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KANTOWSKI SACHS MACROSCOPIC BODY COSMOLOGICAL MODELS IN SCALAR TENSOR THEORIES OF GRAVITATION

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Abstract. In this paper, we have obtained the field equations in scalar-tensor theories of gravitation proposed by Brans-Dicke [5] and Saez-Ballester [28] with the aid of Kantowski Sachs Space time in the presence of macroscopic body. Both theories' field equations can be solved definitively by using transformation.The present discussion explores into the expression involving energy density, energy flow vector, stress tensor, and specific physical features of both of these models.

AMS Classification: 54H20.

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1. Introduction

Scalar-tensor theories are considered essential in explaining gravitational interactions in the range of the Plank scale. Many theories, such as string theory, higher order theories in the Ricci scalar, extended inflation and others demand the presence of a scalar field. Scalar-tensor theories of gravity are widely regarded as the most natural extensions of general relativity. These theories propose that the force of gravity is expressed not only by the metric of space-time, but also by a scalar field. The gravitational constant is a dynamical variable that is determined by the

scalar field in scalar-tensor theories. It should be emphasised from a theoretical perspective that some scalar-tensor theories naturally appear as a low-energy limit of string theory. In cosmology, scalar-tensor theories have attracted a lot of attention. Finding and researching exact solutions is strongly related to progress in our understanding of scalar-tensor theories of gravity. Exact solutions are necessary for a theoretical discussion of many aspects of the early cosmos, gravitational waves, gravitational collapse and the structure of the compact objects within the framework of scalar-tensor theories. Due to its complexity in the general scenario, solving scalar-tensor theory equations in the presence of a source is an impossible task. In comparison to Einstein equations, scalar-tensor gravity equations tend to be substantially more difficult. The source-less scalar-tensor equations are transformed into Einstein equations with a minimally related scalar field in the Einstein frame. In this scenario, there has been significant advancement in the search for precise homogeneous and inhomogeneous cosmological solutions. For particular combinations of the Brans-Dicke, Saez-Ballester, Barker, and Bekenstein theories, specific classes of solutions were provided.
Scalar and tensor fields with combined Brans-Dicke [5] field equations are

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}{\phi'}^{k}) - \phi^{-1}(\phi_{i;j} - g_{ij}\Box\phi) \tag{1}$$

$$\Box\phi = \phi'^{k}_{;k} = 8\pi\phi^{-1}(3+2\omega)^{-1}T \tag{2}$$

and Saez and Ballester [27] field equations are

$$G_{ij} - \omega\phi^{n}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}{\phi'}^{k}) = -8\pi T_{ij} \tag{3}$$

$$2\phi^{n}\phi^{i}_{;i} + n\phi_{,k}{\phi'}^{k} = 0 \tag{4}$$

where, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor, R_{ij} is the Ricci tensor, R is the scalar curvature, n an arbitrary constant, ω is a dimensionless coupling constant and T_{ij} is the matter energy momentum tensor. Here comma and semicolon denote partial and covariant differentiation respectively.
The energy conservation equation

$$T^{ij}_{;j} = 0, \tag{5}$$

is a consequence of field equations in both the theories.
The study of cosmological models in the frame work of scalar-tensor theories has

been the active area of research for the last few decades. Singh et al [32, 33] gives a detailed discussion of Brans-Dicke cosmological models andinvestigated several aspects of cosmological model in Saez-Ballester scalar-tensor theory of gravitation. Ram et al [18], Rao et al [21,22], BhaskaraRao et al [4], Barrow et al [3], Adhav et al [1], Katore et al [7],Raju et al [17],Reddy et al [24,26],Suresh Kumar et al [35],Aditya et al [2], Tirandari et al [36],Sahu et al [28], Trivedi et al [37], Shanthi et al [29],Recently , Vinutha et al [38],Mishra et al [9],Shukla et al [31], Peracaula et al [16],Mitsotoulos et al [10], Shaikh et al [30], Rasouli et al [23], Rani et al [19],Yazadjiev [39], Nimkar et al [11], Pawar et al[15] etc. are some of the authors who have investigated various aspects of Saez-Ballester and Brans-Dicke theories of gravitation. Nimkar et al [13] investigated microscopic body cosmological model in self creation theory. Reddy et al [25] Genly. Leon et al [6], SubaRao et al [34] have studied Kantowski-Sachs cosmological model. Nimkar et al [12], Pawar et al [14] examined observational parameter of cosmological model.

