),$$

by using the initial conditions (3.4), solving the previous differential equations, we get

$$\lambda_y(\vartheta) = c\vartheta^2 + c_2\langle W_y, \eta''(0)\rangle;\ \mu_y(\vartheta) = 2c_3(\vartheta - \sinh\vartheta); \rho_y(\vartheta) = c_2\vartheta + 2c_3\cosh\vartheta,$$

where $c, c_3, c_2 \in \mathbb{R}_0^+$. Hence the theorem is proved. □

4. The rotational surfaces generated by the $y-$magnetic curve in $\mathbf{Q}^2 \subset E_1^3$

In this part, surface of rotation generated by $y-$magnetic curve is examined in $\mathbf{Q}^2 \subset E_1^3$ and the some characterizations are given.

Theorem 4.1. *Let η be a $y-$magnetic trajectory as to the killing vector field $W_y = \mp\kappa(\overrightarrow{\eta} + \overrightarrow{y})$ in $\mathbf{Q}^2 \subset E_1^3$. Then the rotated surface $\Lambda^y(\vartheta, t)$ consisted by the Φ^y is given as follows*

i) for $c_2 = c_3$ and $\eta_y(t) = 4bc_2\kappa\,(1, 0, 1)$,

$$\Lambda^y(t, \vartheta) = 4bc_2\kappa\left(\cos\left(\sqrt{2}\kappa\vartheta\right), \kappa\vartheta, \cos\left(\sqrt{2}\kappa\vartheta\right)\right).$$

ii) for $-c_2 = c_3$ and $\eta_y(t) = 4bc_2\kappa^2\,(2t - 1, 0, 2t - 1)$,

$$\Lambda_1^y(t, \vartheta) = 4bc_2\kappa^2\,(2t-1)\left(\cos\left(\sqrt{2}\kappa\vartheta\right), \kappa\vartheta, \cos\left(\sqrt{2}\kappa\vartheta\right)\right). \tag{4.1}$$

ii$_1$) The surface Λ_1^y with $\eta_y(t)$ is a minimal surface.
where the curvature of the curve $\eta(t) = (at^3 + bt^2, 0, at^3 + bt^2)$ is κ.

Proof. We have to research a one-parameter group of Lorentz of transformation which unchangeable all points on the $\eta y-$plane. This necessitates the Killing vector field satisfying $W_y(\vartheta) = \pm\kappa\left(\overrightarrow{\eta(\vartheta)} + \overrightarrow{y(\vartheta)}\right)$. Thus, if 3×3 matrix Φ^y is used, we can write one-parameter group of homomorphism $\Phi_\vartheta(\eta, \beta, y)$ given by $\Phi'_\vartheta(\eta) =$

$\Phi^y\Phi_\vartheta(\eta)$. So, we find $\Phi_\vartheta(\eta) = e^{\vartheta\Phi_\vartheta}\eta$ and calculating the matrix exponential, we get

$$\Pi^y(\vartheta) = \begin{bmatrix} \frac{\left(\cos\left(\sqrt{2}\kappa\vartheta\right)+1\right)}{2} & \frac{-\sin\left(\sqrt{2}\kappa\vartheta\right)}{2\sqrt{2}} & \frac{\cos\left(\sqrt{2}\kappa\vartheta\right)}{2} \\ & -\frac{\kappa\vartheta}{2} & -\frac{1}{2} \\ \frac{\kappa\vartheta}{2} & & \frac{\kappa\vartheta}{2} \\ +\frac{\sin\left(\sqrt{2}\kappa\vartheta\right)}{2\sqrt{2}} & -\cos\left(\sqrt{2}\kappa\vartheta\right) & -\frac{\sin\left(\sqrt{2}\kappa\vartheta\right)}{2\sqrt{2}} \\ \frac{\cos\left(\sqrt{2}\kappa\vartheta\right)}{2} & \frac{-\sin\left(\sqrt{2}\kappa\vartheta\right)}{2\sqrt{2}} & \frac{\left(\cos\left(\sqrt{2}\kappa\vartheta\right)+1\right)}{2} \\ -\frac{1}{2} & -\frac{\kappa\vartheta}{2} & \end{bmatrix}. \tag{4.2}$$

Here, we are rotating a curve taking the rotation matrix (3.1), the plane of rotation is written by $W_y(\vartheta) = \pm\kappa\left(\overrightarrow{\eta(\vartheta)} + \overrightarrow{y(\vartheta)}\right)$.