The purpose of present investigation is to construct macroscopic body cosmological models in scalar-tensor theories of gravitation proposed by Brans-Dicke [5] and Saez-Ballester [27]. The present paper is structured as follows: Section 2 is dedicated to the Metric and Energy Momentum Tensor. Section 3 presents the derivation of the field equations and solutions within the framework of Brans-Dicke theory, utilising the Kantowski-Sachs space-time. Section 4 pertains to the physical and kinematic features of cosmological models within the framework of Brans-Dicke theory. Section 5 presents the derivation of the field equations and solutions within the framework of Saez-Ballester theory. Section 6 pertains to the physical and kinematic characteristics of cosmological models within the frameworkof Saez-Ballester theory. Section 7 primarily focuses on the observational parameters of cosmologicalmodels in both theories. The final section of the paper consists of the conclusions drawn from the preceding analysis.

2. Metric and Energy Momentum Tensor

The Kantowski-Sach spacetime is expressed in the following form:

$$ds^2 = dt^2 - R^2 dr^2 - S^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

R and S are functions that depend only on time t.

The energy momentum-tensor for a macroscopic body (Landue L.D.and Lifshitz

E.M,[8]) is given by

(7) $$T^{ij} = (p+\epsilon)u^i u^k - pg^{ik}$$

Here p is the pressure, ϵ is the energy density and u_i is the four velocity vectors of the distribution respectively.
Using Equation (7), we have
$T_1^1 = T_2^2 = T_3^3 = -p$ and $T_4^4 = \epsilon$
The equation that gives the energy-momentum tensor's trace is as follows:

(8) $$T = T_1^1 + T_2^2 + T_3^3 + T_4^4 = -3p + \epsilon$$

3. Solutions of Field Equations (Brance-Dicke)

With the assistance of Eqs. (2) - (8), the field equation (1) reduces to

(9) $$\frac{2S_{44}}{S} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} + \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_4}{\phi}\frac{2S_4}{S} + \frac{\phi_{44}}{\phi} = -8\pi p\phi^{-1}$$

(10) $$\frac{R_{44}}{R} + \frac{S_{44}}{S} + \frac{R_4 S_4}{RS} + \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_4}{\phi}\left(\frac{R_4}{R} + \frac{S_4}{S}\right) + \frac{\phi_{44}}{\phi} = -8\pi p\phi^{-1}$$

(11) $$\frac{2R_4 S_4}{RS} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} - \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_4}{\phi}\left(\frac{R_4}{R} + \frac{2S_4}{S}\right) = 8\pi\epsilon^{-1}$$

(12) $$\left(\frac{R_4}{R} + \frac{2S_4}{S}\right)\phi_4 + \phi_{44} = \frac{8\pi\phi^{-1}}{(3+2\omega)}(-3p+\epsilon)$$

(13) $$\epsilon_4 + \left(\frac{R_4}{R} + \frac{2S_4}{S}\right)(\epsilon+p) - \frac{p\cot\theta}{S} = 0$$

Where field variable suffix 4 indicates ordinary differentiation with respect to time t.
Five independent equations with five unknowns R, S, p, ϵ and ϕ constitute equations (9) to (13) In order to get deterministic solution we introduce the transformation (Rao, V.U.M. and Neelima, D.[20]) $dt = RS^2 dT$
the above field equations can be written as

(14) $$2\frac{S''}{S} - \frac{3S'^2}{S^2} - \frac{2R'S'}{RS} + R^2S^2 + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 + \frac{\phi''}{\phi} - \frac{\phi'}{\phi}\frac{R'}{R} = -8\pi p\phi^{-1}R^2S^4$$

$$\frac{R''}{R}+\frac{S''}{S}-\frac{2R'S'}{RS}-\frac{R'^2}{R^2}-\frac{2S'^2}{S^2}+\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+\frac{\phi''}{\phi}-\frac{\phi'S'}{\phi S}=-8\pi p\phi^{-1}R^2S^4 \tag{15}$$

$$\frac{2R'S'}{RS}+\frac{S'^2}{S^2}+R^2S^2-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+\frac{\phi'}{\phi}\left(\frac{R'}{R}+\frac{2S'}{S}\right)=8\pi\epsilon\phi^{-1}R^2S^4 \tag{16}$$

$$\phi''=\frac{8\pi\phi^{-1}}{(3+2\omega)}(-3p+\epsilon)R^2S^4 \tag{17}$$

$$\epsilon'+\left(\frac{R'}{R}+\frac{2S'}{S}\right)(\epsilon+p)-p\cot\theta RS=0 \tag{18}$$

From hereafter the overhead dash denotes differentiation with respect to T.

The above equations admits an equation

$$\frac{2S''}{S}-\frac{4S'^2}{S^2}+\alpha^2S^4=\frac{-\omega\phi'^2}{3\phi^2} \tag{19}$$

The scalar field is expressed as follows:

$$\phi=K_1T+K_2 \tag{20}$$

Using equations (19) and (20) we get

$$\frac{du}{dT}-\frac{u^2}{2}+K_4\exp^{2uT}=-\frac{\omega K_1^2}{3(K_1T+K_2)^2} \tag{21}$$

Above equation does not solvable. So the average scale factor A and spatial volume V in the Kantowski-Sachs cosmological model are given by

$$A(t)=(RS^2)^{\frac{1}{3}} \tag{22}$$

$$V=A^3=RS^2 \tag{23}$$

The Hubble's parameter for this model, as a function of direction is

$$H=\frac{1}{3}(H_1+H_2+H_3) \tag{24}$$

The directional Hubble parameters in the direction of x, y, z axes are

$$H_1=\frac{R_4}{R}, H_2=\frac{S_4}{S}, H_3=\frac{S_4}{S} \tag{25}$$

The parameter describing the anisotropic expansion of the universe is formally defined as follows

$$A_m=\frac{1}{3}\sum_{i=1}^{3}(\frac{\Delta H_i}{H})^2 \qquad where \Delta H_i=H_i-H \tag{26}$$

The utilisation of the anisotropic parameter enables an examination of the potential anisotropic expansion of the universe when the anisotropic parameter possesses a non-zero value, while the universe may expand isotropicallyif $A_m = 0$
The deceleration parameter is a measure used in cosmology to describe the rate at which the expansion of the universe is slowing down .

$$q = \frac{-AA_{44}}{A_4^2} \tag{27}$$

The expansion scalar θ and shear scalar σ are given as follows.

$$\theta = 3H \tag{28}$$

$$= \frac{R_4}{R} + 2\frac{S_4}{S} \tag{29}$$

Using (24) and (25), the average anisotropy parameter can be decreased to a lower value.

$$A_m = \frac{2}{9H^2}\left[\frac{R_4}{R} - \frac{S_4}{S}\right]^2 \tag{30}$$

$$\sigma^2 = \frac{3}{2}A_m H^2 \tag{31}$$

Using (30)-(31) we get,

$$\sigma^2 = \frac{1}{3}\left[\frac{R_4}{R} - \frac{S_4}{S}\right]^2 \tag{32}$$

The solution to the highly nonlinear differential equations (9)-(12) involves five unknowns R, S, ϕ, P and ϵ In order to put in additional constraints, we employ the power law relationship between the average scale factor and scalar field ϕ. This relationship is expressed as follows :

$$\phi \propto A^k$$
$$\phi = k_1 A^k \tag{33}$$

The shear scalar σ is directly proportional to expansion scalar θ is

$$\sigma \propto \theta$$
$$\sigma = k_2\theta \tag{34}$$

Using equation (29), (32) and (34) we get,

$$\frac{1}{\sqrt{3}}\left[\frac{R_4}{R}-\frac{S_4}{S}\right]=k_2\left[\frac{R_4}{R}+2\frac{S_4}{S}\right] \tag{35}$$

On integrating, above equation we get,

$$R=k_3^{\frac{\sqrt{3}}{1-\sqrt{3}k_2}}S^{\frac{2\sqrt{3}k_2+1}{1-\sqrt{3}k_2}} \tag{36}$$

The average scale factor is

$$A(t)=k_4 S^{\frac{1}{3}\left(\frac{2\sqrt{3}k_2+1}{1-\sqrt{3}k_2}+2\right)} \tag{37}$$

From equation (33) and (37) Brans-Dicke field ϕ leads to

$$\phi=k_5 S^{\frac{k}{3}\left(\frac{2\sqrt{3}k_2+1}{1-\sqrt{3}k_2}+2\right)} \tag{38}$$