Let η be as $\eta(t) = (at^3+bt^2, 0, at^3+bt^2)$, and using the curve $\eta_y(\vartheta)$, we can give some statements:

i) for $c_2 = c_3$, $\eta_y(t) = 4bc_2\kappa\,(1,0,1)$,

ii) for $-c_2 = c_3$, $\eta_y(t) = 4bc_2\kappa^2\,(2t-1,0,2t-1)$.

i) Barely, an arbitrary point in $\mathbf{Q}^2$ can be thought to be on the $\eta y-$plane using some notation, so we suppose that the $\eta_y(\vartheta)$ curve lies in $\eta y-$plane. Thus, we can write one of its parametrizations as

$$\eta_y(t) = 4bc_2\kappa\,(1,0,1)\,, b, c_2 \in \mathbb{R}_0.$$

Hence, the rotational surface $\Lambda^y_{W_y}$ round W_y can be parametrized by

$$\Lambda^y\,(t,\vartheta) = \Pi^y(\vartheta) \times \begin{bmatrix} 4bc_2\kappa \\ 0 \\ 4bc_2\kappa \end{bmatrix},$$

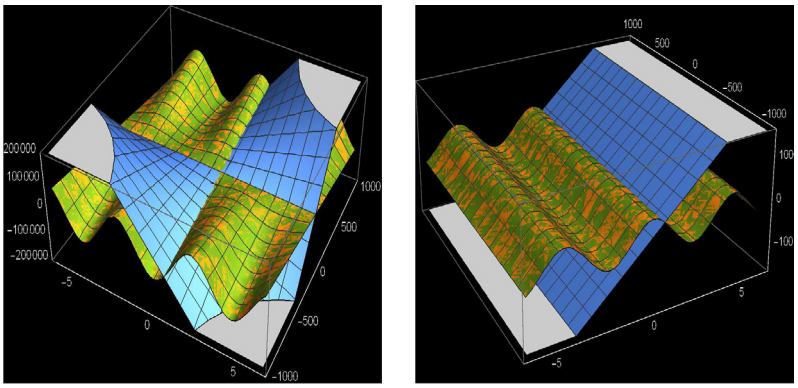
$$\Lambda^y(t,\vartheta) = 4bc_2\kappa\left(\cos\left(\sqrt{2}\kappa\vartheta\right), \kappa\vartheta, \cos\left(\sqrt{2}\kappa\vartheta\right)\right).$$

ii) Let η be a curve as

$$\eta_y(t) = 4bc_2\kappa^2\,(2t-1,0,2t-1)\,, b, c_2 \in \mathbb{R}_0.$$

Thus, the rotated surface ${}_{W_y}\Lambda^y_1$ around W_y can be parametrized by

$$\Lambda^y_1\,(t,\vartheta) = \Pi^y(\vartheta) \times \begin{bmatrix} 4bc_2\kappa^2\,(2t-1) \\ 0 \\ 4bc_2\kappa^2\,(2t-1) \end{bmatrix}$$



(A) The surface Λ^y (B) The surface Λ_1^y

FIGURE 1. The graphics of surfaces Λ^y and Λ_1^y

$$\Lambda_1^y(t,\vartheta) = 4bc_2\kappa^2(2t-1)\left(\cos\left(\sqrt{2}\kappa\vartheta\right), \kappa\vartheta, \cos\left(\sqrt{2}\kappa\vartheta\right)\right),$$

where $t \in I$. For the first and the second fundamental forms, we get

$$E(t,\vartheta) = 4b^2c_2^2\kappa^4(3\kappa'(\vartheta)\vartheta + \kappa(\vartheta))^2(2t-1)^2,$$
$$F(\vartheta) = 8bc_2\kappa^5\vartheta(3\kappa'(\vartheta)\vartheta + \kappa(\vartheta)),$$
$$G(\vartheta) = 8bc_2\kappa^3\vartheta; L = M = N = 0.$$

So, since $K, H = 0$, the surface Λ_1^y with $\eta_y(t)$ is a minimal surface. □

4.1. **The Clairaut's theorem on rotational surface formed by y−magnetic curve in $\mathbf{Q}^2 \subset E_1^3$.** We expressed Clairaut's theorem on rotational surface by using $y-$magnetic curve in $\mathbf{Q}^2 \subset E_1^3$. Moreover, we gave the general equations of geodesic in $\mathbf{Q}^2$.

Theorem 4.2. *Let $\Lambda_1^y(\vartheta,t)$ be the rotational surface formed by $y-$magnetic curve in $\mathbf{Q}^2$. Then, one holds:*

(1) For the parameters $\vartheta = v; t = ue^{-v}$ satisfying the following equation

$$E(u,v)\,ue^{-v} = 2F(u,v),$$

then the Lagrange equation on $\Lambda_1^y(t,\vartheta)$ is written as follows

$$E(u,v)\,e^{-2v}\dot{u}^2 + G(v)\,\dot{v}^2 = L.$$

(2) The curve $\eta(s) = \Lambda_1^y(u(s), v(s))$ *is a geodesic on* $\Lambda_1^y(u, v)$ *if and only if the following equations supply*

$$2\int G(v)\,dv = c_{10}s + c_{11};\ E(u,v) = c_{13}e^{2v};$$
$$\int \frac{du}{\sqrt{E(u,v)}} = c_{14}s + c_{15}; v = \int \sin\theta ds + c_{16},$$

where $c_i, \in \mathbb{R}_0, i \in I$.

Proof. Let $\Lambda_1^y(t, \vartheta)$ be the surface formed by $y-$magnetic curve given as

$$\Lambda_1^y(t,\vartheta) = 4bc_2\kappa^2(2t-1)\left(\cos\left(\sqrt{2}\kappa\vartheta\right), \kappa\vartheta, \cos\left(\sqrt{2}\kappa\vartheta\right)\right).$$

Thus, the fundamental forms are written as

$$I = \begin{pmatrix} E(t,\vartheta) & F(\vartheta) \\ F(\vartheta) & G(\vartheta) \end{pmatrix}, II = 0. \tag{4.3}$$

and, we get $K = H = 0$. Thus, the surface is a minimal surface which defined by a zero mean curvature. Since $\tau \neq 0$ the first fundamental form is two variable parameter. Moreover, it is considerable to note that, the coordinates of parametrization are orthogonal, since the first fundamental form is diagonal. So, Lagrangian equation is written as

$$E(t,\vartheta)\,d^2t + 2F(\vartheta)\,d\vartheta dt + G(\vartheta)\,d^2\vartheta = L \text{ or}$$
$$E(t,\vartheta)\,\dot{t}^2 + 2F(\vartheta)\,\dot{\vartheta}\dot{t} + G(\vartheta)\,\dot{\vartheta}^2 = L \tag{4.4}$$

and from Euler Lagrangian equations and using following equation

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial t}{\partial s}}\right) = \frac{\partial L}{\partial t},$$

which gives

$$\frac{\partial E(t,\vartheta)}{\partial t}\dot{t}^2 + 2E(t,\vartheta)\,\ddot{t} + 2\frac{\partial F}{\partial \vartheta}\dot{\vartheta}^2 + 2F(\vartheta)\,\ddot{\vartheta} = 0,$$

and

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \vartheta}{\partial s}}\right) = \frac{\partial L}{\partial \vartheta} = \frac{\partial E(t,\vartheta)}{\partial \vartheta}\dot{t}^2 + 2\frac{\partial F(t,\vartheta)}{\partial \vartheta}\dot{\vartheta}\dot{t} + G(\vartheta)\,\dot{\vartheta}^2 \neq 0.$$