In radiation,

$$-3p+\epsilon=0 \tag{39}$$

Use equation (39) in equation (12) we get

$$\phi_4=\frac{k_6}{RS^2} \tag{40}$$

Solving equation (40) and (38) we get

$$S=k_9(k_7t+k_8)^{\frac{3}{(k+3)(N+2)}} \tag{41}$$

$$R=k_{10}(k_7t+k_8)^{\frac{3N}{(k+3)(N+2)}} \tag{42}$$

Where, $N=\frac{2\sqrt{3}k_2+1}{1-\sqrt{3}k_2}$ And scalar field is given by

$$\phi=k_{11}(k_7t+k_8)^{\frac{k}{k+3}} \tag{43}$$

The energy density and pressure is given by

(44)
$$\epsilon=\frac{1}{(k_7t+k_8)^{k+6}}[\frac{k_{11}k_7^2}{8\pi(N+2)^2(k+3)^2}(9(2N+1)+\frac{(N+2)^2(k+3)^2}{k_7^2k_9^2(k_7t+k_8)^{\frac{6}{(N+2)(k+3)}-2}}$$
$$-\frac{\omega k^2(N+2)^2}{2}+3(N+2)^2k)]$$

(45)
$$p=\frac{1}{3(k_7t+k_8)^{k+6}}[\frac{k_{11}k_7^2}{8\pi(N+2)^2(k+3)^2}(9(2N+1)+\frac{(N+2)^2(k+3)^2}{k_7^2k_9^2(k_7t+k_8)^{\frac{6}{(N+2)(k+3)}-2}}$$
$$-\frac{\omega k^2(N+2)^2}{2}+3k(N+2)^2)].$$

Using equation (41) and (42) kantowski-sachs cosmological model in equation (6) take the form

$$ds^2 = dt^2 - \left(k_{10}(k_7 t + k_8)^{\frac{3N}{(k+3)(N+2)}}\right)^2 dr^2 - \left(k_9(k_7 t + k_8)^{\frac{3}{(N+2)(k+3)}}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{46}$$

4. The Physical and Kinematical Properties

Some physical parameter of present model spatial volume, Shear Scalar, Hubble parameter, Expansion Scalar and average scale factor is given by

$$V = k_{12}(k_7 t + k_8)^{\frac{3}{k+3}} \tag{47}$$

$$\sigma^2 = \frac{k_{14}k_7^2(N-1)^2}{(N+2)^2(k_7 t + k_8)^2} \tag{48}$$

$$H = \frac{k_7}{(k+3)(k_7 t + k_8)} \tag{49}$$

$$\theta = \frac{3k_7}{(k+3)(k_7 t + k_8)} \tag{50}$$

$$A(t) = k_{13}(k_7 t + k_8)^{\frac{1}{k+3}} \tag{51}$$

Also, the expression for the energy density W, energy flow vector S and stress tensor $\sigma_{\alpha\beta}$ are

$$W = \left(\frac{1 + \frac{\nu^2}{3c^2}}{1 - \frac{\nu^2}{c^2}}\right) \frac{1}{(k_7 t + k_8)^{k+6}} [\frac{k_{11}k_7^2}{8\pi(N+2)^2(k+3)^2}(9(2N+1) + \frac{(N+2)^2(k+3)^2}{k_7^2 k_9^2 (k_7 t + k_8)^{\frac{6}{(N+2)(k+3)}-2}} - \frac{\omega}{2}k^2(N+2)^2 + 3k(N+2)^2)] \tag{52}$$

$$S = \frac{1}{3}\left(\frac{4\nu}{1 - \frac{\nu^2}{c^2}}\right) \frac{1}{(k_7 t + k_8)^{k+6}} [\frac{k_{11}k_7^2}{8\pi(N+2)^2(k+3)^2}(9(2N+1) + \frac{(N+2)^2(k+3)^2}{k_7^2 k_9^2 (k_7 t + k_8)^{\frac{6}{(N+2)(k+3)}-2}} - \frac{\omega}{2}k^2(N+2)^2 + 3k(N+2)^2)] \tag{53}$$

$$\sigma_{\alpha\beta} = \left(\frac{4\nu_\alpha \nu_\beta}{c^2(1 - \frac{\nu^2}{c^2})} + \delta_{\alpha\beta}\right) \frac{1}{3(k_7 t + k_8)^{k+6}} [\frac{k_{11}k_7^2}{8\pi(N+2)^2(k+3)^2}(9(2N+1) + \frac{(N+2)^2(k+3)^2}{k_7^2 k_9^2 (k_7 t + k_8)^{\frac{6}{(N+2)(k+3)}-2}} - \frac{\omega}{2}k^2(N+2)^2 + 3k(N+2)^2)] \tag{54}$$