Hence, we say that $\frac{\partial L}{\frac{\partial \vartheta}{\partial s}}$ is not constant. Now, to solve this problem we can change the coordinate of this equation into (u, v), therefore this coordinate can be chosen as

$$\vartheta = v; t = ue^{-v}. \tag{4.5}$$

Also, we can write previous equations taking derivatives according to s parameter as follows

$$\dot{\vartheta} = \dot{v}; \dot{t} = e^{-v}\left(\dot{u} - u\dot{v}\right),$$

and using following equations

$$E\left(u,v\right)ue^{-v} = 2F\left(u,v\right). \tag{4.6}$$

Lagrangian equation can be written as

$$E\left(u,v\right)e^{-2v}\dot{u}^2 + G\left(v\right)\dot{v}^2 = L. \tag{4.7}$$

For the following equations

$$2\int G\left(v\right)dv = c_{10}s + c_{11}, \tag{4.8}$$

$$E(u,v) = c_{13}e^{2v}, \tag{4.9}$$

$$\int \frac{du}{\sqrt{E(u,v)}} = c_{14}s + c_{15}, \tag{4.10}$$

which is a constant along the geodesic given by

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u} = \frac{\partial E}{\partial u}e^{-2v}\dot{u}^2 \text{ and } \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} = 0,$$

which means

$$\frac{\partial L}{\frac{\partial v}{\partial s}} = 2G(v)\dot{v}$$

is constant along the geodesic.

Furthermore, for following the equation

$$\int \frac{du}{\sqrt{E(u,v)}} = c_{14}s + c_{15},$$

one has $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u} = \frac{\partial E(u,v)}{\partial u} \neq 0$.

For the following equation

$$E(u,v) = c_{13}e^{2v},$$

by using equations $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} = 0$, one has $\frac{\partial}{\partial s}\left(2G\left(v\right)\dot{v}\right) = 0$, which means

$$2\int G\left(v\right)dv = c_{10}s + c_{11}$$

is constant along the geodesic. Let η_y be a geodesic on the surface of $\Lambda_1^y\left(u,v\right)$, so it is written as $(u(s), v(s))$. Then, let θ be the angle between $\dot{\eta}_y$and a meridian; n_u

$= \Lambda_{1_u}^y(u,v)$ is the vector along parallel of Λ_1^y, and $n_v = \Lambda_{1_v}^y(u,v)$ is the vector along meridians of Λ_1^y. Then, using orthonormal basis $\{n_u, n_v\}$ a unit vector tangent $\dot{\delta}$ on $\Lambda_1^y(u,v)$ can be written

$$\dot{\eta} = n_u \cos\theta + n_v \sin\theta = \dot{u}\Lambda_{1_u}^y + \dot{v}\Lambda_{1_v}^y = \dot{u}n_u + \dot{v}n_v.$$

Then, since $\dot{v} = \sin\theta$, we can write

$$2G(v)\,\dot{v} = 2G(v)\sin\theta$$

and this equations is constant along η. Conversely, η is a $y-$magnetic curve providing the equation $2G(v)\,\dot{v} = 2G(v)\sin\theta$ =constant. Hence, we have

$$v = \int \sin\theta ds + c_{16}. \tag{4.11}$$

□

Theorem 4.3. *The general equation of geodesics on the surfaces formed by the $y-$magnetic curves in $\mathbf{Q}^2$ is written as*

$$\frac{du}{dv} = c_{17}e^v \sqrt{\frac{G(v)}{E(u,v)}\left(4\varepsilon G(v) - c_{18}\right)}, \tag{4.12}$$

where $c_i \in \mathbb{R}_0$.

Proof. By consider the Euler-Lagrange equations for (4.7), $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u} = \frac{\partial E(u,v)}{\partial u} \neq 0$, the general equation of geodesics can be find. Then, solving of the differential equations in $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}$, we get

$$2\int G(v)\,dv = c_{10}s + c_{11}, v = \int \sin\theta ds + c_{16},$$

and by using equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}$, we have $\frac{\partial}{\partial s}\left(2G(v)\,\dot{v}\right) = 0$, which means

$$\frac{dv}{ds} = \frac{c_{10}}{2G(v)}; c_{10} \in \mathbb{R}.$$

If we put the value of $\dot{v}$ at the equation $E(u,v)\,e^{-2v}\dot{u}^2 + G(v)\,\dot{v}^2 = \varepsilon$, we get

$$E(u,v)\,e^{-2v}\left(\frac{du}{dv}\frac{dv}{ds}\right)^2 + G(v)\left(\frac{dv}{ds}\right)^2 = \varepsilon,$$

and then this equation can be written as follow

$$\frac{du}{dv} = c_{17}e^v\sqrt{\frac{G(v)\left(4\varepsilon G(v) - c_{18}\right)}{E(u,v)}}.$$

□

5. The physical approach on surface generated by the $y-$magnetic curve in $\mathbf{Q}^2$

Let $\Lambda_1^y(t(s),\vartheta(s))$ be a parametrized curve on the rotational surface given by

$$\Lambda_1^y(t(s),\vartheta(s)) = 4bc_2\kappa^2(2t-1)\left(\cos\left(\sqrt{2}\kappa\vartheta\right),\kappa\vartheta,\cos\left(\sqrt{2}\kappa\vartheta\right)\right).$$

Also, to solve this problem, according to the parameters $\vartheta = v; t = ue^{-v}$, the Lagrange equation on the surface $\Lambda_1^y(t,\vartheta)$ can be written as follows

$$E(u,v)\,e^{-2v}\dot{u}^2 + G(v)\,\dot{v}^2 = L.$$

By using the chain rule, we can write the tangent vector of this curve as

$$\begin{aligned}\dot{\eta} &= \frac{d\Lambda_1^y(u(s),v(s))}{ds} = \frac{du(s)}{ds}\Lambda_{1_u}^y + \frac{dv(s)}{ds}\Lambda_{1_v}^y \\ &= n_u\cos\theta + n_v\sin\theta = \dot{u}n_u + \dot{v}n_v = \dot{u}\Lambda_{1_u}^y + \dot{v}\Lambda_{1_v}^y\end{aligned}$$

Furthermore, one gets

$$\overrightarrow{T} = \frac{d\Lambda_1^y(u(s),v(s))}{ds} = T^u\Lambda_{1_u}^y + T^v\Lambda_{1_v}^y \tag{5.1}$$

and we might write two component vector notation as to the basis vectors $\Lambda_{1_i}^y$ and we say that the speed is called by its magnitude $T = \left\langle \overrightarrow{T}, \overrightarrow{T} \right\rangle^{1/2}$ and we can write this magnitude as follows

$$T = \sqrt{g_{ij}\frac{dz^i}{ds}\frac{dz^j}{ds}}. \tag{5.2}$$

$T^{u^*} = \sqrt{E(u,v)}e^{-v}T^u = T\cos\theta$ being along the first axis is the radial velocity and T^v is the horizontal angular velocity; $T^{v^*} = \sqrt{G(v)}T^v = T\sin\theta$ is the vertical component of the velocity vector, and one writes following equation using T^{u^*} and T^{v^*}

$$\left\langle T^{u^*}, T^{v^*} \right\rangle = T\left\langle \cos\theta, \sin\theta \right\rangle.$$