When the velocity v of macroscopic motion is significantly lower than the velocity of light, the resulting value can be calculated to approximately $S = (p+\epsilon)\nu$
The quantity $\frac{S}{c^2}$ represents the momentum density while $\frac{(p+\epsilon)}{c^2}$ serves as the mass density of the object.
From the expression (7), we get

$$T_i^i = \epsilon - 3p \tag{55}$$

$$\text{But,} \qquad T_i^i = \sum_a m_a c^2 \sqrt{1 - \frac{\nu_a^2}{c^2}}\,\delta(r - r_a) \tag{56}$$

The comparison between relation (55) and the general formula (56), which was established as applicable to any system, is warranted. Given that we are currently examining a macroscopic entity, it is necessary to compute the average of expression (56) across all possible values of r within a unit volume.
A result is obtained.

$$\epsilon - 3p = \sum_a m_a c^2 \sqrt{1 - \frac{\nu_a^2}{c^2}}$$

The summation in dispute involves all particles within a unit volume. In the ultra-relativistic limit, the equation of state of matter $p = \frac{\epsilon}{3}$tends towards zero on the right side of this equation.

5. Solutions of Field Equations (Saez-Ballester)

Now with the assistance of Eqs. (4) - (8), the field equat ion (3) reduces to

$$\frac{2S_{44}}{S} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} - \frac{\omega}{2}\phi^n \phi_4^2 = -8\pi p \tag{57}$$

$$\frac{R_{44}}{R} + \frac{S_{44}}{S} + \frac{R_4 S_4}{RS} - \frac{\omega}{2}\phi^n \phi_4^2 = -8\pi p \tag{58}$$

$$\frac{2R_4 S_4}{RS} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} + \frac{\omega}{2}\phi^n \phi_4^2 = 8\pi\epsilon \tag{59}$$

$$\left(\frac{R_4}{R} + \frac{2S_4}{S}\right)\phi_4 + \phi_{44} + \frac{n}{2}\frac{\phi_4^2}{\phi} = 0 \tag{60}$$

$$\epsilon_4 + \left(\frac{R_4}{R} + \frac{2S_4}{S}\right)(\epsilon + p) - \frac{p\cot\theta}{S} = 0 \tag{61}$$

The addition of the suffix "4" after the field variable indicates an ordinary differentiation with regard to time t.

The equations (57) to (61) form a system of independent equations in which each equation includes five unknown variables R, S, p, ϵ and ϕ . In order to get deterministic solution we introduce the transformation (V.U.M. Rao and D. Neelima,[20]) $dt = RS^2 dT$

the above field equations can be written as

$$\frac{2S''}{S} - \frac{3S'^2}{S^2} - \frac{2R'S'}{RS} + R^2S^2 - \frac{\omega}{2}\phi^n\phi'^2 = -8\pi pR^2S^4 \tag{62}$$

$$\frac{R''}{R} + \frac{S''}{S} - \frac{2S'^2}{S^2} - \frac{R'^2}{R^2} - \frac{2R'S'}{RS} - \frac{\omega}{2}\phi^n\phi'^2 = -8\pi pR^2S^4 \tag{63}$$

$$\frac{2R'S'}{RS} + \frac{S'^2}{S^2} + R^2S^2 + \frac{\omega}{2}\phi^n\phi'^2 = 8\pi\epsilon R^2S^4 \tag{64}$$

$$\phi'' + \frac{n}{2}\frac{\phi'^2}{\phi} = 0 \tag{65}$$

$$\epsilon' + \left(\frac{R'}{R} + \frac{2S'}{S}\right)(\epsilon + p) - p\cot\theta RS = 0 \tag{66}$$

The equations admits an equation

$$\frac{2S''}{S} - \frac{4S'^2}{S^2} + \alpha^2S^4 = \frac{1}{3}\omega k^2 \tag{67}$$