Here, according to the direction $\Lambda_{1_{u^*}}^y$ the direction of the velocity is given the angle θ in this plane. Hence, the speed is constant along the geodesic. Also, from the references [22, 23], the specific quantities found by partitioning out the mass

will be used instead the some physical properties as energy and momentum. Hence, we write the specific kinetic energy as follows

$$E = \frac{1}{2}T^2 = \frac{1}{2}\left(T^2\cos^2\theta + T^2\sin^2\theta\right) \tag{5.3}$$

$$= \frac{1}{2}E\left(u,v\right)e^{-2v}\left(\frac{du}{ds}\right)^2 + \frac{1}{2}G\left(v\right)\left(\frac{dv}{ds}\right)^2, \tag{5.4}$$

where this equations is called as specific kinetic energy. Hence, speed $T = \sqrt{2E}$ have to be constant along a geodesic.

Theorem 5.1. *Let $\Lambda_1^y\left(u,v\right)$ be the surface $\Lambda_1^y\left(u,v\right)$ generated by a $y-$magnetic curve, the specific kinetic energy and the specific angular momentum are constant due to its motion in $\mathbf{Q}^2$ under some conditions and the following statements hold:*

(1) For the parameter $v = \int \sin\theta ds + c_{16}$ and the equations $2\int G\left(v\right)dv = c_{10}s + c_{11}$, $\int \frac{du}{\sqrt{E(u,v)}} = c_{14}s + c_{15}$ and $E(u,v) = c_{13}e^{2v}$, the specific angular momentum ℓ is given by

$$\ell = \sqrt{G\left(v\right)}T\sin\theta. \tag{5.5}$$

(2) The specific kinetic energy E is given by

$$E = \frac{1}{2}\left(E(u,v)e^{-2v}\left(\frac{du}{ds}\right)^2 + \frac{\ell^2}{G\left(v\right)}\right), \tag{5.6}$$

where T is the tangent vector of the geodesic curve.

Proof. Previously, for the following equations

$$\int \frac{du}{\sqrt{E(u,v)}} = c_{14}s + c_{15}; E(u,v) = c_{13}e^{2v}; 2\int G\left(v\right)dv = c_{10}s + c_{11},$$

we write $2G\left(v\right)\dot{v} = 2G\left(v\right)\sin\theta$ being constant along η. Also, we may express with respect to motion around an axis with radius $\left\|\overrightarrow{R}\right\| = \sqrt{G\left(v\right)}$. Hence, $T^{v^*} = \sqrt{G\left(v\right)}T^v = T\sin\theta = \sqrt{G\left(v\right)}\frac{dv}{ds}$ is the angular direction where $\sqrt{G\left(v\right)}$ is radius of the circle. Therefore, the specific angular momentum ℓ may be expressed by

$$\ell = \overrightarrow{e_3}.\left(\overrightarrow{R}\times\overrightarrow{T}\right) = \sqrt{G\left(v\right)}T\sin\theta,$$

since $T^{v^*} = T\sin\theta = \sqrt{G\left(v\right)}\frac{dv}{ds}$, $\sqrt{G\left(v\right)}T\sin\theta = G\left(v\right)\frac{dv}{ds}$ is constant along η, and we say that the specific angular momentum ℓ is constant along a geodesic and one has

$$\ell = G\left(v\right)\frac{dv}{ds} \Rightarrow \frac{dv}{ds} = \frac{\ell}{G\left(v\right)}.$$

This expression may be taken to rewrite as dv/ds in (5.6). Then, the specific energy E is written as

$$E = \frac{1}{2}\left(E(u,v)e^{-2v}\left(\frac{du}{ds}\right)^2 + \frac{\ell^2}{G(v)}\right).$$

□

6. Corollary

In this study, the new type surface of rotation formed by $y-$magnetic curves is given, and the some results of the geodesics on this surface is expressed. Then, the specific energy and specific angular momentum obtained on this surface are expressed in $\mathbf{Q}^2$. Also, we have explored the conditions of being geodesic using the Clairaut's theorem in $\mathbf{Q}^2$.

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