We use same method of scalar factor. From equation (60) we get

$$\phi_4\phi^{\frac{n}{2}} = \frac{C_1}{RS^2} \tag{68}$$

Where C_1 is constant

Solving equation (38) and (60) we get same exact solution such as

$$S = k_{17}(k_{15}t + k_{16})^{\frac{6}{(N+2)(k(n+2)+6)}} \tag{69}$$

$$R = k_{18}(k_{15}t + k_{16})^{\frac{6N}{(N+2)(k(n+2)+6)}} \tag{70}$$

$$\phi = k_{19}(k_{15}t + k_{16})^{\frac{2k}{(k(n+2)+6)}} \tag{71}$$

The energy density and pressure are expressed by the following equations.

$$\epsilon = \frac{k_{15}^2}{8\pi(k_{15}t+k_{16})^2}\Big[\frac{1}{(N+2)^2(k(n+2)+6)^2}(36(2N+1) + \frac{(N+2)^2(k(n+2)+6)^2}{k_{17}^2k_{15}^2(k_{15}t+k_{16})^{\frac{12}{(N+2)(k(n+2)+6)}-2}} + \frac{\omega}{2}4k^2(N+2)^2k_{19}^n(k_{15}t+k_{16})^{\frac{2nk-4k}{k(n+2)+6}})\Big] \tag{72}$$

$$p = \frac{k_{15}^2}{24\pi(k_{15}t+k_{16})^2}\Big[\frac{1}{(N+2)^2(k(n+2)+6)^2}(36(2N+1) + \frac{(N+2)^2(k(n+2)+6)^2}{k_{17}^2k_{15}^2(k_{15}t+k_{16})^{\frac{12}{(N+2)(k(n+2)+6)}-2}} + \frac{\omega}{2}4k^2(N+2)^2k_{19}^n(k_{15}t+k_{16})^{\frac{2nk-4k}{k(n+2)+6}})\Big] \tag{73}$$

Using equation (69) and (70) Kantowski-Sach cosmological model (6) take the form

$$ds^2 = dt^2 - [k_{18}(k_{15}t+k_{16})^{\frac{6N}{(N+2)(k(n+2)+6)}}]^2dr^2 - [k_{17}(k_{15}t+k_{16})^{\frac{6}{(N+2)(k(n+2)+6)}}]^2(d\theta^2+\sin^2\theta d\phi^2). \tag{74}$$

6. The physical and kinematical properties

The model's some physical parameters are specified by.

$$V = k_{20}(k_{15}t+k_{16})^{\frac{6}{(k(n+2)+6)}} \tag{75}$$

$$\sigma^2 = \frac{k_{20}k_{15}^2(N-1)^2}{(k_{15}t+k_{16})^2(N+2)^2} \tag{76}$$

$$H = \frac{2k_{15}}{(k(n+2)+6)(k_{15}t+k_{16})} \tag{77}$$

$$\theta = \frac{6k_{15}}{(k(n+2)+6)(k_{15}t+k_{16})} \tag{78}$$

$$A(t) = k_{21}(k_{15}t+k_{16})^{\frac{2}{(k(n+2)+6)}} \tag{79}$$

Using the value of average scale factor of Saez-Ballester theory we obtain,

$$\phi = k_{22}(k_{15}t+k_{16})^{\frac{-12(N+1)}{(N+2)(k(n+2)+6)}} \tag{80}$$

The mathematical formulations for the energy density W, energy flow vector S, and stress tensor $\sigma_{\alpha\beta}$ are as follows

$$W = \left(\frac{1+\frac{\nu^2}{3c^2}}{1-\frac{\nu^2}{c^2}}\right)\frac{k_{15}^2}{(N+2)^2(k(n+2)+6)^2 8\pi(k_{15}t+k_{16})^2}\Big[36(2N+1) + \frac{(N+2)^2(k(n+2)+6)^2}{k_{17}^2k_{15}^2(k_{15}t+k_{16})^{\frac{12}{(N+2)(k(n+2)+6)}-2}} + \frac{\omega}{2}4k^2(N+2)^2k_{19}^n(k_{15}t+k_{16})^{\frac{2k(n-2)}{k(n+2)+6}}\Big] \tag{81}$$

$$S = \frac{4\nu}{3(1-\frac{\nu^2}{c^2})} \frac{k_{15}^2}{(N+2)^2(k(n+2)+6)^2 8\pi(k_{15}t+k_{16})^2}[36(2N+1) + \frac{(N+2)^2(k(n+2)+6)^2}{k_{17}^2 k_{15}^2 (k_{15}t+k_{16})^{\frac{12}{(N+2)(k(n+2)+6)}-2}} + \frac{\omega}{2}4k^2(N+2)^2 k_{19}^n (k_{15}t+k_{16})^{\frac{2k(n-2)}{k(n+2)+6}}] \tag{82}$$

$$\sigma_{\alpha\beta} = \left(\frac{4\nu_\alpha\nu_\beta}{c^2(1-\frac{\nu^2}{c^2})} + \delta_{\alpha\beta}\right) \frac{k_{15}^2}{24\pi(k_{15}t+k_{16})^2(N+2)^2(k(n+2)+6)^2}[36(2N+1) + \frac{(N+2)^2(k(n+2)+6)^2}{k_{17}^2 k_{15}^2 (k_{15}t+k_{16})^{\frac{12}{(N+2)(k(n+2)+6)}-2}} + \frac{\omega}{2}4k^2(N+2)^2 k_{19}^n (k_{15}t+k_{16})^{\frac{2k(n-2)}{k(n+2)+6}}] \tag{83}$$

The physical and kinematical parameters in Brance-Dicke and Saez-Ballester scalar-tensor theory exhibit a high degree of similarity, though with distinct constants. Graphs are constructed to represent specific values of physical parameters and additional integration constants.

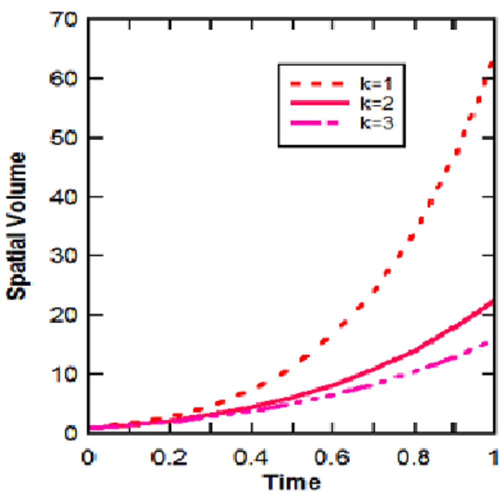

FIGURE 1. Plot of Spatial Volume Vs. Time for $K_7 = K_8 = K_12 = 1$

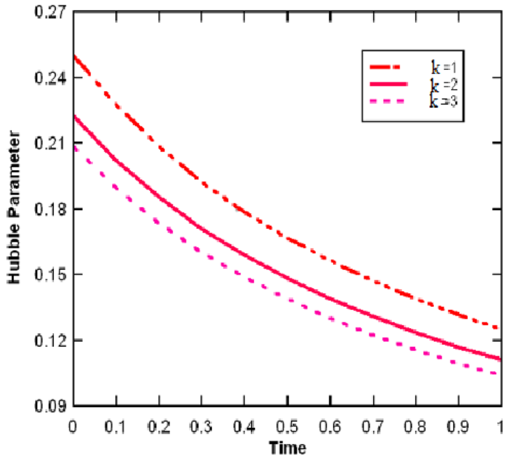

FIGURE 2. Plot of Hubble Parameter Vs. Time for $K_7 = K_8 = 1$

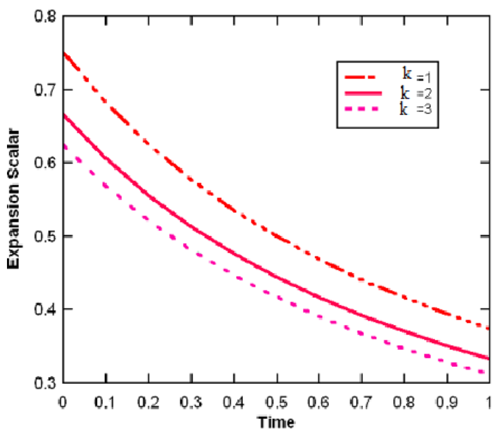

FIGURE 3. Plot of Expansion Scalar Vs. Time for $K_7 = K_8 = 1$

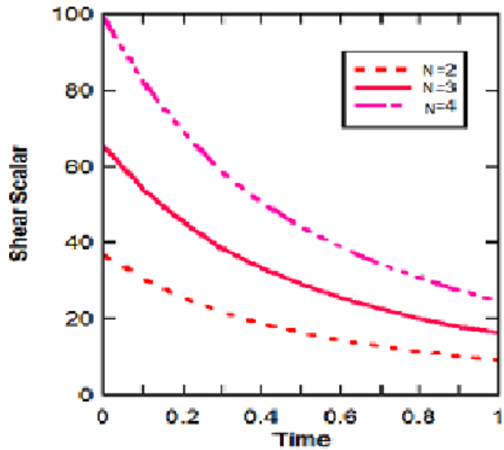

FIGURE 4. Plot of Shear Scalar Vs. Time for $K_7 = K_8 = K_{14} = 1$

7. SOME OBSERVATIONAL PARAMETERS (BRANCE-DICKE AND SAEZ-BALLESTER THEORY

This section focuses on examining the uniformity of our model by utilising observational parameters such as redshift, look-back time, and luminosity distance. This study will focus on the astrophysical significance of the formation of the universe. The term "look-back time" refers to the duration between the current age of the universe t_0 and the age of the universe denoted as t ,when a specific light ray was emitted at a given redshift z It depends on the dynamics of the universe.

$$t_L = t_0 - t, \tag{84}$$

The present age of the universe $t_0$can be determined by measuring the redshift of light from far distant objects, such as galaxies. The letter z is used to denote this redshift, which is a well-measured quantity. The redshift of light is caused by the expansion of the universe.The present scale factor of the universe $a_0(t)$ is related to the average scale factor of the universe $a(t)$for redshift z given by

$$1+z=\frac{a_0(t)}{a(t)}, \tag{85}$$

From equations (49), (51) and (84) gives

$$t_1=\frac{k_7}{H_0(k+3)}[1-\frac{1}{(1+z)^(k+3)}] \tag{86}$$

And, Luminosity distance is given by,

$$H_0dL=\frac{k_7}{(k+2)}[1-(1+z)^{-k+2)}](1+z) \tag{87}$$

Equations (86) and (87) gives the required Look-back time and Luminosity distance in Brans-Dicke theory respectively.

Also, From equations (77), (79) and (84) gives

$$t_L=\frac{2k_{15}}{H_0(k(n+2)+6)}[1-\frac{1}{(1+z)^{\frac{(k(n+2)+6)}{2}}}] \tag{88}$$

And, Luminosity distance is given by,

$$t_L=\frac{2k_{15}}{(k(n+2)+4)}[1-(1+z)^{\frac{-(k(n+2)+4)}{2}}](1+z) \tag{89}$$

Equations (88) and (89) gives the required Look-back time and Luminosity distance in Saez-Ballester theory respectively.

8. Conclusion

The present study examines the Kantowski-Sachs cosmological model within the frameworks of Brans-Dicke and Saez Ballester gravitational theories. while solving the Kantowski-Sachs space-time, a collection of extremely chaotic Einstein field equations was obtained. In order to obtain a precise solution to the field equations, a collection of supplementary constraints were employed, which are to be resolved in accordance with the given initial and boundary values. Specifically, these values must indicate the accurate transmission of limitations. The utilisation of the weak correlation between the scalar field and the mean scale factor A has been employed to derive a precise cosmological model due to the inadequacy of the transformation method utilised. Both theories consider physical parameters, such as spatial volume, directional mean Hubble parameter, expansion scalar, and shear scalar, to be dependent on cosmic time and exhibit similar behaviour. The kinematical properties, namely energy density W, energy flow vector S, and stress tensor $\sigma_{\alpha\beta}$, remain consistent, although with a differing value for the variable ϕ. According to the current model, the volume of the universe is observed to increase as cosmic time progresses, indicating that the universe is undergoing expansion. It is important to point out that with the gradual increase of cosmic time, the scalar expansion, shear scalar, and Hubble parameter exhibit a decrease. Observational parameters are computed for the model that has been acquired. Redshift is utilised to determine the rate at which the universe is expanding. The effects of redshift are extensive, encompassing the origins of the universe, its present behaviour, and its potential fate. The concept of look-back time refers to the temporal gap between the current age of the Universe (at the moment of observation) denoted as t0 and the age of the Universe at the moment of photon emission from the object, denoted as t. Evolutionary models, such as passive stellar evolution, are utilised to forecast the characteristics of high-redshift entities. The luminosity distance is a suitable approximation for the Euclidean notion of distance for objects that are in close range.

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